Solutions Assignment 6

6.16 Using the hint given in the problem, the distance between two points on a sphere of radiuis R is

$$L = \int_1^2 \sqrt{R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2} = R\int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \phi'^2} d\theta.$$

Since in this form the integrand is independent of ϕ the Euler-Lagrange equation reduces to

$$\frac{\partial}{\partial \phi'}\sqrt{1 + \sin^2\theta\phi'^2} = \frac{\sin^2\theta\phi'}{\sqrt{1 + \sin^2\theta\phi'^2}} = \alpha.$$

Now with full generality we can assume that the curve originates at $\theta = 0$, i.e. choose the initial point to be the North Pole. This forces $\alpha = 0$. Now for arbitrary θ the equation reduces to $\phi' = 0$ or

$$\phi = \phi_o.$$

This is corresponds to a line of longitude all of which are great circles.

6.18 Solution (i): In plane polar coordinates the distance between points 1 and 2 is

$$L = \int_1^2 \sqrt{dr^2 + r^2 d\phi^2} = \int_1^2 \sqrt{1 + r^2\phi'^2} dr.$$

Since the integrand is independent of ϕ, the Euler-Lagrange equation reduces to

$$\frac{\partial}{\partial \phi'}\sqrt{1 + r^2\phi'^2} = \frac{r^2\phi'}{\sqrt{1 + r^2\phi'^2}} = -a.$$

Solving for ϕ' yields

$$\phi' = -\frac{a}{r\sqrt{r^2 - a^2}} \rightarrow \phi - \phi_o = -\int \frac{a}{r\sqrt{r^2 - a^2}} dr.$$

Let $r = a/\sin\theta$ so that $r^2 - a^2 = a^2\cot^2\theta$ and $dr = -a\cos\theta d\theta/\sin^2\theta$. The integral now becomes

$$\phi - \phi_o = \int \frac{\sin\theta}{a^2\cot\theta}\frac{a^2\cos\theta}{\sin^2\theta} d\theta = \int d\theta = \theta.$$

Taking the sine of both sides yields

$$r\sin\theta = r\sin(\phi - \phi_o) = a.$$

Expanding $\sin(\phi - \phi_o)$ and recognizing that $x = r\cos\phi$ and $y = r\sin\phi$ yields

$$y\cos\phi_o - x\sin\phi_o = a \rightarrow y = x\tan\phi_o + a/\cos\phi_o = x\tan\phi_o + b,$$

which is the equation of a straight line.

Solution (ii):

$$L=\int_1^2\sqrt{dr^2+r^2d\phi^2}=\int_1^2\sqrt{r'^2+r^2}d\phi.$$

Since the integrand is independent of ϕ we can use the first integral of the resulting Euler Lagrange equation with the result,

$$\begin{aligned}
&\sqrt{r'^2+r^2}-r'\frac{\partial}{\partial r'}\sqrt{r'^2+r^2}=\sqrt{r'^2+r^2}-\frac{r'^2}{\sqrt{r'^2+r^2}}=-1/a\\
&r^2=a\sqrt{r'^2+r^2}\rightarrow r^2\left(r^2-a^2\right)=a^2r'^2\rightarrow\frac{d\phi}{dr}=-\frac{a}{r\sqrt{r^2-a^2}}\\
&\phi-\phi_o=-\int\frac{adr}{r\sqrt{r^2-a^2}}.
\end{aligned}$$

This is the same integral obtained in part (i).

6.22 To find the maximum area enclosed above the x axis by a string of length ℓ that originates at the origin and teminates somewhere along the positive x axis we consider the integral

$$A=\int y\left(x\right)dx.$$

But we need to rewrite this integral in terms of $ds=\sqrt{dx^2+dy^2}$ with the length of the string fixed at ℓ. Solving for dx yields

$$dx=\sqrt{ds^2-dy^2}=\sqrt{1-\left(dy/ds\right)^2}ds.$$

Now y is a function of s and the area integral is

$$A=\int_0^\ell y\left(s\right)\sqrt{1-\left(dy/ds\right)^2}ds=\int_0^\ell y\sqrt{1-y'^2}ds.$$

Now $f\left(y,y'\right)$ is independent of s, hence we can exploit the first integral expression for the Euler Lagrange equation with the result

$$\begin{aligned}
&f-y'\frac{\partial f}{\partial y'}=y\sqrt{1-y'^2}+y'y\frac{y'}{\sqrt{1-y'^2}}=a\\
&y\left(1-y'^2\right)+yy'^2=y=a\sqrt{1-y'^2}\\
&a^2-y^2=a^2\left(dy/ds\right)^2\rightarrow\sqrt{a^2-y^2}=ady/ds\\
&\frac{ady}{\sqrt{a^2-y^2}}=ds.
\end{aligned}$$

Define $y=a\sin\theta\rightarrow dy=a\cos\theta d\theta$ and the integral becomes

$$s-s_o=\int\frac{a^2\cos\theta d\theta}{a\cos\theta}=a\theta\rightarrow a\sin\frac{s-s_o}{a}=a\sin\theta=y.$$

Since the curve starts at the origin ($x = y = s = 0$), $s_o = 0$ and the expression for $y(s)$ becomes

$$y = a \sin \frac{s}{a}.$$

Hence the curve not only intersects the x axis at the origin but again when $s = \pi a = \ell$. Now

$$\begin{aligned} \left(\frac{dy}{ds}\right)^2 &= 1 - \left(\frac{dx}{ds}\right)^2 = \cos^2 \frac{s}{a} \\ \frac{dx}{ds} &= \sin \frac{s}{a}. \end{aligned}$$

To satisfy the condition $x(s = 0) = 0$ we find $x = a(1 - \cos s/a)$. This means that $(x - a)^2 + y^2 = a^2 = \ell^2/\pi^2$ and in general $s = a\theta$. So this curve is a semicircle of radius a centered at $x = a$ with a length $\ell = \pi a$. Note that we are measuring θ as positive in the clock wise direction starting from a radial line that originates from $(a, 0)$ and ending at the origin.

6.25 The time to fall from from rest at P_0 toe the bottom of the cycloid, P, is

$$T = \int_0^P \frac{\sqrt{dx^2 + dy^2}}{v} = \frac{1}{\sqrt{2g}} \int_0^P \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y - y_0}}.$$

For the cycloid

$$y = a(1 - \cos\theta), \text{ and } x = a(\theta - \sin\theta),$$

so that

$$dy = a \sin\theta d\theta, \text{ and } dx = a(1 - \cos\theta) d\theta.$$

Substituting these results into the integral for the time yields

$$\begin{aligned} T &= \frac{1}{\sqrt{2g}} \int_{\theta_0}^{\pi} \frac{a\sqrt{(1 - \cos\theta)^2 + \sin^2\theta}}{\sqrt{a}\sqrt{(1 - \cos\theta) - (1 - \cos\theta_0)}} d\theta \\ T &= \sqrt{\frac{a}{2g}} \int_{\theta_0}^{\pi} \frac{\sqrt{2 - 2\cos\theta}}{\sqrt{\cos\theta_0 - \cos\theta}} d\theta = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1 - \cos\theta}}{\sqrt{\cos\theta_0 - \cos\theta}} d\theta. \end{aligned}$$

Rewritting the integral as

$$T = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{1 - \cos\theta}}{\sqrt{(1 + \cos\theta_0) - (1 + \cos\theta)}} d\theta,$$

and making use of half angle relations yields

$$T = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sqrt{\sin^2\theta/2}}{\sqrt{\cos^2\theta_0/2 - \cos^2\theta/2}} d\theta = \sqrt{\frac{a}{g}} \int_{\theta_0}^{\pi} \frac{\sin\theta/2}{\sqrt{\cos^2\theta_0/2 - \cos^2\theta/2}} d\theta.$$

Define $u = \cos\theta/2 \rightarrow 2du = -(\sin\theta/2)\, d\theta$. With this substitution we find

$$T = \sqrt{\frac{a}{g}} \int_{u_0}^{0} \frac{-2}{\sqrt{u_0^2 - u^2}} du = 2\sqrt{\frac{a}{g}} \int_{0}^{u_0} \frac{du}{\sqrt{u_0^2 - u^2}}.$$

With one more change in variables, $u = u_0 \sin\phi \rightarrow du = u_0 \cos\phi d\phi$, we find

$$T = 2\sqrt{\frac{a}{g}} \int_0^{\pi/2} \frac{u_0 \cos\phi d\phi}{u_0 \cos\phi} = \pi\sqrt{\frac{a}{g}}.$$

This result is independent of θ_0. So the period is constant whereas with a simple pendulum the period is only constant for small amplitude oscillations.

7.3 The Lagrangian for this problem is

$$\mathcal{L} = T - U = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - \frac{1}{2} k \left(x^2 + y^2 \right).$$

The two Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} &= m\ddot{x} + kx = 0 \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} &= m\ddot{y} + ky = 0. \end{aligned}$$

This represents periodic oscillations of the same period in both the x and y directions. The resulting motion will be ellipses like those shown in figure 5.8.

7.8 (a) The potential energy of the spring with natural length ℓ in this configuration is

$$U = \frac{1}{2} k \left(x_1 - x_2 - \ell \right)^2.$$

Hence the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) - \frac{1}{2} k \left(x_1 - x_2 - \ell \right)^2$$

(b,c) Consider the variables

$$X = \frac{1}{2}(x_1 + x_2) \text{ and } x = x_1 - x_2 - \ell.$$

The time derivatives are

$$2\dot{X} = \dot{x}_1 + \dot{x}_2, \text{ and } \dot{x} = \dot{x}_1 - \dot{x}_2.$$

Solving for $\dot{x}_1$ and $\dot{x}_2$ in terms of $\dot{X}$ and $\dot{x}$ results in

$$\dot{x}_1 = \dot{X} + \dot{x}/2, \text{ and } \dot{x}_2 = \dot{X} - \dot{x}/2.$$

Substituting these coordinates into the Lagrangian yields

$$\mathcal{L} = \frac{1}{2}m\left(\left(\dot{X}+\dot{x}/2\right)^2+\left(\dot{X}-\dot{x}/2\right)^2\right)-\frac{1}{2}kx^2$$

$$\mathcal{L} = \frac{1}{2}m\left(2\dot{X}^2+\dot{x}^2/2\right)-\frac{1}{2}kx^2=\frac{1}{2}\left(2m\dot{X}^2+\frac{m}{2}\dot{x}^2\right)-\frac{1}{2}kx^2.$$

The two Lagrange equations and their solutions are

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{X}}-\frac{\partial\mathcal{L}}{\partial X} = 2m\ddot{X}=0\rightarrow\dot{X}=const$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}-\frac{\partial\mathcal{L}}{\partial x} = \frac{m}{2}\ddot{x}+kx=0\rightarrow x=A\cos\left(\omega t-\delta\right)\ \text{where}\ \omega^2=2k/m.$$

The coordinate X is the center of mass coordinate. Since there are no external forces (only the internal interaction of the spring) the momentum and/or velocity of the center of mass is constant. On the other hand the coordinate for the extension of the spring (relative coordinate between the two masses) undergoes simple harmonic oscillation with an angular frequency resulting from $\omega^2 = 2k/m$. To understand where the factor of 2 comes from we first note that in the frame of the center of mass the center of the spring is fixed. If either mass moves by an amount δx then the mass on the other end of the spring must move in the opposite direction an amount $-\delta x$. Hence the spring is stretched twice the amount that either mass moves. Hence the effective spring constant $2k$.

7.10 For the cone the z coordinate is expressed as $\rho = z\tan\alpha$. Hence the Cartesian coordinates for an object confined to the surface of this cone are

$$x=\rho\cos\phi,\ y=\rho\sin\phi,\ z=\rho/\tan\alpha.$$

The coordinates ρ and ϕ are

$$\rho=\sqrt{x^2+y^2}=z\tan\alpha,\ \phi=\tan^{-1}y/x.$$

7.14 The moment of inertia for a uniform disk rotating about its center is $I = mR^2/2$. The kinetic energy for the disk is

$$T=\frac{1}{2}m\dot{x}^2+\frac{1}{2}I\omega^2=\frac{1}{2}m\dot{x}^2+\frac{1}{4}mR^2\omega^2=\frac{3}{4}m\dot{x}^2,$$

where we have used the nonslip condition for the yoyo, $\dot{x} = R\omega$. The potential energy of the yoyo is $U = -mgx$. Hence the Lagrangian is

$$\mathcal{L}=\frac{3}{4}m\dot{x}^2+mgx,$$

with the resulting Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}}-\frac{\partial\mathcal{L}}{\partial x}=\frac{3}{2}m\ddot{x}-mg=0\rightarrow\ddot{x}=2g/3.$$

Integrand of the form $f(y, y', y'', x)$ Assume that we want to find the path for which the integral, I, is stationary, where the integrand depends on y, y', and y'', as well as the independent parameter x,

$$I = \int_1^2 f(y, y', y'', x)\, dx.$$

Again if we have a small deviation, $\eta(x)$, from the stationary path then

$$\delta I = \int_1^2 \left[\frac{\partial}{\partial y} f(y, y', y'', x)\, \eta(x) + \frac{\partial}{\partial y'} f(y, y', y'', x)\, \eta'(x) + \frac{\partial}{\partial y''} f(y, y', y'', x)\, \eta''(x) \right] dx = 0.$$

The derivation here for the first two terms is identical to that shown in class. So we will concentrate on the third term. Integrating the third terms by parts yields

$$\begin{aligned} \int_1^2 \frac{\partial}{\partial y''} f(y, y', y'', x)\, \eta''(x)\, dx &= -\int_1^2 \left(\frac{d}{dx} \frac{\partial}{\partial y''} f(y, y', y'', x) \right) \eta'(x)\, dx \\ &\quad + \left[\frac{\partial}{\partial y''} f(y, y', y'', x)\, \eta'(x) \right]_{x_1}^{x_2}. \end{aligned}$$

Since the constraints now include

$$\eta'(x_1) = \eta'(x_2) = 0,$$

we see that the last term in this equation vanishes and we are left with

$$\begin{aligned} \delta I &= \int_1^2 \left[\frac{\partial}{\partial y} f(y, y', y'', x) - \left(\frac{d}{dx} \frac{\partial}{\partial y'} f(y, y', y'', x) \right) \right] \eta(x)\, dx \\ &\quad - \int_1^2 \frac{d}{dx} \frac{\partial}{\partial y''} f(y, y', y'', x)\, \eta'(x)\, dx. \end{aligned}$$

Integrating the last term by parts one more time subject to the usual constraints,

$$\eta(x_1) = \eta(x_2) = 0,$$

yields

$$\delta I = \int_1^2 \left[\frac{\partial}{\partial y} f(y, y', y'', x) - \frac{d}{dx} \frac{\partial}{\partial y'} f(y, y', y'', x) + \frac{d^2}{dx^2} \frac{\partial}{\partial y''} f(y, y', y'', x) \right] \eta(x)\, dx = 0.$$

Since $\eta(x)$ is arbitrary we have the expression

$$\frac{\partial}{\partial y} f(y, y', y'', x) - \frac{d}{dx} \frac{\partial}{\partial y'} f(y, y', y'', x) + \frac{d^2}{dx^2} \frac{\partial}{\partial y''} f(y, y', y'', x) = 0$$

as the equation that f must satisfy in order for I to be a “stationary” integral.