PHYSICS 201 (FALL 2009)

MIDTERM EXAM 1

Given: October 23, 4:00 pm

Due: October 24, 12:00 noon

Dear students:

To let you have the benefit of all possible aids at your disposal, this exam is a take-home one. But please note that you CANNOT have ANY consultation with ANY other person --------- NONE AT ALL!

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1. The real part $u(x,y)$ of an analytic function $f(z)$ is given to be

$$\sin 2x \,/\, (\cos 2x + \cosh 2y).$$

Knowing this, determine the imaginary part of this function.

What, do you think, is the form of the function $f(z)$ in terms of z?

2. Locate the singularities of the function $\cot(\pi z) / (z - a)^2$, where a is a real number, and evaluate the residues of this function at those singularities.

3. Using the calculus of residues, evaluate the integral

$$\int_0^{2\pi} \cos\{n\theta - \sin\theta\} . \exp(\cos\theta)\, d\theta$$

for all *integral* values of n.

4. Show that, for $-1 < p < 1$ and $-\pi < \lambda < \pi$, the integral

$$\int_0^\infty \frac{x^p\,dx}{1 + 2x\cos\lambda + x^2} = \frac{\pi}{\sin \pi p}\,\frac{\sin \lambda p}{\sin \lambda}. \qquad \text{(Euler)}$$

Hint: it will be helpful to write $cos\lambda$ as $\frac{1}{2}\left(e^{i\lambda} + e^{-i\lambda}\right)$.

5. By integrating a suitable function around a closed contour in the z–plane, evaluate the integral

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx,$$

a being a real number.

6. Evaluate the Inverse Laplace Transform of the function

$$F(s) = s/(s^4 - a^4)$$

and check the limiting case $a \to 0$.

Midterm 1 solutions

(Total = 55)

1. Given $u(x,y) = \dfrac{\sin 2x}{\cos 2x + \cosh 2y}$. (1)

To get $v(x,y)$, we make use of Cauchy-Riemann conditions.
We start with

$$\frac{\partial u}{\partial y} = \frac{-\sin 2x \cdot 2\sinh 2y}{(\cos 2x + \cosh 2y)^2} = -\frac{\partial v}{\partial x}, \text{ so that}$$

$$v(x,y) = \frac{\sinh 2y}{\cos 2x + \cosh 2y} + \underbrace{\text{a function of } y \text{ only.}}_{?} \quad (2)$$

Next,

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x}{\cos 2x + \cosh 2y} + \frac{-\sin 2x \cdot (-2\sin 2x)}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2\cos^2 2x + 2\cos 2x \cosh 2y + 2\sin^2 2x}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2\cosh^2 2y + 2\cos 2x \cosh 2y - 2\sinh^2 2y}{(\cos 2x + \cosh 2y)^2}$$

$$= \frac{2\cosh 2y}{\cos 2x + \cosh 2y} - \frac{2\sinh^2 2y}{(\cos 2x + \cosh 2y)^2} = \frac{\partial v}{\partial y},$$

so that

$$v(x,y) = \frac{\sinh 2y}{\cos 2x + \cosh 2y} + \underbrace{\text{a function of } x \text{ only.}}_{?} \quad (3)$$

Comparing (2) & (3), we infer that

$$v(x,y) = \frac{\sinh 2y}{\cos 2x + \cosh 2y} + \underbrace{\text{a constant at most.}}_{?} \qquad (4)$$

$$\therefore\ f(z) = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} + iC. \qquad (5)$$

The form of the function $f(z)$

For $y=0$, we readily get $\dfrac{\sin 2x}{\cos 2x + 1} + iC = \tan x + iC$.

By analytic continuation, $f(z) = \tan z + iC.$

Alternatively, one may utilize the fact that

$$\frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} = \frac{\sin 2x + \sin i2y}{\cos 2x + \cos i2y}$$

$$= \frac{\cancel{2} \sin(x+iy)\, \cancel{\cos(x-iy)}}{\cancel{2} \cos(x+iy)\, \cancel{\cos(x-iy)}}$$

$$= \tan(x+iy).$$

2. $f(z) = \dfrac{\cot \pi z}{(z-a)^2}$ has poles at $z = n = 0, \pm 1, \pm 2, \ldots$ & at $z = a$.

Residues

A. If a is non-integral, then $z = n$ is a simple pole with residue $\dfrac{1}{\pi (n-a)^2}$, while $z = a$ is a double pole with residue $-\pi \operatorname{cosec}^2 \pi a$ {EXCEPT for the case when a is half-odd-integral, in which case $z = a$ is also a simple pole with residue $-\pi$}.

B. If a is integral, say $a = m$, then $z = n \neq m$ is again a simple pole with residue $\dfrac{1}{\pi (n-m)^2}$. However, $z = m$ is now a pole of order 3, with Laurent expansion

$$f(m+\varepsilon) = \frac{\cot \pi (m+\varepsilon)}{(m+\varepsilon-m)^2} = \frac{\cot \pi \varepsilon}{\varepsilon^2} = \frac{1}{\varepsilon^2}\left(\frac{1}{\pi \varepsilon} - \frac{1}{3}\pi \varepsilon - \ldots\right)$$

$$= \frac{1}{\pi \varepsilon^3} - \frac{1}{3}\frac{\pi}{\epsilon} - \ldots,$$

with residue $-\frac{1}{3}\pi$.

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3. $I = \int_0^{2\pi} \cos(n\theta - \sin\theta)\, e^{\cos\theta}\, d\theta = \mathrm{Re} \int_0^{2\pi} e^{i(n\theta - \sin\theta) + \cos\theta}\, d\theta$

$$= \mathrm{Re} \int_0^{2\pi} \exp(e^{-i\theta} + in\theta)\, d\theta.$$

Substituting $e^{i\theta} = z$, we get

$$I = \mathrm{Re} \oint_C \exp\left(\frac{1}{z}\right) z^n \frac{dz}{iz}, \quad \left. \begin{array}{l} C \text{ being a unit circle} \\ \text{centered at } 0. \end{array} \right\}$$

The integrand has an essential singularity at $z = 0$, with Laurent expansion $\sum_{m=\infty}^{0} \frac{(1/z)^m}{m!} \cdot \frac{z^{n-1}}{i}$ and residue $\frac{1}{n!\,i}$ for

$n = 0, 1, 2, \ldots$ & zero for $n = -1, -2, -3, \ldots$

It follows that

$$I = \left. \begin{array}{ll} \frac{2\pi}{n!} & \text{for } n = 0, 1, 2, \ldots \\ 0 & \text{for } n = -1, -2, -3, \ldots \end{array} \right\} \checkmark$$

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In passing, we have also shown that

$$I = \int_0^{2\pi} \sin(n\theta - \sin\theta)\, e^{\cos\theta}\, d\theta = 0 \quad \text{for } \underline{\text{all}}\ n.$$

4. $$I = \int_0^\infty \frac{x^p\,dx}{(x+e^{i\lambda})(x+e^{-i\lambda})} \qquad (-1<p<1,\ -\pi<\lambda<\pi).$$

For this, we consider the contour integral

$$I_c = \oint_C \frac{z^p\,dz}{(z+e^{i\lambda})(z+e^{-i\lambda})},$$ where C is the contour used for integrals of category 3 — the one with a branch point at the origin and a cut-line along the positive real axis, so that $0<\theta<2\pi$.

Now, as $R\to\infty$, $I_R \sim R^{p-1} \to 0$ because $p<1$ and
as $r\to 0$, $I_r \sim r^{p+1} \to 0$ because $p>-1$,
with the result that $I_c \longrightarrow (1-e^{2\pi i p})I$.

By the calculus of residues,

$I_c = 2\pi i \sum$ residues at $z=-e^{i\lambda}$, i.e. at $e^{i(\pi+\lambda)}$ and
at $z=-e^{-i\lambda}$, i.e. at $e^{i(\pi-\lambda)}$.

Note that both $\pi\pm\lambda$ lie in the desired range $(0,2\pi)$.

$$\therefore\ I_c = 2\pi i\left(\frac{e^{i(\pi+\lambda)p}}{-e^{i\lambda}+e^{-i\lambda}} + \frac{e^{i(\pi-\lambda)p}}{-e^{-i\lambda}+e^{i\lambda}}\right),$$ so that

$$I = 2\pi i\,\frac{e^{i\pi p}}{1-e^{2\pi i p}}\cdot\frac{-e^{i\lambda p}+e^{-i\lambda p}}{e^{i\lambda}-e^{-i\lambda}}$$

$$= 2\pi i\,\frac{1}{-2i\sin\pi p}\cdot\frac{-2i\sin\lambda p}{2i\sin\lambda} = \frac{\pi}{\sin\pi p}\cdot\frac{\sin\lambda p}{\sin\lambda}.$$ ✓

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5. $I = \int_0^\infty \frac{\sin x}{x(x^2+a^2)}\,dx = \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+a^2)}\,dx.$

For this, we consider the contour integral

$I_C = \oint_C \frac{e^{iz}}{z(z^2+a^2)}\,dz$, with simple poles at $z=0, \pm ia$.

For C, take a "semi-circle in the upper-half plane" so that $I_R \to 0$ as $R \to \infty$. We then get, assuming $a>0$,

$$I_C = (\pi i \times \text{res. at } z=0 + 2\pi i \times \text{res. at } z = ia)$$

$$= \pi i \cdot \frac{1}{a^2} + 2\pi i \frac{e^{i.ia}}{ia \cdot 2ia}$$

$$= \frac{\pi i}{a^2}\left(1-e^{-a}\right).$$

Hence, $I = \frac{\pi}{2a^2}\left(1-e^{-a}\right)$. ✓

If $a<0$, we'll get $\frac{\pi}{2a^2}\left(1-e^{a}\right)$, so that in general

(8) $$I = \frac{\pi}{2a^2}\left(1-e^{-|a|}\right).$$ ✓

6. Given: $F(s) = \frac{s}{s^4 - a^4}$, with simple poles at $s = \pm a, \pm ia$.

The corresponding $f(t)$ is given by the "sum of the residues of the function $e^{st}F(s)$ at the poles of $F(s)$", i.e.

$$\sum_{(\text{poles})} \frac{s e^{st}}{4s^3} = \frac{e^{at}}{4a^2} + \frac{e^{-at}}{4a^2} + \frac{e^{iat}}{-4a^2} + \frac{e^{-iat}}{-4a^2}$$

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$$= \frac{1}{2a^2}(\cosh at - \cos at). \checkmark$$

The limit $a \to 0$ gives

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$$\frac{1}{2a^2}\left[\left(1 + \frac{a^2t^2}{2} + \cdots\right) - \left(1 - \frac{a^2t^2}{2} + \cdots\right)\right] = \frac{1}{2}t^2, \checkmark$$

which indeed is the I.L.T. of $1/s^3$.