Solutions Assignment 9

Verify the relations in (8.52) The semimajor axis is found from

$$\begin{aligned} 2a &= r_{\max} + r_{\min} = \frac{r_o}{1-\epsilon} + \frac{r_o}{1+\epsilon} = \frac{2r_o}{1-\epsilon^2} \\ a &= \frac{r_o}{1-\epsilon^2} \end{aligned}$$

The semiminor axis is found from

$$b = y_{\max}.$$

Since $y = r\sin\phi$ the maximum value for y is determined by setting $dy/d\phi = 0$. We find

$$\begin{aligned} \frac{dy}{d\phi} &= r_o \frac{d}{d\phi}\frac{\sin\phi}{1+\epsilon\cos\phi} = r_o\left(\frac{\cos\phi}{1+\epsilon\cos\phi} + \frac{\epsilon\sin^2\phi}{(1+\epsilon\cos\phi)^2}\right) \\ \frac{dy}{d\phi} &= r_o\left(\frac{\cos\phi\,(1+\epsilon\cos\phi) + \epsilon\left(1-\cos^2\phi\right)}{(1+\epsilon\cos\phi)^2}\right) = r_o\left(\frac{\cos\phi+\epsilon}{(1+\epsilon\cos\phi)^2}\right). \end{aligned}$$

Setting this expression to zero yields $\cos\phi_o = -\epsilon$. Hence

$$y_{\max} = \frac{r_o\sin\phi_o}{1+\epsilon\cos\phi_o} = \frac{r_o\sqrt{1-\epsilon^2}}{1-\epsilon^2} = \frac{r_o}{\sqrt{1-\epsilon^2}} = b$$

The offset, d, is simply

$$\begin{aligned} d &= r_{\max} - a = \frac{r_o}{1-\epsilon} - \frac{r_o}{1-\epsilon^2} \\ d &= \frac{r_o}{1-\epsilon}\left(1 - \frac{1}{1+\epsilon}\right) = \frac{\epsilon r_o}{1-\epsilon^2} = \epsilon a. \end{aligned}$$

8.19 At perigee and apogee respectively,

$$r_p = y_p + R_e = \frac{\ell^2/GM_e}{1+\epsilon}, \text{ and } r_a = y_a + R_e = \frac{\ell^2/GM_e}{1-\epsilon}.$$

Solving for the eccentricity we find

$$\begin{aligned} \frac{r_p}{r_a} &= \frac{1-\epsilon}{1+\epsilon} \to (1+\epsilon)\, r_p/r_a = 1-\epsilon \\ \epsilon &= \frac{1-r_p/r_a}{1+r_p/r_a} = \frac{r_a-r_p}{r_a+r_p} = \frac{y_a-y_p}{y_a+y_p+2R_e} \\ \epsilon &= \frac{2700}{3300+2\times 6400} = .1677 \end{aligned}$$

From the statement of the problem, as the satellite crosses the y axis $\phi = \pi/2$ and

$$\begin{aligned} r_y &= y + R_e = \ell^2/GM_e = (1+\epsilon)(y_p + R_e) \\ y &= (1+\epsilon)(y_p + R_e) - R_e = (1+\epsilon)\, y_p + \epsilon R_e \\ y &= 1.1677 \times 300 + .1677 \times 6400 = 1424km \end{aligned}$$

8.20 The expression for the orbit is

$$r(\phi) = \frac{\ell^2/GM_\odot}{1 + \epsilon\cos\phi}.$$

At apogee

$$r_{\rm max} = \frac{\ell^2/GM_\odot}{1-\epsilon} \rightarrow r_{\rm max}(1-\epsilon) = \ell^2/GM_\odot.$$

Therefore if we hold $r_{\rm max}$ fixed and let $\ell \rightarrow 0$ then ϵ must approach 1. For $\epsilon = 1$,

$$r_{\rm min} = \frac{\ell^2/GM_\odot}{1+\epsilon} = \ell^2/2GM_\odot.$$

As $\ell \rightarrow 0$, $r_{\rm min} \rightarrow 0$ as well. With this analysis it is clear that if $r_{\rm max}$ is fixed and ℓ is small then the eccentricity is close to 1. This is an orbit for which the semimajor axis is much larger than the semiminor axis. The semimajor axis is expressed as

$$2a = r_{\rm max} + r_{\rm min} = \frac{\ell^2/GM_\odot}{1-\epsilon} + \frac{\ell^2/GM_\odot}{1+\epsilon} = r_{\rm max} + \frac{1-\epsilon}{1+\epsilon} r_{\rm max} = r_{\rm max}\frac{2}{1+\epsilon}.$$

For ϵ very close to 1, $a \simeq r_{\rm max}/2$.

8.21 (b) Kepler's third law states

$$\tau^2 = \frac{4\pi^2}{GM_\odot} a^3.$$

For the case described in 8.20, $a \simeq r_{\rm max}/2$. Hence in terms of $r_{\rm max}$ Kepler's third law becomes

$$\tau^2 = \frac{\pi^2}{2GM_\odot} r_{\rm max}^3.$$

(c) For this orbit the total time to fall from $r = r_{\rm max}$ (where its total energy is just its potential energy) is

$$\begin{aligned} T &= -\sqrt{\frac{\mu}{2}}\int_{r_{\rm max}}^{0} \frac{dr}{\sqrt{U(r_{\rm max}) - U(r)}} = -\sqrt{\frac{\mu}{2}}\int_{r_{\rm max}}^{0} \frac{dr}{\sqrt{-G\mu M_\odot/r_{\rm max} + G\mu M_\odot/r}} \\ T &= -\sqrt{\frac{1}{2GM_\odot}}\int_{r_{\rm max}}^{0} \frac{dr}{\sqrt{-1/r_{\rm max} + 1/r}} = -\sqrt{\frac{1}{2GM_\odot}}\int_{r_{\rm max}}^{0} \frac{\sqrt{r}dr}{\sqrt{1 - r/r_{\rm max}}}. \end{aligned}$$

The minus sign is used as the radial velocity is negative (the radius is decreasing). Defining

$$r = r_{\max} \cos^2 \theta,$$

the integral becomes

$$\begin{aligned} T &= \sqrt{\frac{1}{2GM_\odot}} r_{\max}^{3/2} \int_0^{\pi/2} \frac{\cos\theta \left(2 \sin\theta \cos\theta d\theta\right)}{\sin\theta} = \sqrt{\frac{2}{GM_\odot}} r_{\max}^{3/2} \int_0^{\pi/2} \cos^2 \theta d\theta \\ T &= \sqrt{\frac{2}{GM_\odot}} r_{\max}^{3/2} \frac{\pi}{4} = \sqrt{\frac{1}{8GM_\odot}} \pi r_{\max}^{3/2}. \end{aligned}$$

This comet approaches the Sun on a nearly radial line. As it reaches the Sun it makes a U turn and returns.

(d,e) The period for this orbit is

$$\tau = 2T = \sqrt{\frac{1}{2GM_\odot}} \pi r_{\max}^{3/2}.$$

Squaring both sides yields

$$\tau^2 = \frac{\pi^2}{2GM_\odot} r_{\max}^3,$$

which is in exact agreement with part (b).

8.23 (a,b) The potential energy for a particle of mass m in the force field

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3}$$

is

$$U(r) = -\frac{k}{r} + \frac{\lambda}{2r^2}.$$

If the particle moves with an angular momentum L then the expression for the conservation of energy is

$$\begin{aligned} E &= \frac{1}{2} m\dot{r}^2 + U(r) + \frac{L^2}{2mr^2} = \frac{1}{2} m\dot{r}^2 - \frac{k}{r} + \frac{\lambda}{2r^2} + \frac{L^2}{2mr^2} \\ E &= \frac{1}{2} m\dot{r}^2 - \frac{k}{r} + \frac{L^2 + m\lambda}{2mr^2}. \end{aligned}$$

As usual we $r = 1/u$ or equivalently $u = 1/r$. Also we note that $d/dt = \dot{\phi} d/d\phi$. First consider the radial kinetic energy term

$$\begin{aligned} \dot{r} &= \frac{dr}{du}\frac{du}{dt} = -\frac{1}{u^2}\dot{\phi}\frac{du}{d\phi} = -\frac{Lu^2}{m}\frac{1}{u^2}\frac{du}{d\phi} = -\frac{L}{m}\frac{du}{d\phi} \\ \dot{r}^2 &= \left(\frac{L}{m}\right)^2 \left(\frac{du}{d\phi}\right)^2. \end{aligned}$$

Substituting this result into the conservation of energy in the special case where $k = 0$ we find

$$\begin{aligned}
\frac{L^2}{2m}\left(\frac{du}{d\phi}\right)^2 + \frac{L^2 + m\lambda}{2m}u^2 &= E, \\
\left(\frac{du}{d\phi}\right)^2 + \left(1 + m\lambda/L^2\right)u^2 &= \frac{2m}{L^2}E, \\
\frac{1}{1 + m\lambda/L^2}\left(\frac{du}{d\phi}\right)^2 + u^2 &= \frac{2m}{L^2 + m\lambda}E.
\end{aligned}$$

From observation the solution to this nonlinear differential equation is

$u = u_0 \cos\beta\phi,$ with $\beta = \sqrt{1 + m\lambda/L^2}$ and $u_0 = \sqrt{2mE/\left(L^2 + m\lambda\right)} = \sqrt{2mE/\beta^2 L^2}$.

With k nonzero the expression for the conservation of energy is

$$\frac{L^2}{2m}\left(\frac{du}{d\phi}\right)^2 + \frac{L^2 + m\lambda}{2m}u^2 - ku = E$$

Going through the same procedure as that with $k = 0$ we find

$$\begin{aligned}
\frac{1}{1 + m\lambda/L^2}\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2mk}{L^2 + m\lambda}u &= \frac{2m}{L^2 + m\lambda}E \\
\frac{1}{\beta^2}\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2mk}{\beta^2 L^2}u &= \frac{2m}{\beta^2 L^2}E.
\end{aligned}$$

Completing the square results in

$$\begin{aligned}
\frac{1}{\beta^2}\left(\frac{du}{d\phi}\right)^2 + u^2 - \frac{2mk}{\beta^2 L^2}u + \frac{m^2k^2}{\beta^4 L^4} &= \frac{2m}{\beta^2 L^2}E + \frac{m^2k^2}{\beta^4 L^4}, \\
\frac{1}{\beta^2}\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{mk}{\beta^2 L^2}\right)^2 &= \frac{2m}{\beta^2 L^2}E + \frac{m^2k^2}{\beta^4 L^4}.
\end{aligned}$$

From the result for the case with $k = 0$ we see that the solution is now

$$\begin{aligned}
u &= \frac{mk}{\beta^2 L^2} + \sqrt{\frac{2m}{\beta^2 L^2}E + \frac{m^2k^2}{\beta^4 L^4}}\cos\beta\phi, \\
u &= \frac{mk}{\beta^2 L^2}\left(1 + \sqrt{1 + 2\beta^2 L^2 E/mk^2}\cos\beta\phi\right),
\end{aligned}$$

where again

$$\beta = \sqrt{1 + m\lambda/L^2}.$$

Inverting this expression we find

$$r(\phi) = \frac{\beta^2 L^2/mk}{1 + \sqrt{1 + 2\beta^2 L^2 E/mk^2}\cos\beta\phi}$$

This is of the form

$$r(\phi) = \frac{c}{1 + \epsilon \cos \beta\phi},$$

where

$$c = \beta^2 L^2/mk \text{ and } \epsilon = \sqrt{1 + 2\beta^2 L^2 E/mk^2}.$$

(c) This orbit is closed whenever $\beta = n/m$, a rational number. Note that as $\lambda \to 0$ the parameter $\beta \to 1$ and the solution is that for a Kepler orbit.

8.29 The kinetic energy of the Earth would remain unchanged. However the potential energy would immediately be halved. In a circular orbit the virial applies not just on average but for all time. Hence prior to the Sun losing its mass

$$T = \frac{n}{2}U = -\frac{1}{2}U, \text{ and } E = T + U = U/2.$$

Since U is negative this is the energy of a bound particle. If the Sun lost half its mass then relative to its new potential energy $T = -U$. Now the total energy is

$$E = T + U = 0.$$

The Earth is just barely unbound.

8.35 Assume that the initial radius for the circular orbit is R_1. After a backward thrust given by $\lambda = v_2/v_1 < 1$, the orbit will become an ellipse with the rocket located at the apogee. Hence

$$R_1 = \ell_1^2/GM = \frac{\ell_2^2/GM}{1 - \epsilon_2} = \frac{\lambda^2 \ell_1^2/GM}{1 - \epsilon_2} = \frac{\lambda^2}{1 - \epsilon_2} R_1.$$

This implies that

$$\lambda^2 = 1 - \epsilon_2 \to \epsilon_2 = 1 - \lambda^2.$$

At the perigee the distance from the Sun is R_3 given by

$$R_3 = \frac{\ell_2^2/GM}{1 + \epsilon_2} = \frac{\lambda^2 \ell_1^2/GM}{2 - \lambda^2} = \frac{\lambda^2}{2 - \lambda^2} R_1.$$

Solving for λ^2 we find

$$\left(2 - \lambda^2\right) R_3 = \lambda^2 R_1 \to \lambda^2 = \frac{2R_3}{R_1 + R_3}.$$

Since $R_3 = R_1/4$, $\lambda = \sqrt{2/5} = 0.6325$.

To obtain a circular orbit at this radius an additional backward thrust is required at the perigee. Since R_3 is held fixed we find

$$\begin{aligned} R_3 &= \frac{\ell_2^2/GM}{1 + \epsilon_2} = \ell_3^2/GM = \lambda'^2 \ell_2^2/GM \\ \lambda'^2 &= \frac{1}{1 + \epsilon_2} = \frac{1}{2 - \lambda^2} = \frac{R_1 + R_3}{2R_1}. \end{aligned}$$

Again $R_3 = R_1/4$ so that $\lambda' = \sqrt{5/8} = 0.7906$.

The final velocity is

$$\begin{aligned} v_3 &= \lambda' \frac{v_2\,(per)}{v_2\,(apo)} \lambda v_1 = \lambda' \frac{\ell_2/R_3}{\ell_2/R_1} \lambda v_1 = \sqrt{\frac{R_1+R_3}{2R_1}} \frac{R_1}{R_3} \sqrt{\frac{2R_3}{R_1+R_3}} v_1 \\ v_3 &= \sqrt{R_1/R_3} v_1 = 2v_1. \end{aligned}$$

11.6

(a) The Lagrangian for this system ($m_1 = m_2 = m$, $k_1 = 3k$, and $k_2 = 2k$) is

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) - \frac{1}{2} 3kx_1^2 - \frac{1}{2} 2k\,(x_2 - x_1)^2 .$$

Hence the equations of motion are

$$\begin{aligned} m\ddot{x}_1 &= -3kx_1 - 2k\,(x_1 - x_2) = -5kx_1 + 2kx_2 \\ m\ddot{x}_2 &= -2k\,(x_2 - x_1) \end{aligned}$$

Assuming a solution of the form

$$\mathbf{z} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{i\omega t}$$

where $\mathbf{x} = \operatorname{Re} \mathbf{z}$, we find

$$\begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A nontrivial solution requires the secular equation,

$$\det \begin{bmatrix} 5k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{bmatrix} = m^2\omega^4 - 7km\omega^2 + 6k^2 = 0,$$

to be satisfied. The normal mode frequencies are

$$\omega_1^2 = k/m, \text{ and } \omega_2^2 = 6k/m.$$

(b) To find the ratios of a_1 and a_2 for ω_1 we find

$$\begin{bmatrix} 5-1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

Hence

$$2a_1 = a_2.$$

In this mode the oscillations are in phase with the amplitude of x_2 being twice that of x_1.

To find the ratios of a_1 and a_2 for ω_1 we find

$$\begin{bmatrix} 5-6 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0.$$

Hence

$$a_1 = -2a_2.$$

In this mode the oscillations are exactly out of phase with the amplitude of x_1 being twice that of x_2.

11.9 (a) The equations of motion when $m_1 = m_2 = m$ and $k_1 = k_2 = k_3 = k$ are

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 \\ m\ddot{x}_2 &= -2kx_2 + kx_1. \end{aligned}$$

The normal coordinates are

$$\xi_1 = (x_1 + x_2)/2 \quad \text{and} \quad \xi_2 = (x_1 - x_2)/2.$$

Hence

$$x_1 = \xi_1 + \xi_2 \quad \text{and} \quad x_2 = \xi_1 - \xi_2.$$

Substituting this result into the equations of motion leads to

$$\begin{aligned} m\left(\ddot{\xi}_1 + \ddot{\xi}_2\right) &= -2k(\xi_1 + \xi_2) + k(\xi_1 - \xi_2) = -k\xi_1 - 3k\xi_2 \\ m\left(\ddot{\xi}_1 - \ddot{\xi}_2\right) &= -2k(\xi_1 - \xi_2) + k(\xi_1 + \xi_2) = . - k\xi_1 + 3k\xi_2 \end{aligned}$$

Adding and subtracting these two expressions results in

$$\begin{aligned} m\ddot{\xi}_1 &= -k\xi_1, \\ m\ddot{\xi}_2 &= -3k\xi_2. \end{aligned}$$

(b) The solutions are

$$\xi_1 = A_1 \cos(\omega_1 t - \delta_1) \quad \text{and} \quad \xi_2 = A_2 \cos(\omega_2 t - \delta_2)$$

where $\omega_1^2 = k/m$ and $\omega_2^2 = 3k/m$. The general solution for the displacements is

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \xi_1(t) + \xi_2(t) \\ \xi_1(t) - \xi_2(t) \end{bmatrix} \\ \mathbf{x}(t) &= A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\omega_2 t - \delta_2). \end{aligned}$$