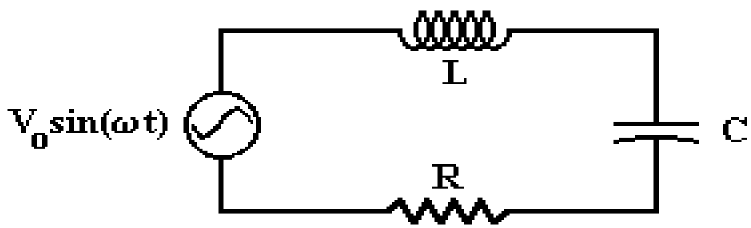
Experiment 3: Resonance in LRC Circuits Driven by Alternating Current

Goal: examine LRC series circuit driven by AC

1. Determine quality factor, resonance frequency, and bandwidth

2. Measure phase shift (extra credit)



Complex Impedance

The complex generalization of resistance is <u>impedance</u> $Z = \dfrac{\tilde{V}}{\tilde{I}}$

$\tilde{V} = V_0 e^{i\omega t} = Z\,\tilde{I}$ $\tilde{I} = I_0 e^{i(\omega t+\phi)}$ ϕ is the amount by which $\tilde{V}$ and $\tilde{I}$ are out of phase

a capacitor

$$V_C = \frac{Q}{C} = \frac{1}{C}\int I dt \qquad \frac{dV_C}{dt} = \frac{I}{C}$$

$$\begin{aligned}\tilde{I} &= C\frac{d}{dt}\left(V_{C0}e^{iwt}\right)\\ &= i\omega C\left(V_{C0}e^{iwt}\right)\\ &= i\omega C\left(Z_C\,\tilde{I}\right)\end{aligned}$$

$$\boxed{Z_C = \frac{1}{i\omega C}}$$

an inductor

$$V_L = L\frac{dI}{dt}$$

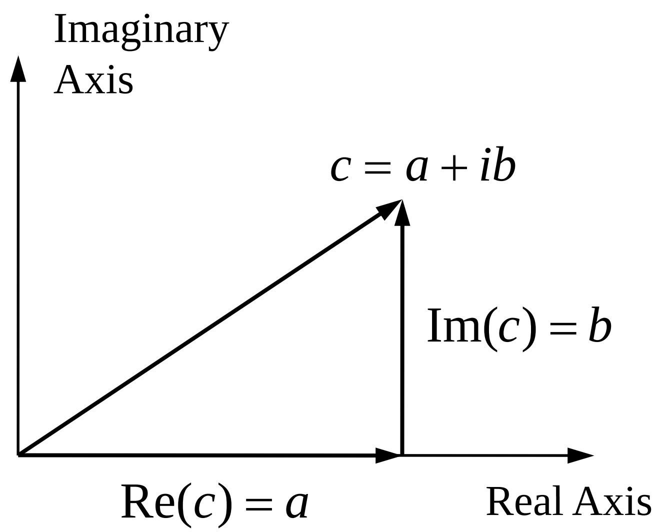
$$\begin{aligned}\tilde{V}_L &= L\frac{d}{dt}\left(\frac{\tilde{V}_L}{Z_L}\right)\\ &= \frac{i\omega L}{Z_L}V_{L0}e^{i\omega t}\\ &= \frac{i\omega L}{Z_L}\tilde{V}_L\end{aligned}$$

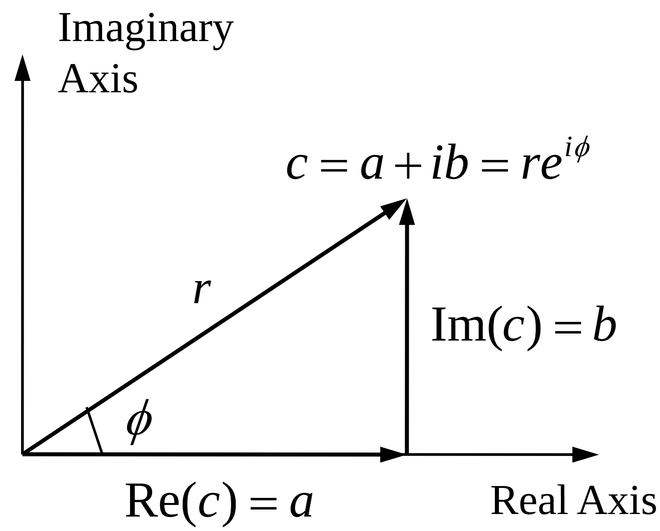
$$\boxed{Z_L = i\omega L}$$

Complex Numbers

$c = a + ib$

a and b are real numbers, $i = \sqrt{-1}$



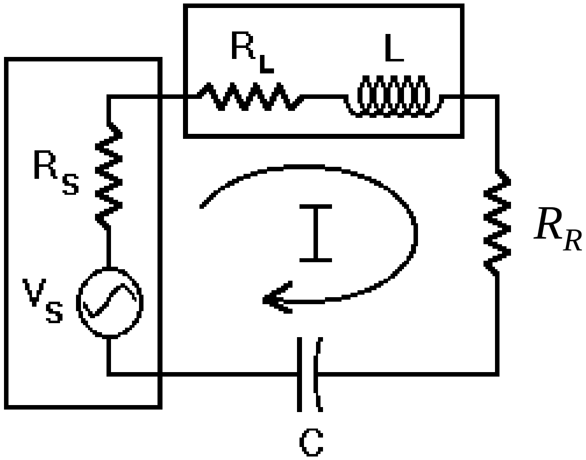


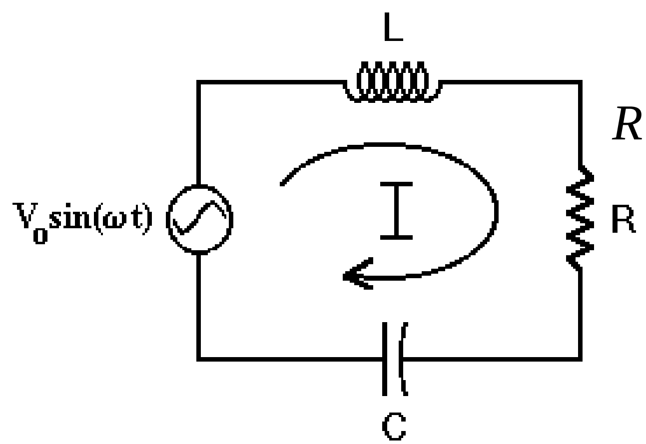
relationships between Cartesian and polar coordinates

$$r = \sqrt{a^2 + b^2} \qquad a = r\cos(\theta)$$

$$\theta = \arctan\left(\frac{b}{a}\right) \qquad b = r\sin(\theta)$$

LRC Series Circuit Driven by AC





$$R = (R_S + R_L + R_R)$$

R is the total resistance in the loop

R_R is the resistance of only the resistor

Kirchhoff's Law $\quad V_S + V_L + V_R + V_C = 0$

$$V_S - I(Z_L + Z_R + Z_C) = 0 \rightarrow V_S = I\left(i\omega L + R + {}^{1}\!/\!_{(i\omega C)} \right)$$

$$c = a + ib = re^{i\phi}$$

$$r = \sqrt{a^2 + b^2}$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

$$V_S - I(Z_L + Z_R + Z_C) = 0 \rightarrow V_S = I\left(i\omega L + R + 1/(i\omega C) \right)$$

the total impedance

$$\left[R + i\left(\omega L - \frac{1}{\omega C} \right) \right] = \sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega LC} \right)^2} \exp\left[i \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC} \right) \right]$$

$$\left| Z_{Total} \right| = \sqrt{R^2 + L^2\left(\omega - \frac{1}{\omega LC} \right)^2} \quad = R\sqrt{1 + Q^2\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}, \text{ and } Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{\omega_0 RC}$$

$$\phi = \arctan\left(\frac{\omega L}{R} - \frac{1}{\omega RC} \right) = \arctan\left[Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right) \right]$$

$$\tilde{V} = V_0 e^{i\omega t}$$

$$\tilde{V} = \tilde{I} Z$$

$$V_0 e^{i\omega t} = \tilde{I} \left| Z_{total} \right| e^{i\phi}$$

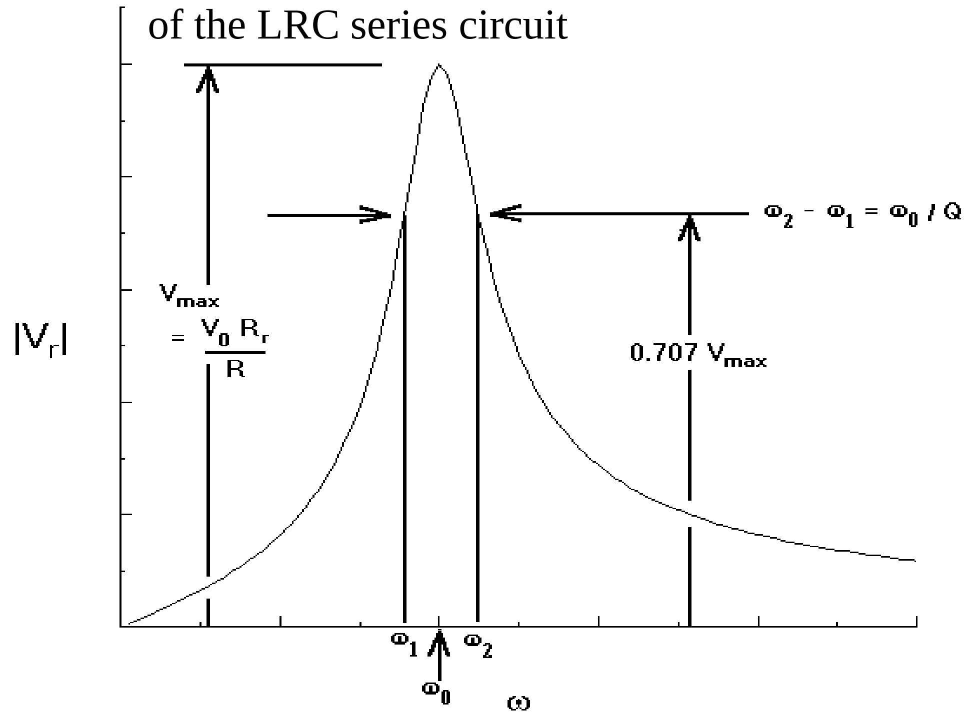
$$\tilde{I} = \frac{V_0}{\left| Z_{total} \right|} e^{i(\omega t - \phi)}$$

Voltage over the resistor

$$|V_R| = |I||Z_R| = V_0 \frac{|Z_R|}{|Z_{Total}|}$$

$$= \frac{V_0 R_R}{R\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}}$$

the response function of the LRC series circuit



V_R is greatest when $\omega = \omega_0$

↑ resonance frequency

power $P \propto V^2$

the half power points occur when the voltage decreases to $1/\sqrt{2}$ of its peak value, which happens at $\omega = \omega_1$ and $\omega = \omega_2$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}} \rightarrow Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2 = 1 \longrightarrow \boxed{Q = \frac{\omega_0}{\omega_2-\omega_1}}$$

← bandwidth

↖ way to measure Q

The Q multiplier

$$|Z_{total}| = R\sqrt{1+Q^2\left(\frac{\omega}{\omega_0}-\frac{\omega_0}{\omega}\right)^2}$$

$$|Z_{total}| = R \quad \text{at } \omega = \omega_0$$

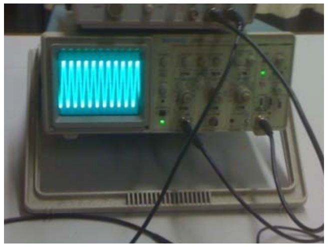
voltage over the capacitor
at $\omega = \omega_0$

$$V_{C0} \equiv |V_C(\omega_0)| = \frac{V_0}{|Z_{total}|}\frac{1}{\omega_0 C} = \frac{V_0}{R}\frac{1}{\omega_0 C} = \frac{V_0}{R}\sqrt{\frac{L}{C}} = V_0 Q$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

$$Q = \frac{V_{C0}}{V_0}$$

way to measure Q

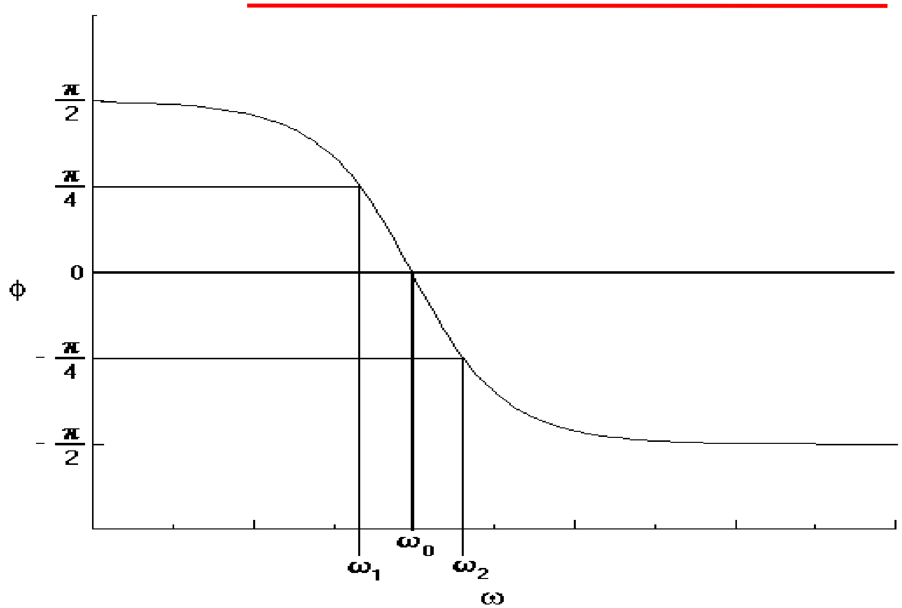


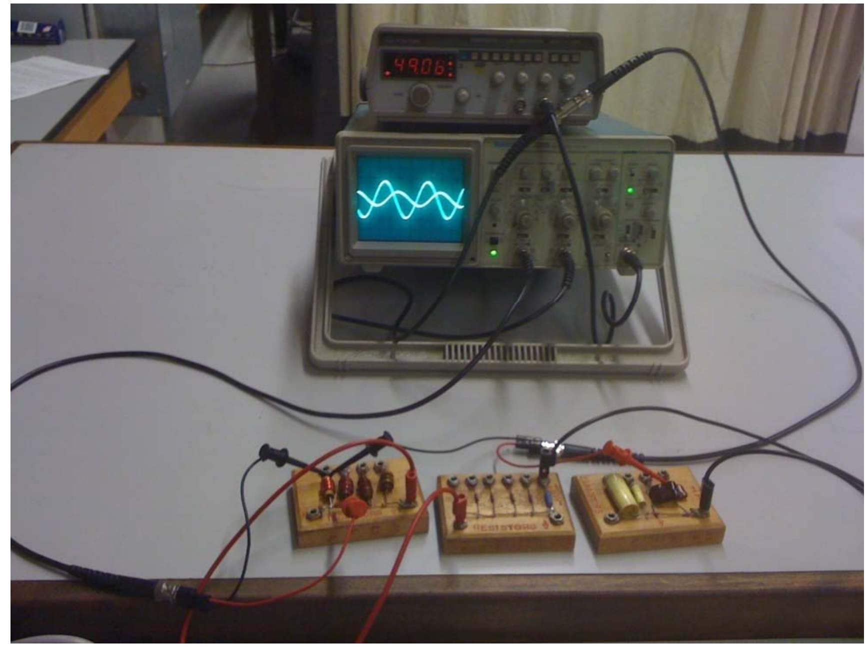
Voltage Phase Offsets of Circuit Elements

for a resistor

$$\tilde{V}_R = \tilde{I} Z_R = \tilde{I} R_R = \frac{V_0}{|Z_{total}|} R_R e^{i(\omega t - \phi)}$$

where $\phi = \phi_R(\omega) \equiv -\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right)$





$$\phi_R(\omega_1) \equiv -\arctan(-1) = \frac{\pi}{4}$$

$$\phi_R(\omega_0) \equiv 0$$

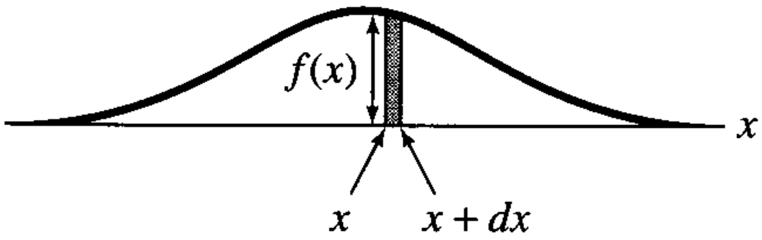
$$\phi_R(\omega_2) \equiv -\arctan(+1) = -\frac{\pi}{4}$$

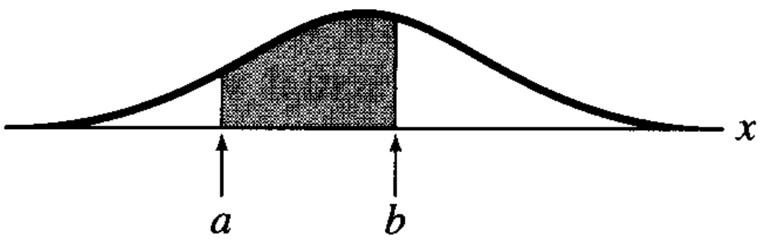
following same procedure:

$$\phi_L(\omega) \equiv -\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) + \frac{\pi}{2}$$

$$\phi_C(\omega) \equiv -\arctan\left(Q\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)\right) - \frac{\pi}{2}$$

Limiting Distributions



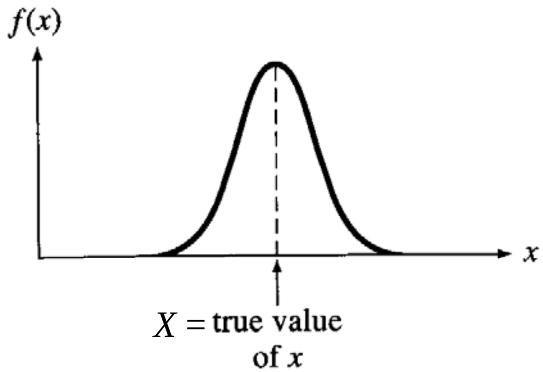


$f(x)\,dx$ = fraction of measurements that fall between x and $x+dx$
= probability that any measurement will give an answer between x and $x+dx$

$\int_a^b f(x)dx$ = fraction of measurements that fall between $x=a$ and $x=b$
= probability that any measurement will give an answer between $x=a$ and $x=b$

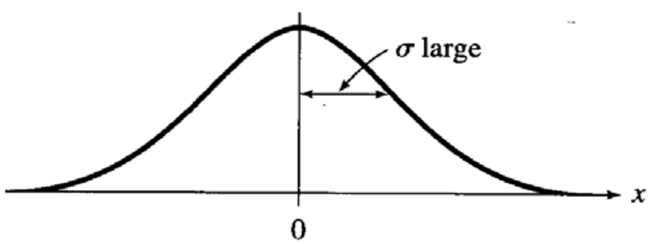
$$\int_{-\infty}^{+\infty} f(x)dx = 1$$ normalization condition

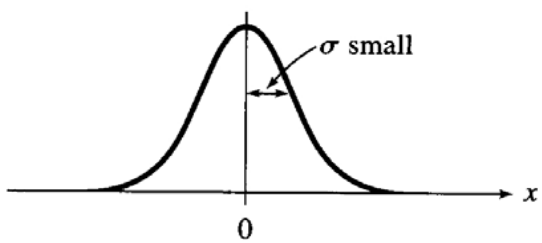
The Gauss, or Normal Distribution



the limiting distribution for a measurement subject to many small random errors is bell shaped and centered on the true value of *x*

the mathematical function that describes the bell-shape curve is called the <u>normal distribution</u>, or <u>Gauss function</u>





prototype function

$$e^{-x^2/2\sigma^2}$$

$$e^{-(x-X)^2/2\sigma^2}$$

σ – width parameter

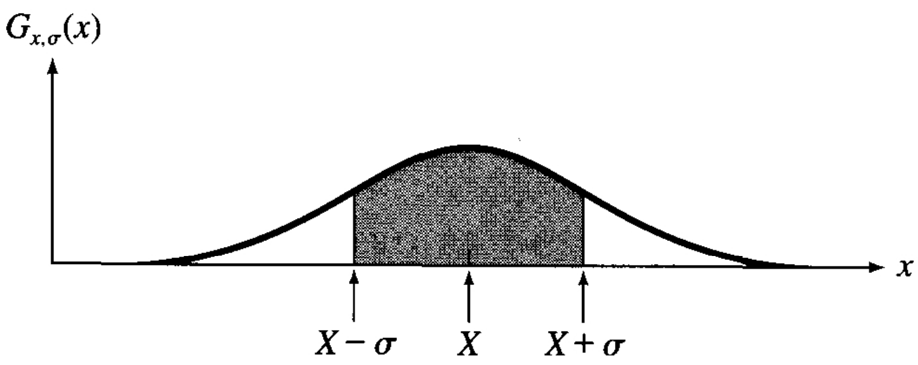
X – true value of x

The Gauss, or Normal Distribution

normalize $e^{-(x-X)^2/2\sigma^2}$ $\longrightarrow$ $\int_{-\infty}^{+\infty} f(x)dx = 1$

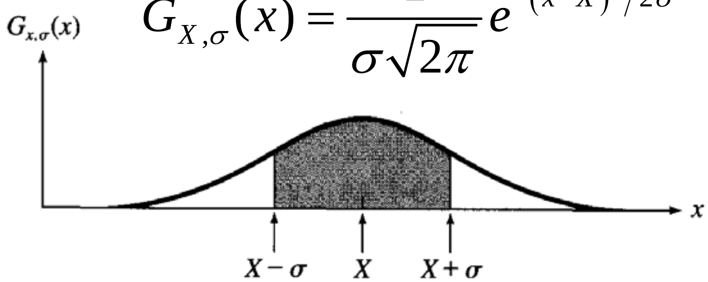
$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$

standard deviation σ_x = width parameter of the Gauss function σ
the mean value of x = true value X



The standard Deviation as 68% Confidence Limit

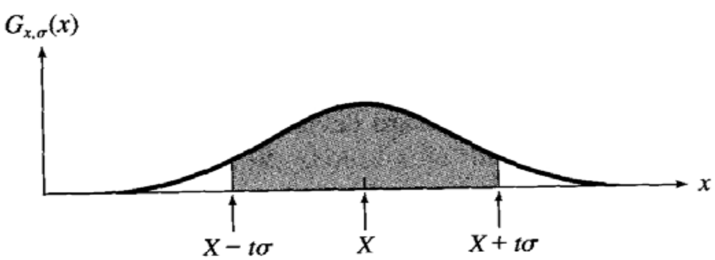
$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-X)^2/2\sigma^2}$$



$$\text{Prob(within } \sigma) = \int_{X-\sigma}^{X+\sigma} G_{X,\sigma}(x)dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{X-\sigma}^{X+\sigma} e^{-(x-X)^2/2\sigma^2} dx$$

$$(x - X)/\sigma = z$$



$$\text{Prob(within } \sigma) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-z^2/2} dz$$

$$\text{Prob(within } t\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-z^2/2} dz$$

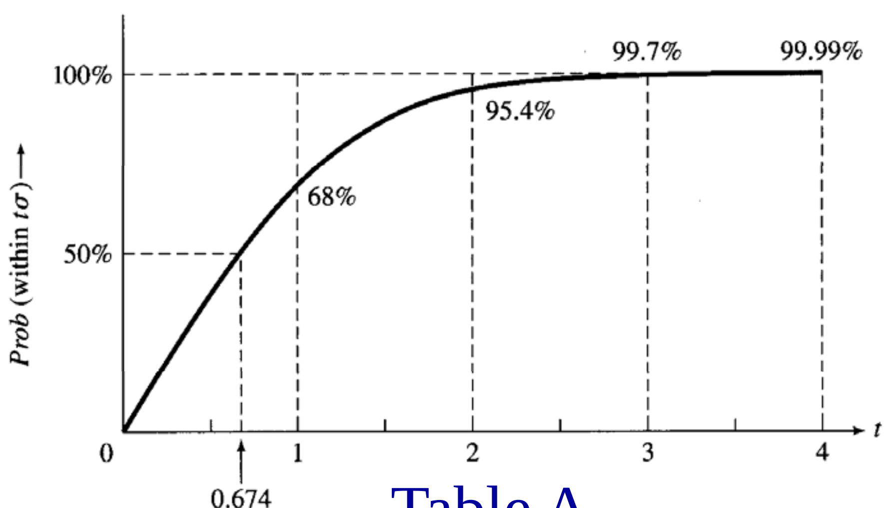


Table A.

← *erf(t)* – error function

the probability that a measurement will fall within one standard deviation of the true answer is 68 %

$x = x_{best} \pm \delta x \qquad \delta x = \sigma$