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PHYSICS 140A
MIDTERM EXAM 2
Tuesday November 10, 2009

$$T\ dS = C_V\ dT + T\frac{\beta}{\kappa}\ dV$$

$$T\ dS = C_p\ dT - VT\ \beta\ dp$$

$$T\ dS = C_V\frac{\kappa}{\beta}\ dp + \frac{C_p}{\beta V}\ dV$$

1. In this problem, we consider 3 heat engines. The 3 cycles of the working substance are drawn in the S-T plane below. Cycle 1 is $ABEHA$; cycle 2 is $ABCDEHA$; and cycle 3 is $ABEFGHA$. Assume all 3 cycles are reversible. Also, assume $T'' < T' < T$ and $S'' < S' < S$.

(a) What is the efficiency η_1 of heat engine 1?

(b) What is the efficiency η_2 of heat engine 2?

(c) What is the efficiency η_3 of heat engine 3?

(d) Compare the 3 efficiencies η_1, η_2 and η_3.

(e) Since the cycles are reversible, we can consider the combination consisting of cycle 3 and the reverse of cycle 1. What is the efficiency η of this new cycle? What can one say about the efficiencies η_1, η_3 and η relative to one another?

(a)

$$\eta_1 = 1 - \frac{T'}{T}$$

(b)

$$\eta_2 = 1 - \frac{T'}{T}$$

(c)

$$\eta_3 = 1 - \frac{T''}{T}$$

(d)

$$\eta_3 > \eta_1 = \eta_2$$

(e)

$$\eta = 1 - \frac{T''}{T'}$$

$$\eta_3 > \eta_1$$

$$\eta_3 > \eta$$

One cannot determine whether η_1 or η is larger without more information about relative sizes of ratios T'/T and T''/T'

2. Consider the reversible, closed cycle of an ideal gas, which consists of

(i) an adiabatic expansion from state A to state B

(ii) an isobaric compression from state B to state C

(iii) an isochoric transformation from state C to state A

The cycle is drawn in the V-p plane below. The coordinates of states A, B and C are (V_A, p_A), (V_B, p_B), and (V_A, p_B), respectively. Assume that the heat capacity of the ideal gas at constant volume is $C_V = \alpha n R$, for α some constant.

(a) Compute the change in entropy S for each transformation of the cycle. (Compute S_{BA}, S_{CB} and S_{AC}.) Check your result by verifying that the total change in entropy around the closed cycle is what you expect.

(b) Compute the heat Q for each transformation of the cycle. (Compute Q_{BA}, Q_{CB} and Q_{AC}.) What is the total heat Q_{TOT} for the cycle?

(c) Compute the work W for each transformation of the cycle. (Compute W_{BA}, W_{CB} and W_{AC}.) What is the total work W_{TOT} for the cycle?

(d) Compare Q_{TOT} and W_{TOT}? What do you conclude about the change in internal energy U for the cycle?

(a)

$$T\ dS = C_V \frac{\kappa}{\beta}\ dp + \frac{C_p}{\beta V}\ dV$$

For ideal gas, $C_V = \alpha n R$, $C_p = (\alpha+1) n R$, $\beta = \frac{1}{T}$, $\kappa = \frac{1}{p}$, $pV = nRT$.

$$T\ dS = \alpha V\ dp + (\alpha+1) p\ dV$$

$$dS = \alpha n R\ \frac{dp}{p} + (\alpha+1) n R\ \frac{dV}{V}$$

$$S_{BA} = 0$$

$$S_{CB} = (\alpha+1)\ nR\ \ln\left(\frac{V_A}{V_B}\right)$$

$$S_{AC} = \alpha\ nR\ \ln\left(\frac{p_A}{p_B}\right)$$

$$S_{\text{TOT}} = (\alpha + 1)\ nR\ \ln\left(\frac{V_A}{V_B}\right) + \alpha\ nR\ \ln\left(\frac{p_A}{p_B}\right)$$

$A \to B$ is an adiabatic expansion, so

$$p_A V_A^{\gamma} = p_B V_B^{\gamma}$$

$$\frac{p_A}{p_B} = \left(\frac{V_B}{V_A}\right)^{\frac{\alpha+1}{\alpha}}$$

Thus,

$$S_{\text{TOT}} = (\alpha + 1)\ nR\ \ln\left(\frac{V_A}{V_B}\right) + \alpha\ nR\ \ln\left(\frac{V_B}{V_A}\right)^{\frac{\alpha+1}{\alpha}} = 0$$

as expected.

(b)

$$đQ_{\text{rev}} = T\ dS = \alpha V\ dp + (\alpha + 1)p\ dV$$

$$Q_{BA} = 0$$

for adiabatic transformation.

$$Q_{CB} = (\alpha + 1)\ p_B \int_{V_B}^{V_A} dV = (\alpha + 1)\ p_B\,(V_A - V_B) < 0$$

$$Q_{AC} = \alpha V_A \int_{p_B}^{p_A} dp = \alpha V_A\,(p_A - p_B) > 0$$

$$Q_{\text{TOT}} = (\alpha + 1)\ p_B\,(V_A - V_B) + \alpha V_A\,(p_A - p_B)$$

(c)

$$đW_{\text{rev}} = pdV$$

$$W_{BA} = \frac{1}{(1-\gamma)}\,(p_B V_B - p_A V_A) = \alpha\,(p_A V_A - p_B V_B) > 0$$

since $\gamma = \frac{\alpha+1}{\alpha} \Rightarrow \frac{1}{1-\gamma} = -\alpha$

$$W_{CB} = p_B\,(V_A - V_B)$$

$$W_{AC} = 0$$

$$W_{\text{TOT}} = \alpha p_A V_A - (\alpha + 1) p_B V_B + p_B V_A$$

(d)

$$Q_{\text{TOT}} = W_{\text{TOT}}$$

so $U_{\text{TOT}} = 0$ for the closed cycle as expected.

3. Short problems.

(a) Prove for a general thermodynamic system that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V .$$

(b) If two thermodynamic systems A and B are in thermodynamic equilibrium, what do you know about them?

(c) A liquid consists of a mixture of 3 different liquids in thermodynamic equilibrium. How many independent variables are required to describe the system? List the independent variables.

(d) The solid and liquid phases of a given substance are in thermodynamic equilibrium. What do you know about the specific Gibbs free energies of the solid and the liquid?

(a)

$$S = -\left(\frac{\partial F}{\partial T}\right)_V$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$\left(\frac{\partial S}{\partial V}\right)_T = -\frac{\partial^2 F}{\partial V \partial T} = -\frac{\partial^2 F}{\partial T \partial V} = \left(\frac{\partial p}{\partial T}\right)_V$$

(b)

$$T_A = T_B$$

$$p_A = p_B$$

$$\mu_A = \mu_B$$

(c)

$$k = 3 \text{ and } \pi = 1 \qquad \blacksquare$$

$$f = k - \pi + 2 = 3 - 1 + 2 = 4$$

Independent variables: T, p, and 2 different mole fractions $x_1 = \frac{n_1}{n_1+n_2+n_3}$, $x_2 = \frac{n_2}{n_1+n_2+n_3}$
Note that $x_3 = 1 - x_1 - x_2$, so one of the three mole fractions is not independent.

(d) The specific Gibbs free energies of the solid and liquid phases are equal. $\blacksquare$

4. A gas (NOT an ideal gas) has Helmholtz free energy

$$F(T,V,n) = -nRT\left\{\ln T^{3/2} + \ln V - \ln n + 1 + c\right\} + n^2RT\frac{B(T)}{V}$$

where c is a constant and $B(T)$ is an arbitrary function of temperature T.

(a) Find the entropy S of the gas.

(b) Find the pressure p of the gas.

(c) Find the chemical potential μ of the gas.

(d) Find C_V, the heat capacity at constant volume.

(e) Find the internal energy U of the gas.

(f) Find the Gibbs free energy G of the gas.

(a)

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,n}$$

$$S = nR\left\{\ln T^{3/2} + \ln V - \ln n + 1 + c\right\} + \frac{3}{2}nR - n^2R\frac{B(T)}{V} - n^2RT\frac{B'(T)}{V}$$

(b)

$$p = -\left(\frac{\partial F}{\partial V}\right)_{T,n}$$

$$p = \frac{nRT}{V} + \frac{n^2RTB(T)}{V^2}$$

$$p = \frac{nRT}{V}\left(1 + \frac{nB(T)}{V}\right)$$

(c)

$$\mu = \left(\frac{\partial F}{\partial n}\right)_{T,V}$$

$$\mu = -RT\left\{\ln T^{3/2} + \ln V - \ln n + 1 + c\right\} + RT + 2nRT\frac{B(T)}{V}$$

$$\mu = -RT\left\{\ln T^{3/2} + \ln V - \ln n + c\right\} + 2nRT\frac{B(T)}{V}$$

(d)

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,n}$$

$$\left(\frac{\partial S}{\partial T}\right)_{V,n} = \frac{3nR}{2T} - 2n^2R\frac{B'(T)}{V} - n^2RT\frac{B''(T)}{V}$$

$$C_V = T\left(\frac{\partial S}{\partial T}\right)_{V,n} = \frac{3}{2}nR - 2n^2RT\frac{B'(T)}{V} - n^2RT^2\frac{B''(T)}{V}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,n}$$

(e)

$$U = F + TS$$

$$U = \frac{3}{2}nRT - n^2RT^2\frac{B'(T)}{V}$$

(f)

$$G = F + pV$$

$$G = -nRT\left\{\ln T^{3/2} + \ln V - \ln n + 1 + c\right\} + n^2RT\frac{B(T)}{V} + nRT + \frac{n^2RTB(T)}{V}$$

$$G = -nRT\left\{\ln T^{3/2} + \ln V - \ln n + c\right\} + 2n^2RT\frac{B(T)}{V}$$

$$G = \mu n$$