Solutions Assignment 4

4.35 (a) Assuming that the length of the string is ℓ, then ignoring the size of the pulley the gravitational potential energy is

$$U(x) = -m_1 g x - m_2 g (\ell - x).$$

The kinetic energy including the rotational energy of the pulley is

$$\begin{aligned} T &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I \omega^2, \\ T &= \frac{1}{2} \left(m_1 + m_2 + I/R^2 \right) \dot{x}^2. \end{aligned}$$

In this last step we included the nonslip condition, $\omega = \dot{x}/R$. Hence, the total energy is

$$E = \frac{1}{2} \left(m_1 + m_2 + I/R^2 \right) \dot{x}^2 - m_1 g x - m_2 g (\ell - x).$$

(b) To find the EOM we differentiate this expression and find

$$\begin{aligned} \left(m_1 + m_2 + I/R^2 \right) \dot{x}\ddot{x} &= (m_1 g - m_2 g) \dot{x}, \\ \left(m_1 + m_2 + I/R^2 \right) \ddot{x} &= (m_1 - m_2) g. \end{aligned}$$

The three separate EOM are

$$\begin{aligned} m_1 \ddot{x} &= m_1 g - T_R, \\ m_2 \ddot{x} &= -m_2 g + T_L, \\ I \ddot{\phi} &= (T_R - T_L) R. \end{aligned}$$

Since $\ddot{\phi} = \ddot{x}/R$, dividing the last equation by R we see that if we add them up we obtain the expression we obtained by differentiating the expression for the conservation of energy.

4.36 (a) The potential energy for this arrangement of masses and pulley is

$$U = -mgh - MgH.$$

The total length of the string is given by $\ell = H + \sqrt{b^2 + h^2}$, where $\sqrt{b^2 + h^2} = b/\sin\theta$ and $h = b/\tan\theta$. This allows us to rewrite the potential as a function of θ,

$$U(\theta) = -mgb/\tan\theta - Mg(\ell - b/\sin\theta).$$

Ignoring the constant $(-Mg\ell)$ we can write the potential as

$$U(\theta) = gb(M/\sin\theta - m/\tan\theta).$$

(b) To find a position of possible equilibrium we take the derivative of $U(\theta)$ and find

$$\frac{dU(\theta)}{d\theta} = gb\left(-\frac{M\cos\theta}{\sin^2\theta} + \frac{m}{\tan^2\theta}\frac{1}{\cos^2\theta}\right) = \frac{gb}{\sin^2\theta}(m - M\cos\theta).$$

The condition for equilibrium is

$$\cos\theta = m/M.$$

The equilibrium condition can only be satisfied if $m \leq M$.. When $m = M$, equilibrium occurs at $\theta = 0$ which requires an infinitely long string. However if $m < M$ then the equilibrium condition is possible when $\theta = \cos^{-1} m/M$.

(c) Simply balancing the vertical forces for each of the masses leads to

$$Mg = T, \text{ and } mg = T\cos\theta.$$

Substituting for T yields

$$\cos\theta = m/M,$$

which is consistent with the solution in (b).

4.37 (a) To within a constant (that depends on the length of the string) the potential energy relative to the position at $\phi = 0$ is

$$U(\phi) = MgR(1 - \cos\phi) - mgR\phi.$$

(b) Equilibrium occurs when

$$\frac{dU(\phi)}{d\phi} = MgR\sin\phi - mgR = 0 \rightarrow \sin\phi = m/M.$$

As in problem 4.36 this can only occur if $m < M$. To determine the stability we examine $d^2U/d\phi^2$ at the equilibrium position. This results in

$$\frac{d^2U\left(\phi = \sin^{-1} m/M\right)}{d\phi^2} = MgR\cos\left(\sin^{-1} m/M\right) = \pm gR\sqrt{M^2 - m^2}.$$

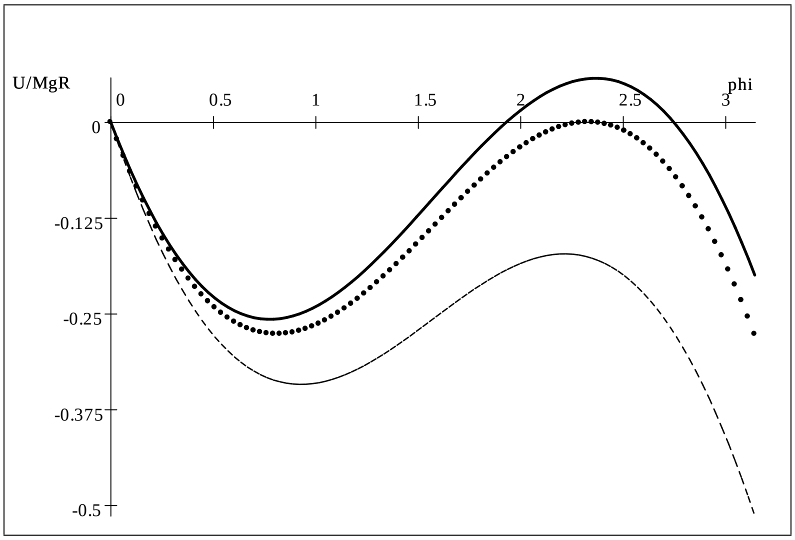
The plus sign applies when ϕ_+ is in the first quadrant, while the minus sign means that ϕ_- is in the second quadrant, i.e. $\phi_- = \pi - \phi_+$.

Determining equilibrium from balancing torques (about the axle),

$$MgR\sin\phi = mgR \rightarrow \sin\phi = m/M,$$

yields the identical result.

(c) The plots of the potential energy, U/MgR, for $m/M = .7$ (solid line) and $m/M = .8$ (dashed line)are shown below.



If the system is released from $\phi = 0$ for the case $m/M = .7$, the wheel passes through equilibrium and comes to rest at $\phi \approx 1.9rad$ and returns to $\phi = 0$. For a heavier m, $m/M = .8$, the wheel passes through equilibrium and passes over the top, $\phi = \pi$, and continues on.

(d) The critical value of m/M corresponds to the solution for unstable equilibrium occurring when $U = 0$. This relation is given by

$$\begin{aligned} U\left(\phi_-\right) &= MgR\left(1-\cos\phi_-\right) - mgR\phi_- \\ U\left(\phi_-\right) &= MgR\left(1+\sqrt{1-m^2/M^2} - \left(\pi - \sin^{-1} m/M\right) m/M\right) = 0 \end{aligned}$$

The solution to this transcendental equation is $m/M = .7246$. The plot of the potential energy at this value of m/M is shown above as a dotted line.

4.39 (a) The energy for an oscillating pendulum is

$$E = \frac{1}{2}m\left(\ell\dot{\phi}\right)^2 + mg\ell\left(1-\cos\phi\right) = \frac{1}{2}m\left(\ell\dot{\phi}\right)^2 + 2mg\ell\sin^2\phi/2.$$

The maximum amplitude for ϕ, $\phi = \Phi$, occurs when $\dot{\phi}$ is zero and $E = 2mg\ell\sin^2\Phi/2$. Solving for $\dot{\phi}$ we find

$$\begin{aligned} \dot{\phi}^2 &= 4\frac{g}{\ell}\left(\sin^2\Phi/2 - \sin^2\phi/2\right), \\ \dot{\phi} &= 2\sqrt{\frac{g}{\ell}\left(\sin^2\Phi/2 - \sin^2\phi/2\right)}. \end{aligned}$$

Integrating this equation we find

$$\int dt = t = \frac{1}{2}\sqrt{\frac{\ell}{g}}\int_0^{\phi}\frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}}.$$

Since the period is four times the time it takes for ϕ to go from 0 to Φ we have

$$\tau = 2\sqrt{\frac{\ell}{g}}\int_0^\Phi \frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}} = \frac{\tau_o}{\pi}\int_0^\Phi \frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}},$$

where as we saw in problem 4.34, $\tau_o = 2\pi\sqrt{\ell/g}$. Using the substitution $Au = u\sin\Phi/2 = \sin\phi/2$, yields

$$Adu = \frac{d\phi}{2}\cos\phi/2 = \frac{d\phi}{2}\sqrt{1-\sin^2\phi/2} = \frac{d\phi}{2}\sqrt{1-A^2u^2}.$$

The period can then be expressed as

$$\tau = \frac{2\tau_o}{\pi}\int_0^1 \frac{du}{\sqrt{1-A^2u^2}\sqrt{1-u^2}} = \frac{2\tau_o}{\pi}K\left(A^2\right),$$

where K is a *complete elliptic integral of the first kind*. The values of K are tabulated.

(b) In the limit $\Phi << 1$, then the expression for τ becomes

$$\begin{aligned}\tau &= 2\sqrt{\frac{\ell}{g}}\int_0^\Phi \frac{d\phi}{\sqrt{\sin^2\Phi/2 - \sin^2\phi/2}} \simeq 2\sqrt{\frac{\ell}{g}}\int_0^\Phi \frac{2d\phi}{\sqrt{\Phi^2-\phi^2}} \\ \tau &\simeq 4\sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\frac{\Phi\cos\theta d\theta}{\Phi\cos\theta} = 2\pi\sqrt{\frac{\ell}{g}}.\end{aligned}$$

Here we have used the substitution $\phi = \Phi\sin\theta$ to perform the integration. The period in this limit is the same as that in the small amplitude pendulum problem.

(c) If the amplitude is small but not very small we can assume

$$\frac{1}{\sqrt{1-A^2u^2}} \simeq 1 + \frac{1}{2}A^2u^2.$$

The integral for the period becomes reduces to

$$\tau \simeq \frac{2\tau_o}{\pi}\int_0^1 \frac{du}{\sqrt{1-u^2}} + \frac{2\tau_o}{\pi}\frac{A^2}{2}\int_0^1 \frac{u^2du}{\sqrt{1-u^2}}.$$

Making the usual substitution $u = \sin\theta$, we find

$$\tau \simeq \frac{2\tau_o}{\pi}\int_0^{\pi/2} d\theta + \frac{\tau_o}{\pi}A^2\int_0^{\pi/2}\sin^2\theta d\theta = \tau_o\left(1 + \frac{1}{4}\sin^2\Phi/2\right).$$

This shows that the correction to the period for small amplitudes is second order. For example when $\Phi = \pi/4$ we find $\tau = 1.037\tau_o$ as compared to the numerical result $\tau = 1.040\tau_o$. This shows that the expansion is very accurate even for Φ as large as $\pi/4$.

4.46 In an elastic collision between two masses, m_1 and m_2, both the kinetic energy and momentum are conserved. This is expressed as

$$\begin{aligned} m_1 v_1^2 &= m_1 v_1'^2 + m_2 v_2'^2, \\ m_1 \vec{v}_1 &= m_1 \vec{v}'_1 + m_2 \vec{v}'_2, \end{aligned}$$

where the prime denotes the velocities after the collision. Taking the scaler product of the conservation of momentum with itself we find

$$m_1^2 v_1^2 = m_1^2 v_1'^2 + m_2^2 v_2'^2 + 2 m_1 m_2 \vec{v}'_1 \cdot \vec{v}'_2.$$

Subtracting the conservation of energy (after multiplying through by m_1) and solving for $\vec{v}'_1 \cdot \vec{v}'_2$ yields

$$\vec{v}'_1 \cdot \vec{v}'_2 = \frac{m_1 - m_2}{2m_1} v_2'^2.$$

From this expression we see that if $m_1 > m_2$ then $\cos\theta > 0$ and $\theta < \pi/2$. On the other hand if $m_1 < m_2$ then $\cos\theta < 0$ and $\theta > \pi/2$.

4.47 (i) There are a couple of ways to do this. I prefer transforming the system to a frame in which the center of mass is stationary, the CM frame. The velocity of this frame is

$$v_{cm} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}.$$

The initial velocities of the two particles in the center of mass frame are

$$u_1 = v_1 - v_{cm} \quad \text{and} \quad u_2 = v_2 - v_{cm}.$$

Note that the total momentum of the two particles vanishes in this frame ($m_1 u_1 + m_2 u_2 = 0$). In an elastic collision the kinetic energy as well as the momentum is conserved (in all frames). Hence in the center of mass frames the velocities in the center of mass frame are simply reversed from their values prior to the collision as this conserves both energy and momentum. Hence,

$$u_1' = -u_1 = v_{cm} - v_1 \quad \text{and} \quad u_2' = -u_2 = v_{cm} - v_2.$$

Transforming back to the original frame requires that we add v_{cm} to these velocities with the result

$$v_1' = 2v_{cm} - v_1 \quad \text{and} \quad v_2' = 2v_{cm} - v_2.$$

If we examine the relative velocities after the collision we find

$$v_1' - v_2' = -\left(v_1 - v_2\right),$$

which is the desired result.

(ii) The usual approach is to simultaneously satisfy conservation of momentum and energy,

$$\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1 v_1' + m_2 v_2' \\ m_1 v_1^2 + m_2 v_2^2 &= m_1 v_1'^2 + m_2 v_2'^2 . \end{aligned}$$

Rearranging these equations so that all of the terms involving m_1 are on one side and those with m_2 are on the opposite side leaves

$$\begin{aligned} m_1 \left(v_1 - v_1'\right) &= m_2 \left(v_2' - v_2\right) \\ m_1 \left(v_1^2 - v_1'^2\right) &= m_2 \left(v_2'^2 - v_2^2\right) . \end{aligned}$$

Now dividing the expression that satisfies the conservation of momentum into that which satisfies the conservation of energy yields

$$\begin{aligned} v_1 + v_1' &= v_2' + v_2 \quad \text{or} \\ v_1' - v_2' &= -\left(v_1 - v_2\right) , \end{aligned}$$

which is the desired result.

The primary difference in these two approaches is that the former yields the solutions for v_1' and v_2' prior to determining the relationship between the relative velocities. In the second approach all we have learned is that one relationship.

5.4 Assume that the length of the string when it is hanging freely from the edge of the cylinder (the equilibrium position) is ℓ_o. Define the angle between the horizontal and a radial line to the point where the string breaks contact with the cylinder to be ϕ. This is the same as the angle that the string makes with the vertical beyond the contact point. The length of the string beyond the contact point is $\ell = \ell_o - R\phi$. The height below the contact point is $\ell \cos\phi = (\ell_o - R\phi)\cos\phi$. Additionally the contact point is $R\sin\phi$ below the equilibrium position at $\phi = 0$. Defining $U = 0$ to correspond to the elevation of the mass at equilibrium, the potential energy of the mass as a function of ϕ is

$$U(\phi) = -mg\left(\ell_o - R\phi\right)\cos\phi - mgR\sin\phi - \left(-mg\ell_o\right) .$$

When $\phi << 1$ we can expand this potential up to second order (ignore third order and higher order terms) in ϕ as

$$\begin{aligned} \lim_{\phi << 1} U(\phi) &= -mg\left(\ell_o - R\phi\right)\left(1 - \phi^2/2\right) - mgR\phi + mg\ell_o , \\ \lim_{\phi << 1} U(\phi) &= mg\ell_o \phi^2/2. \end{aligned}$$

The first derivative of this term shows that equilibrium occurs at $\phi = 0$ (which we had already assumed) and the second derivative results in an effective spring constant of $k = mg\ell_o$. The expansion to this order in ϕ results in no change in the usual pendulum equation.

5.13 Given the potential

$$U(r) = U_o \left(\frac{r}{R} + \lambda^2 \frac{R}{r} \right),$$

the equilibrium position, r_o, is found from

$$\begin{aligned} \frac{dU}{dr} &= U_o \left(\frac{1}{R} - \lambda^2 \frac{R}{r_o^2} \right) = 0, \\ r_o^2 &= \lambda^2 R^2 \to r_o = \lambda R. \end{aligned}$$

Since r is positive definite, this is the only answer ($-r_o$ is not a solution). The second derivative evaluated at equilibrium is

$$\frac{d^2U}{dr^2} = 2\lambda^2 U_o \frac{R}{r_o^3} = 2\frac{U_o}{\lambda R^2}.$$

For small displacements from equilibrium the potential is given by

$$U(r) = 2\lambda U_o + \frac{1}{2} 2\frac{U_o}{\lambda R^2} (r - \lambda R)^2 + \cdots.$$

Defining the displacement from equilibrium as $x = r - \lambda R$ the potential reduces to

$$U(r) = 2\lambda U_o + \frac{1}{2} k x^2 + \cdots,$$

where $k = 2U_o / \left(\lambda R^2\right)$. Thus the angular frequency is $\omega = \sqrt{k/m} = \sqrt{2U_o/(\lambda m R^2)}$.

Virial Theorem for two interacting particles For two particles the virial G is given by

$$G = \overrightarrow{p}_1 \cdot \overrightarrow{r}_1 + \overrightarrow{p}_2 \cdot \overrightarrow{r}_2.$$

Again we take the time derivative to find

$$\begin{aligned} \frac{dG}{dt} &= \dot{\overrightarrow{p}}_1 \cdot \overrightarrow{r}_1 + \overrightarrow{p}_1 \cdot \dot{\overrightarrow{r}}_1 + \dot{\overrightarrow{p}}_2 \cdot \overrightarrow{r}_2 + \overrightarrow{p}_2 \cdot \dot{\overrightarrow{r}}_2. \\ \frac{dG}{dt} &= \overrightarrow{F}_1 \cdot \overrightarrow{r}_1 + 2T_1 + \overrightarrow{F}_2 \cdot \overrightarrow{r}_2 + 2T_2 \end{aligned}$$

Now we assume that the only interaction is that between the two particles and that this interaction conservative. Hence

$$\begin{aligned} \overrightarrow{F}_1 &= \overrightarrow{F}_{12} = -\nabla_1 U^{int} \left(\overrightarrow{r}_1 - \overrightarrow{r}_2 \right) \quad \text{and} \\ \overrightarrow{F}_2 &= \overrightarrow{F}_{21} = -\overrightarrow{F}_{12} = -\nabla_2 U^{int} \left(\overrightarrow{r}_1 - \overrightarrow{r}_2 \right). \end{aligned}$$

We can now rewrite the time derivative of the virial as

$$\frac{dG}{dt} = \overrightarrow{F}_{12} \cdot \left(\overrightarrow{r}_1 - \overrightarrow{r}_2 \right) + 2T_1 + 2T_2 = -\nabla_1 U^{int} \left(\overrightarrow{r}_1 - \overrightarrow{r}_2 \right) \cdot \left(\overrightarrow{r}_1 - \overrightarrow{r}_2 \right) + 2T,$$

where $T = T_1 + T_2$. Defining the relative coordinate vector $\overrightarrow{r} = \overrightarrow{r}_1 - \overrightarrow{r}_2$ the above expression becomes

$$\frac{dG}{dt} = -\nabla U^{int}\left(\overrightarrow{r}\right) \cdot \overrightarrow{r} + 2T.$$

Again if the interaction is additionally central and of the form

$$U^{int} = kr^n.$$

Again we have

$$\frac{dG}{dt} = -nkr^n + 2T = -nU^{int} + 2T.$$

Taking the time average as we did before and noting that for bound orbits G is bounded leads to

$$\langle T \rangle = \frac{n}{2}\left\langle U^{int} \right\rangle.$$

This expression is identical to that found for a single particle, only now U^{int} is the potential associated with the interaction and T is the sum of the kinetic energies.