Prancing Through Quantum Fields

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Warning

The ideas presented here are **less than rigorous** and their "derivations" are of the **lowest possible quality**. They are meant only to build intuition and serve as analogies. However, the problem with analogies is that, while they enlighten one aspect, they can break down when applied to even closely related ideas.

You've Been Warned!

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Vector Space

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Vector Space

Particles $\sim$ Vectors $\sim |\psi\rangle \in \mathcal{H}$

Physics $\sim$ Operators $\sim \mathcal{O}$

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  - In deriving the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{x},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x},t)\right)\psi(\mathbf{x},t)$$

    we plugged in a non-relativistic Hamiltonian, $H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x},t)$ leading us to an equation that is first-order in time but second order in space.
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  - Position is an operator, whereas time is only a parameter.
- Mass and energy are interchangeable $\implies$ particle number is no longer conserved, but in quantum mechanics, there's no way to take an n-particle state $\bigotimes^n |\psi_i\rangle$, and get out an m-particle state.

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Square both sides $\implies$

$$\boxed{\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi(\mathbf{x}, t) = -\frac{m^2 c^2}{\hbar^2}\psi(\mathbf{x}, t)}$$

Klein-Gordon Equation

Problem: $\rho = |\psi(\mathbf{x}, t)|^2 \ngeq 0$

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Two options for dealing with position as an operator:

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As for the problem of fixed Hilbert space:

- Work in Fock Space

$$\mathcal{F} = \bigotimes_{i=0}^{\infty} \mathcal{H}_i$$

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Quantum field theory says our building blocks are **Fields**, and elementary particles are excitations of these fields.

- Why is this any better?
  - Wave nature no longer a mystery.
  - Indistinguishability of identical particles now becomes obvious.
  - Non-conservation of particle number no longer a problem.

Classical Fields

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We'd like this to fit into a Lagrangian framework, and thus write down a Lagrangian for our field.

$$L = \int d^3x \mathcal{L} \implies S = \int dt L = \int d^4x \mathcal{L}$$

Real Scalar Field Lagrangian

Causality and Lorentz invariance fixes our kinetic term, to have no more than two derivatives:

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Apply Euler-Lagrange Equations:

$$\delta S = 0 \implies \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

Where $\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}(\mathbf{x}, t)$ is canonical momentum conjugate to the field.

Applying The Euler-Lagrange Equations.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi} \implies$$

$$\boxed{\left(\partial_\mu \partial^\mu + m^2\right) \phi(\mathbf{x}, t) = 0}$$

Note: $\phi(\mathbf{x}, t)$ **does not describe the evolution of a wavefunction!**

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Plug in Fourier transform and use reality condition:

$$\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}} e^{ik\cdot x} + a_{\mathbf{k}}^\dagger e^{-ik\cdot x}\right]$$

Where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$ and $k \cdot x = k^\mu x_\mu = \omega_{\mathbf{k}} t - \mathbf{k} \cdot \mathbf{x}$.

Quantization Procedure

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$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

Implies that proper quantization conditions should be

$$\phi(\mathbf{x}, t) \mapsto \hat{\phi}(\mathbf{x}, t) \;,\; \pi(\mathbf{x}, t) \mapsto \hat{\pi}(\mathbf{x}, t)$$
$$[\hat{\phi}(\mathbf{x}, t), \hat{\pi}(\mathbf{y}, t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

Field Operators

By promoting the fields $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{x}, t)$ to operators, this means that our Fourier coefficients must be promoted to operators:

$$\hat{\phi}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[\hat{a}_{\mathbf{k}} e^{ik\cdot x} + \hat{a}_{\mathbf{k}}^\dagger e^{-ik\cdot x}\right]$$
$$\hat{\pi}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[\hat{a}_{\mathbf{k}} e^{ik\cdot x} - \hat{a}_{\mathbf{k}}^\dagger e^{-ik\cdot x}\right]$$

This implies the following algebraic properties for the a-operators.

$$\hat{a}_{\mathbf{k}} = \int d^3x e^{ik\cdot x} \left[-i\hat{\pi}(\mathbf{x}, t) + \omega_{\mathbf{k}} \hat{\phi}(\mathbf{x}, t)\right]$$
$$\hat{a}_{\mathbf{k}}^\dagger = \int d^3x e^{ik\cdot x} \left[i\hat{\pi}^\dagger(\mathbf{x}, t) + \omega_{\mathbf{k}} \hat{\phi}^\dagger(\mathbf{x}, t)\right]$$
$$[a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

Creation & Annihilation Operators

This should look familiar - these are nothing but the creation and annihilation operators from the harmonic oscillator of ordinary quantum mechanics.

$$a = \frac{1}{2}\left(-ip + \omega x\right), a^\dagger = \frac{1}{2}\left(+ip + \omega x\right)$$

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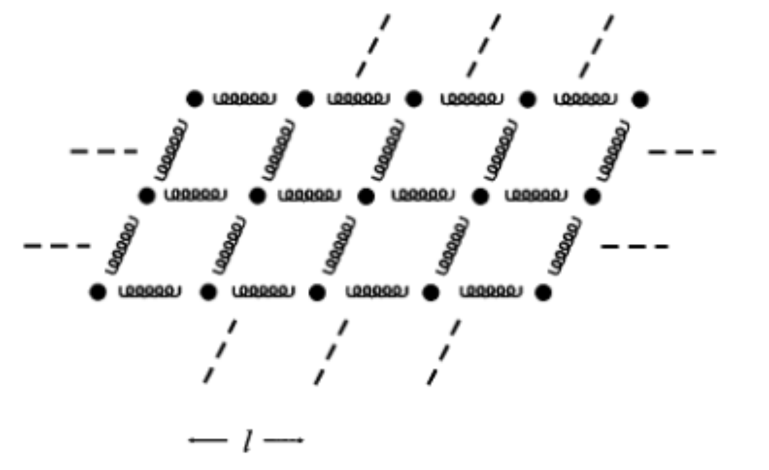
$$[a, a^\dagger] = 1$$

This then means that we can interpret the operator $a^\dagger_{\mathbf{k}}$ as an operator that creates a state of definite momentum $\mathbf{k}$.

$$\boxed{a^\dagger_{\mathbf{k}}|0\rangle = |\mathbf{k}\rangle}$$

$$\boxed{\boxed{\hat{\phi}(\mathbf{x}, t)|0\rangle = |\mathbf{x}\rangle}}$$

The Mattress



The Mattress: Lagrangian

We could write down a Lagrangian for this system, denoting the excitation height as $q_a(t)$

$$L = \frac{1}{2}\left(\sum_a m\dot{q}_a^2 - \sum_{a,b} k_{ab} q_a q_b - \sum_{a,b,c} g_{abc} q_a q_b q_c - \cdots\right)$$

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We can apply the Euler-Lagrange equations and find the resulting equation of motion. Further, we can see what happens in the continuum limit (i.e. $\ell \to 0$). If we do this, the discrete indices on $q_{x,y}(t)$ become continuous variables $\phi(x,y,t)$, and amazingly our equation of motion is

$$\left(\Box + \mu^2\right)\phi = 0 \tag{1}$$

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$$[\hat{q}_{x,y}(t), \hat{p}_{x',y'}(t)] = i\delta_{x,x'}\delta_{y,y'}$$

There are a few interesting facts to note here

- Each of these oscillators will now have zero-point motion ($\frac{1}{2}\hbar\omega$), and since we have an infinite number of them, we have infinite zero-point energy.
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Now, as we shrink further and go to the continuum limit the indices x, y become continuous variables and so our operators then become:

$$\hat{q}_{x,y}(t) \mapsto \hat{\phi}(x,y,t) \ , \ \hat{p}_{x,y}(t) \mapsto \hat{\pi}(x,y,t)$$
$$[\hat{\phi}(x,y,t), \hat{\pi}(x',y',t)] = i\delta^{(2)}(\mathbf{x} - \mathbf{x}')$$

The Mattress: Visualization

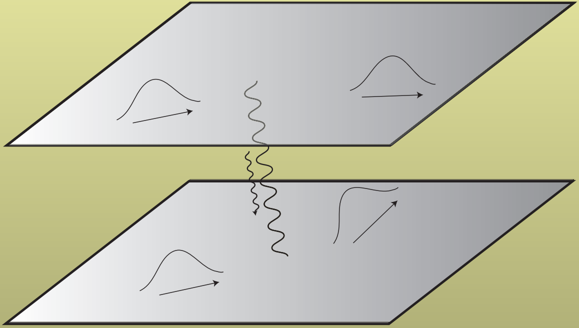

Figure: $\mu^- e^- \to \mu^- e^-$

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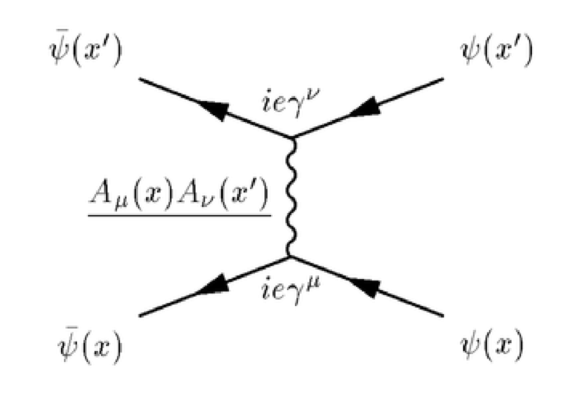

Figure: Feynman Diagram for $\mu^- e^- \rightarrow \mu^- e^-$

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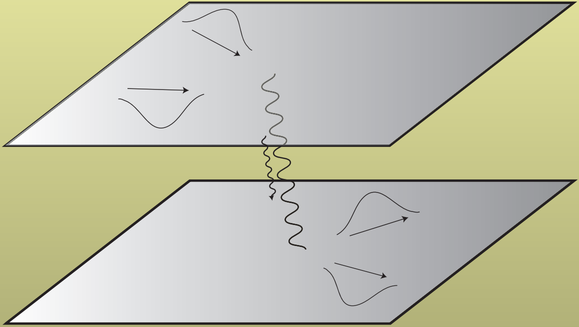

Figure: $e^+e^- \to \mu^+\mu^-$

Why Do We Need Renormalization?

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Empty space isn't empty! It is filled with quantum fluctuations.

Particles can interact with these quantum fluctuations, how can we deal with these interactions?

This is the job of renormalization.

Renormalization: Effective Parameters

Instead of dealing with these interactions explicitly, we can account for them by absorbing their effects into our parameters.

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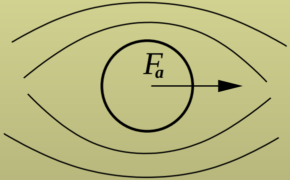
Imagine we have a ball of mass m immersed in a fluid of density ρ.



$$F_a = ma + Sa = m^* a$$
$$m^* = m + S = m + \kappa \rho V$$

κ is a number that depends on the shape of the object.

Renormalization: Running Of Effective Parameters

In further analogy, the effective mass will depend on temperature.

$$\rho \sim \frac{1}{T} \implies \rho(T) = \frac{\rho_0 T_0}{T}$$

Where ρ_0 and T_0 are some reference density and temperature. This implies that our effective mass will also be temperature dependent.

$$m^*(T) = m\left(1 + \frac{S}{m}\right)$$

$$m^*(T) = m\left(1 + \frac{\kappa\rho(T)}{\rho_{body}}\right)$$

$$m^*(T) = m\left(1 + \frac{\kappa\rho_0}{\rho_{body}}\left(\frac{T_0}{T}\right)\right)$$

The mass of an isolated u-quark is $1.5 - 3.0$ MeV, but the mass of a Δ^{++} is 1230 MeV.

Renormalization: Charge Renormalization

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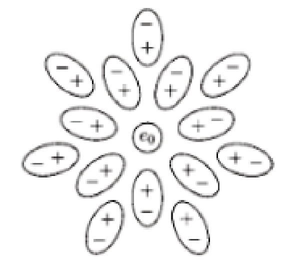
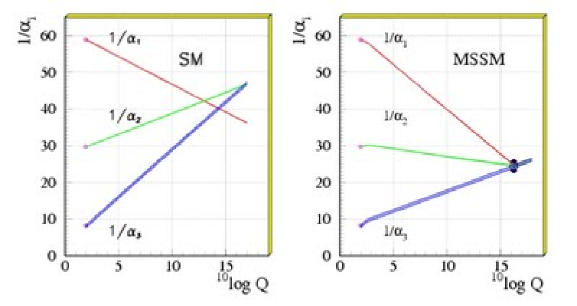


Vacuum fluctuations *screen* charges.

Renormalization: Running Of The Charges



The Higgs Mechanism: Mass Via Interactions

We've seen that the parameters of our Lagrangian are affected by the interactions of our field quanta with vacuum fluctuations. The key point being that these corrections arose due to *interactions*.

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Higgs acquires a VEV:

$$h = \langle h \rangle + \phi$$

Plugging this into our Lagrangian we get terms like

$$\mathcal{L} = y\langle h \rangle\bar{\psi}\psi + g^2\langle h \rangle^2 A_\mu A^\mu + \cdots$$

Just like in our sphere in liquid analogy, if we try and push a particle, we have to push all the Higgses around too, thus giving particles mass, via a *viscous-like force*.

The Higgs Mechanism: Analogy

Figure: To understand the Higgs mechanism, imagine that a room full of physicists chattering quietly is like space filled with the Higgs field ...

The Higgs Mechanism: Analogy

Figure: ... a well-known scientist walks in, creating a disturbance as he moves across the room and attracting a cluster of admirers with each step ...

The Higgs Mechanism: Analogy

Figure: ... this increases his resistance to movement, in other words, he acquires mass, just like a particle moving through the Higgs field...

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Figure: ... it creates the same kind of clustering, but this time among the scientists themselves. In this analogy, these clusters are the Higgs particles.

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We can use our mattress visualization to understand the Unruh effect quite easily. If we start with a state of zero particles (but remember we still have quantum fluctuations!) and we accelerate the mattress, all of the springs will compress. This will add energy to the system and cause some of the quantum fluctuations to become true and honest excitations.