PHYSICS 201 (FALL 2009)
MIDTERM EXAM 2

Given: November 20, 4:00 pm
Due: November 21, 12:00 noon

Dear students:

To let you have the benefit of all possible aids at your disposal, this exam is a take-home one. But please note that, while you may consult any number of resources in print, you CANNOT have consultation with ANY other person ----- DEAD OR ALIVE!

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1. (i) Using the calculus of residues, evaluate the sum

$$S(a,b) = \sum_{n=-\infty}^{\infty} \frac{1}{(n^2+a^2)(n^2+b^2)}.$$

(ii) Now, let a and b go to zero to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. Determine the limiting value of the function $f(n)$, defined by the product

$$f(n) = \prod_{j=1}^{n} \frac{4j^2}{4j^2-1} = \frac{4}{3}\cdot\frac{16}{15}\cdot\frac{36}{35}\cdots\cdots\frac{4n^2}{4n^2-1},$$

as n tends to infinity.

[Hint: Express $f(n)$ in terms of gamma/factorial functions, and use Stirling's formula].

3. Apply Laplace's method to show that, for large ν,

$$I(a,b;\nu) = \int_0^{\infty} e^{-at-\frac{1}{2}bt^2}\cdot t^{\nu}dt \approx \sqrt{\frac{\pi}{b}}\cdot e^{-a\sqrt{\nu/b}+a^2/4b}\cdot\left(\frac{\nu}{be}\right)^{\nu/2}.$$

To be more specific, the derivation here assumes that $(b\nu/a^2) \gg 1$.

4. Using the method of stationary phase, determine the asymptotic behavior of the function

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 - xt\right) dt$$

for $x >> 1$.

5. Using the method of steepest descents, show that, for large n,

$$I_n = \int_{i-\infty}^{i+\infty} e^{-z^2} (1+z)^{-n} dz \approx \sqrt{\frac{\pi}{2e}}\, i^{-n} \left(\frac{n}{2}\right)^{-n/2} e^{n/2 + i\sqrt{2n}}.$$

Here, n may be regarded as running through integral values only (so that you don't have to employ a "branch cut").

Also note that, of the two saddle points you have in this problem, the contour C can be steered ONLY through the one in the upper-half plane --- NOT through the one in the lower-half plane. Why this is so --- I'll explain in the class!

6. Solve the differential equation

$$3y'' + 2y' - 8y = 5 \cos x$$

for both the complementary function and the particular solution.

7. (i) Apply the Frobenius method to obtain the series solutions of the differential equation

$$x^2 y'' + 2xy' + (x^2 - 2)y = 0.$$

(ii) Next, making a substitution such as $y(x) = f(x)\, z(x)$, put the above equation into the form

$$z'' = Q(x)\, z$$

and apply the WKB method to determine the asymptotic behavior of the function $z(x)$, and hence of $y(x)$, as x goes to infinity.

Solutions to MT-2

1. (i) $$S(a,b) = \frac{1}{b^2-a^2} \sum_{n=-\infty}^{\infty} \left[\frac{1}{n^2+a^2} - \frac{1}{n^2+b^2} \right]$$

$$= \frac{1}{b^2-a^2} \left[\frac{\pi \coth \pi a}{a} - \frac{\pi \coth \pi b}{b} \right];$$

see class notes.

(ii) For small a & b,

$$S(a,b) = \frac{1}{b^2-a^2} \left[\frac{\pi}{a} \left(\frac{1}{\pi a} + \frac{\pi a}{3} - \frac{\pi^3 a^3}{45} + \cdots \right) - \frac{\pi}{b} \left(\frac{1}{\pi b} + \frac{\pi b}{3} - \frac{\pi^3 b^3}{45} + \cdots \right) \right]$$

$$= \frac{1}{a^2 b^2} + \text{zero} + \frac{\pi^4}{45} + \text{terms with positive powers of } a \text{ and } b.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{2} \left[S(a,b) - \frac{1}{a^2 b^2} \right] \Big\}_{\lim a,b \to 0} = \frac{\pi^4}{90}.$$

2. $f(n) = \frac{2.2}{1.3} \cdot \frac{4.4}{3.5} \cdot \frac{6.6}{5.7} \cdots\cdots \frac{2n.2n}{(2n-1)(2n+1)}$.

The numerator here is $4^n.(n!)^2$, while the denominator is

$3^2.5^2.7^2 \ldots (2n-1)^2.(2n+1)$, which is equal to

$$\frac{1^2.2^2.3^2.4^2.5^2.6^2.7^2 \ldots (2n-1)^2 (2n)^2}{2^2.4^2.6^2 \ldots\ldots (2n)^2}(2n+1),$$

that is,

$$\frac{[(2n)!]^2}{2^{2n}.(n!)^2}(2n+1).$$

It follows that

$$f(n) = \frac{2^{4n}.(n!)^4}{(2n!)^2\,(2n+1)}.$$

Using Stirling's formula, i.e., $n! \approx \sqrt{2\pi n}\,(n/e)^n$, we get

$$f(n) \approx \frac{2^{4n}.\left[\sqrt{2\pi n}\,(n/e)^n\right]^4}{\left[\sqrt{4\pi n}\,(2n/e)^{2n}\right]^2.\,2n} = \frac{\pi}{2}.$$

This product is the famous Wallis' product (1655) -------- the first formula that expressed a purely geometrical entity in a purely algebraic form!

3. $$I(a,b;\nu) = \int_0^\infty e^{-at-\frac{1}{2}bt^2}\, t^\nu \, dt \qquad (\nu > -1).$$

We'll estimate it for $\nu \gg 1$.

Write $f(t) = -at - \frac{1}{2}bt^2 + \nu \ln t$.

Then $f'(t) = -a - bt + \frac{\nu}{t} \Longrightarrow t_0 = \dfrac{-a+\sqrt{a^2+4b\nu}}{2b}$

& $f''(t) = -b - \frac{\nu}{t^2}$.

$$= \sqrt{\frac{\nu}{b}} - \frac{a}{2b} + \frac{a^2}{8b^2}\sqrt{\frac{b}{\nu}} + \cdots$$

It follows that

$$I(a,b;\nu) \approx e^{-at_0 - \frac{1}{2}bt_0^2 + \nu \ln t_0} \cdot \sqrt{\frac{2\pi}{|f''(t_0)|}} \qquad \text{full-gaussian}$$

$$f''(t_0) \approx -b - \frac{\nu}{\nu/b} = -2b.$$

$$\text{The exponent} = -a\left[\sqrt{\frac{\nu}{b}} - \frac{a}{2b} + \cdots\right]$$

$$-\frac{1}{2}b\left[\frac{\nu}{b} + \frac{a^2}{4b^2} + \cdots - \frac{a}{b}\sqrt{\frac{\nu}{b}} + \frac{a^2}{4b^2} + \cdots\right]$$

$$+\nu\left[\ln\sqrt{\frac{\nu}{b}} + \underbrace{\ln\left\{1 - \frac{a}{2b}\sqrt{\frac{b}{\nu}} + \frac{a^2}{8b^2}\cdot\frac{b}{\nu} + \cdots\right\}}_{= -\frac{a}{2b}\sqrt{\frac{b}{\nu}} + \frac{a^2}{8b\nu} - \frac{1}{2}\cdot\frac{a^2}{4b^2}\cdot\frac{b}{\nu} + \cdots}\right]$$

$$\approx -a\sqrt{\frac{\nu}{b}} + \frac{a^2}{2b} - \frac{1}{2}\nu - \frac{a^2}{8b} + \cancel{\frac{1}{2}a\sqrt{\frac{\nu}{b}}} - \frac{a^2}{8b}$$

$$+\left[\ln\left\{\left(\frac{\nu}{b}\right)^{\frac{1}{2}\nu}\right\} - \cancel{\frac{a}{2}\sqrt{\frac{\nu}{b}}} + \cancel{\frac{a^2}{8b}} - \cancel{\frac{a^2}{8b}}\right]$$

$$\therefore\ I(a,b;\nu) \approx e^{\frac{a^2}{4b} - a\sqrt{\frac{\nu}{b}} - \frac{1}{2}\nu} \cdot \left(\frac{\nu}{b}\right)^{\frac{1}{2}\nu} \sqrt{\frac{\pi}{b}}\,. \quad ✓$$

4. $f(x) = \frac{1}{\pi} Re \int_0^{\infty} e^{ip(x,t)} dt$, where

$p(x,t) = \frac{1}{3}t^3 - xt$, with stationary points at $t_0 = \pm\sqrt{x}$.

The point $-\sqrt{x}$ being out of the range of integration, we have to consider only $+\sqrt{x}$, for which

$$p''(x,t)\Big|_{t=+\sqrt{x}} = 2\sqrt{x}\text{, which is positive!}$$

It follows that

$$f(x) \approx \frac{1}{\pi} Re \left\{ e^{+i\pi/4} \cdot e^{i\left(\frac{1}{3}x^{3/2} - x.x^{1/2}\right)} \cdot \sqrt{\frac{2\pi}{2\sqrt{x}}} \right\}$$

$$= \frac{1}{\sqrt{\pi}\, x^{1/4}} \cos\left(\frac{2}{3}x^{3/2} - \frac{\pi}{4}\right).$$

It may be noted that this function is precisely $Ai(-x)$!

5. Here, $$I_n = \int_{i-\infty}^{i+\infty} e^{-z^2 - n\ln(1+z)}\, dz,$$

with $$f(z) = -z^2 - n\ln(1+z),$$

$$f'(z) = -2z - \frac{n}{1+z} = 0, \text{ i.e., } 2z^2 + 2z + n = 0,$$

so that $$z_0 = \frac{-2 \pm \sqrt{4-8n}}{4} = \frac{-1 \pm i\sqrt{2n-1}}{2}$$

$$= \pm i\sqrt{n/2} - \frac{1}{2} \mp i\frac{1}{4\sqrt{2n}} + \cdots\cdots;$$

while $f''(z_0) = -2 + \dfrac{n}{(1+z_0)^2} \approx -2 + \dfrac{n}{-n/2} = -4$; the latter being real & negative, the phase angle α here is zero.

Now, as I'll explain in the class, your contour C can be steered ONLY through $z_0 = i\sqrt{n/2} - \frac{1}{2} - i\frac{1}{4\sqrt{2n}} + \cdots\cdots$, NOT through $z_0 = -i\sqrt{n/2} - \cdots\cdots$, with the result that

$$f(z_0) = -\left(i\sqrt{\frac{n}{2}} - \frac{1}{2} - i\frac{1}{4\sqrt{2n}} + \cdots\right)^2 - n\ln\left(i\sqrt{\frac{n}{2}} + \frac{1}{2} - i\frac{1}{4\sqrt{2n}} + \cdots\right)$$

$$= -\left(-\frac{n}{2} + \frac{1}{4} - i\sqrt{\frac{n}{2}} + \frac{1}{4} + \cdots\right) - n\ln\left(i\sqrt{\frac{n}{2}}\right) - n\ln\left(1 - i\sqrt{\frac{1}{2n}} - \frac{1}{4n} + \cdots\right).$$

Now, $$\ln\left(1 - i\sqrt{\frac{1}{2n}} - \frac{1}{4n} + \cdots\right) = \left(-i\sqrt{\frac{1}{2n}} \cancel{- \frac{1}{4n}} + \cdots\right) - \frac{1}{2}\left(\cancel{-\frac{1}{2n}} + \cdots\right) + \cdots$$

$$= -i\sqrt{\frac{1}{2n}} + \cdots$$

$$\therefore f(z_0) = \frac{n}{2} + i\sqrt{\frac{n}{2}} - \frac{1}{2} + \ln\left(i\sqrt{n/2}\right)^{-n} + i\sqrt{\frac{n}{2}} + \cdots$$

With $|f''(z_0)| = 4$ and $\alpha = 0$, we finally obtain

$$I_n \approx \sqrt{\frac{2\pi}{|f''(z_0)|}} \cdot e^{i\alpha} \cdot e^{f(z_0)}$$

$$= \sqrt{\frac{\pi}{2e}} \; i^{-n} \left(\frac{n}{2}\right)^{-n/2} e^{\frac{n}{2} + i\sqrt{2n}} \quad . \checkmark$$

6. Given: $3y'' + 2y' - 8y = 5\cos x$.

With constant coefficients, we try for the complementary function $y = e^{rx}$, with the result that $3r^2 + 2r - 8 = 0$, i.e., $r = \frac{4}{3}$ or -2. So,

$$\underbrace{y_1(x) = e^{\frac{4}{3}x}} \quad \& \quad \underbrace{y_2(x) = e^{-2x}},$$

with
$$W(x) = \begin{vmatrix} e^{4/3\, x} & e^{-2x} \\ 4/3 \cdot e^{4/3\, x} & -2e^{-2x} \end{vmatrix} = \underbrace{-\frac{10}{3}e^{-2/3\, x}}.$$

For the particular solution, we write the given diff. eqn. as

$$y'' + \frac{2}{3}y' - \frac{8}{3}y = \frac{5}{3}\cos x.$$

$$\therefore\ y_p(x) = e^{-2x}\int^x \frac{e^{4/3\, x} \cdot \frac{5}{3}\cos x}{-10/3\, e^{-2/3\, x}}\,dx - e^{\frac{4}{3}x}\int^x \frac{e^{-2x} \cdot \frac{5}{3}\cos x}{-\frac{10}{3}e^{-2/3\, x}}\,dx$$

$$= -\frac{1}{2}e^{-2x}\int^x e^{2x}\cos x \; + \frac{1}{2}e^{4/3\, x}\int^x e^{-4/3\, x}\cos x\,dx$$

$$= -\frac{1}{2}\,\frac{2\cos x + \sin x}{5} + \frac{1}{2}\,\frac{-4/3\cos x + \sin x}{25/9}$$

$$= -\frac{1}{5}\cos x - \frac{1}{10}\sin x - \frac{6}{25}\cos x + \frac{9}{50}\sin x$$

$$= \underbrace{\frac{1}{25}\left(2\sin x - 11\cos x\right)}.$$

$$\therefore\ y(x) = c_1 e^{4/3\, x} + c_2 e^{-2x} + \frac{1}{25}\left(2\sin x - 11\cos x\right).$$

Alternatively, one could readily guess that the particular solution here would be of the form

$$y_p(x) = A \sin x + B \cos x.$$

Substitution into the given diff. eqn. then yields

$$3(-A \sin x - B \cos x) + 2(A \cos x - B \sin x) - 8(A \sin x + B \cos x) = 5 \cos x.$$

This requires that

$$-11A - 2B = 0 \quad \text{and} \quad 2A - 11B = 5,$$

whence

$$A = \frac{2}{25} \text{ and } B = -\frac{11}{25}. \checkmark$$

7. (a) Substituting $y = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$ into the D.E. $x^2 y'' + 2xy' + (x^2-2)y = 0$,

we get

$$\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)(k+\lambda-1)\, x^{k+\lambda} + 2\sum_{\lambda=0}^{\infty} a_\lambda (k+\lambda)\, x^{k+\lambda} + \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda+2} - 2\sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda} = 0.$$

Coeff. of x^k is $a_0 k(k-1) + 2a_0 k - 2a_0 = 0$, i.e. $a_0(k^2+k-2) = 0$.

With $a_0 \neq 0$, $k = 1$ or -2. ✓

Coeff. of x^{k+1} is $a_1\{(k+1)^2 + (k+1) - 2\} = a_1(k^2+3k) = 0$.

$\therefore$ For $k=1$ as well as -2, $a_1 = 0$ ✓

Case $k=1$: Our basic equation becomes

$$\sum_{\lambda=0}^{\infty} a_\lambda \{\underbrace{(\lambda+1)\lambda + 2(\lambda+1) - 2}_{=\lambda(\lambda+3)}\} x^{\lambda+1} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda+3} = 0,$$

so that

$$a_{j+2} = -\frac{1}{(j+2)(j+5)}\, a_j.$$

Thus, $a_2 = -\frac{1}{10} a_0$, $a_4 = +\frac{1}{280} a_0$; ----.

$$\therefore\ y_1(x) = \text{const}\left(x - \frac{x^3}{10} + \frac{x^5}{280} - \cdots\right). ✓$$

Case $k=-2$: We now have

$$\sum_{\lambda=0}^{\infty} a_\lambda \{\underbrace{(\lambda-2)(\lambda-3) + 2(\lambda-2) - 2}_{=(\lambda-3)\lambda}\} x^{\lambda-2} + \sum_{\lambda=0}^{\infty} a_\lambda x^{\lambda} = 0,$$

so that

$$a_{j+2} = -\frac{1}{(j-1)(j+2)}\, a_j.$$

Thus, $a_2 = \frac{1}{2} a_0$, $a_4 = -\frac{1}{8} a_0$, ...

$\therefore \; y_2(x) = \text{const.}\left(\frac{1}{x^2} + \frac{1}{2} - \frac{1}{8}x^2 + \dots\right)$.

(b) The desired $f(x)$ is given by

$f(x) \sim \exp \int^x -\frac{1}{2} p_1(x)\, dx$. With $p_1 = \frac{2}{x}$, we get

$f(x) \sim 1/x$.

Substituting $y(x) = \frac{1}{x} z(x)$, we get

$z'' = Q(x) z$, where $\underline{Q(x) = -\left(1 - \frac{2}{x^2}\right)}$.

The WKB formula now gives

$$z(x) \approx \frac{C_\pm}{Q^{1/4}} \exp \int^x \pm\sqrt{Q(x)}\, dx.$$

Note that

$$\sqrt{Q(x)} = \pm i \sqrt{1 - \frac{2}{x^2}} \simeq \pm i \left(1 - \frac{1}{x^2}\right) \qquad (x \gg 1)$$

$$\approx \pm i, \text{ as } x \text{ goes to } \infty.$$

Hence

$$z(x) \approx C_\pm \, e^{\pm i x}$$

and

$$y(x) \approx C_\pm \frac{1}{x} e^{\pm i x} \quad \underline{\text{or}} \; \text{const.} \frac{\cos x}{x} \;\&\; \text{const.} \frac{\sin x}{x}.$$

$\rightarrow$ Note that the actual solutions of this equation are, in fact,
$y_1(x) \sim \left\{\frac{\cos x}{x} - \frac{\sin x}{x^2}\right\}$ and $y_2(x) \sim \left\{\frac{\sin x}{x} + \frac{\cos x}{x^2}\right\}$.