Physics 2D Final Exam solutions Fall 2009

Problem 1

(a) According to Newton, $m_1 + m_2 = M$

momentum conservation: $2v m_1 = v m_2 \Rightarrow 2m_1 = m_2 \Rightarrow$

$\Rightarrow \boxed{m_1 = M/3, \quad m_2 = 2M/3} \Rightarrow m_1 = 333.3\ \mathrm{MeV}/c^2, \; m_2 = 666.6\ \mathrm{MeV}/c^2$

$v = 0.4c$

(b) Energy conservation:

$M = m_1\gamma_1 + m_2\gamma_2 \quad ; \quad \gamma_1 = \gamma(2v), \; \gamma_2 = \gamma(v)$

Momentum conservation:

$m_1\gamma_1 \cdot 2v = m_2\gamma_2 v \Rightarrow 2m_1\gamma_1 = m_2\gamma_2$

$\Rightarrow \quad M = 3m_1\gamma_1 \quad \Rightarrow \quad m_1 = \dfrac{M}{3\gamma_1}, \quad m_2 = \dfrac{2M}{3\gamma_2}$

$\gamma_1 = \dfrac{1}{\sqrt{1 - \frac{4v^2}{c^2}}} = 1.667 \quad ; \quad \gamma_2 = \dfrac{1}{\sqrt{1 - v^2/c^2}} = 1.091$

$\Rightarrow \quad m_1 = 0.20M, \quad m_2 = 0.611M \quad \Rightarrow \boxed{m_1 = 200\ \frac{\mathrm{MeV}}{c^2}, \quad m_2 = 611\ \mathrm{MeV}/c^2}$

$\Rightarrow \quad m_1 + m_2 = 0.811M$

$\Rightarrow$ the mass deficit is

$\boxed{\Delta M = 0.189M = 189\ \frac{\mathrm{MeV}}{c^2}}$

was converted into kinetic energy.

Particle 1 kinetic energy

$K_1 = (\gamma_1 - 1)m_1c^2 = 0.667 \times 200\ \mathrm{MeV}$

$\Rightarrow \boxed{K_1 = 133\ \mathrm{MeV}}$

Particle 2 kinetic energy:

$K_2 = (\gamma_2 - 1)m_2c^2 = 0.091 \times 611\ \mathrm{MeV}$

$\boxed{K_2 = 55.6\ \mathrm{MeV} \approx 56\ \mathrm{MeV}}$

$\boxed{K_1 + K_2 = 189\ \mathrm{MeV} = \Delta Mc^2}$

Problem 2

The maximum kinetic energy of photoemitted electrons is

$$K = hf - \phi$$

with f = frequency of incident light, ϕ = work function.

For electrons to be emitted we need $hf > \phi = 2.28\,eV$

For light of wavelength 6000 Å,

$$hf_0 = \frac{hc}{6000\text{Å}} = \frac{12,400}{6000}\,eV = 2.067\,eV \text{ is too small.}$$

However, if the flashlight is approaching the Na plate, the frequency increases due to the Doppler effect. We need $hf = 2.28\,eV$

$$hf = hf_0\sqrt{\frac{1+v/c}{1-v/c}} = 2.28\,eV \Rightarrow \sqrt{\frac{1+v/c}{1-v/c}} = \frac{f}{f_0} \Rightarrow$$

$$\frac{1+v/c}{1-v/c} = \frac{f^2}{f_0^2} \Rightarrow v/c\left(1 + f^2/f_0^2\right) = \frac{f^2}{f_0^2} - 1 \Rightarrow \frac{v}{c} = \frac{f^2/f_0^2 - 1}{1 + f^2/f_0^2}$$

$$f/f_0 = \frac{hf}{hf_0} = \frac{2.28}{2.067} = 1.103 \Rightarrow \frac{v}{c} = 0.098$$ or larger for electrons to be emitted.

$$\Rightarrow \boxed{v \geq 0.098\,c} \text{ (a)}$$

(b) for $v = 0.8c$, $$hf = hf_0\sqrt{\frac{1+0.8}{1-0.8}} = 3hf_0 = 6.2\,eV$$

$$\Rightarrow \boxed{K = 6.2\,eV - 2.28\,eV = 3.92\,eV}$$

Problem 3

$$U(r) = -\frac{ke^2 Z}{r} = -245\,eV$$

In n-th orbit, $r_n = r_0 n^2 = \frac{a_0 n^2}{Z} \Rightarrow$

$$U(r_n) = -\frac{ke^2}{a_0}\frac{Z^2}{n^2} = -\frac{14.4\,eV\text{Å}}{0.529\,\text{Å}}\frac{Z^2}{n^2} = -27.22\,eV\frac{Z^2}{n^2} = -245\,eV$$

$$\Rightarrow \frac{Z^2}{n^2} = 9 \Rightarrow \boxed{Z = 3n}$$

So we may have $Z=3$ if $n=1$, $Z=6$ if $n=2$, $Z=9$ if $n=3$, etc.

(b) The total energy is

$$E = -\frac{ke^2 Z}{2r_n} = \frac{1}{2}U = -122.5\,eV$$

So ionization energy is $\boxed{I = 122.5\,eV}$ for any of the Z, n values found in (a)

(c) We can find the speed from

$$L = n\hbar = m_e v_n r_n \Rightarrow \frac{v_n}{c} = \frac{n\hbar}{m_e c r_n} = \frac{\hbar c}{m_e c^2}\frac{n}{r_0 n^2} = \frac{\hbar c}{m_e c^2 a_0}\frac{Z}{n}$$

Since $Z/n = 3$, $a_0 = \hbar^2/ke^2 m_e$

$$\frac{v_n}{c} = 3\frac{\hbar c}{m_e c^2 a_0} = 3\frac{\hbar k e^2 m_e}{m_e c \hbar^2} = 3\frac{ke^2}{\hbar c} = \frac{3}{137}$$

$$\Rightarrow \boxed{\frac{v_n}{c} = 0.022}$$

Problem 4

$$U(r) = -\frac{ke^2 a_0^{1/2}}{r^{3/2}}$$

Uncertainty: $\Delta p \Delta x \simeq \hbar$, $\Delta x \sim r \Rightarrow \Delta p \sim \frac{\hbar}{r}$.

Kinetic energy $K = \frac{p^2}{2m_e} \simeq \frac{(\Delta p)^2}{2m_e} = \frac{\hbar^2}{2m_e r^2}$

Total energy:
$$E = K + U = \frac{\hbar^2}{2m_e r^2} - \frac{ke^2 a_0^{1/2}}{r^{3/2}}$$

Minimize

$$\frac{dE}{dr} = -\frac{\hbar^2}{m_e r^3} + \frac{3}{2}\frac{ke^2 a_0^{1/2}}{r^{5/2}} = 0 \Rightarrow r^{1/2} = \frac{2}{3}\frac{\hbar^2}{m_e k e^2 a_0^{1/2}}$$

Since $\frac{\hbar^2}{m_e k e^2} = a_0 \Rightarrow r^{1/2} = \frac{2}{3}\frac{a_0}{a_0^{1/2}} = \frac{2}{3} a_0^{1/2} \Rightarrow$

$$\boxed{r = \frac{4}{9} a_0 = 0.235\,\text{Å}}$$

(b) In a world with that form of $U(r)$, atoms and hence people would be approximately 1/2 the size ($4/9 \simeq 0.5$), or about 1 meter tall. This is because for small r, $r^{3/2}$ is smaller than r, so the $U(r)$ gives a stronger attractive force than the ordinary Coulomb law, making the atoms smaller.

Problem 5

$\Delta x = 0.5\ \mathring{A}$. $\Delta p\, \Delta x \simeq \hbar$, $p = \hbar k$,

$\Delta k\, \Delta x \simeq 1 \Rightarrow \Delta k \sim \frac{1}{\Delta x} \sim 2\ \mathring{A}^{-1}$

If the velocity is $0.1c \Rightarrow$ $p = m_e v = 0.1c \cdot m_e$

$p = \hbar k = 0.1 c m_e \Rightarrow k = \frac{0.1 c\, m_e}{\hbar} = 0.1 \frac{m_e c^2}{\hbar c} = 25.9\ \mathring{A}^{-1}$

So the center of the $a(k)$ should be at $k \sim 26\ \mathring{A}^{-1}$, and the width should be $2\ \mathring{A}^{-1}$ $\Rightarrow$ $\boxed{k_1 \sim 25\ \mathring{A}^{-1},\ k_2 \sim 27\ \mathring{A}^{-1}}$

(b) $\Delta p = \hbar \Delta k = \hbar c\, \Delta k / c = 1973\, eV\mathring{A} \times 2\ \mathring{A}^{-1}/c \Rightarrow$

$\boxed{\Delta p \simeq 3946\ eV/c}$

Problem 6

$$\psi(x) = e^{iax} + 0.4 e^{-iax} \qquad x < 0$$

$$\psi(x) = C e^{ia_2 x} \qquad x \geq 0$$

The reflection coefficient for a wave $Ae^{iax} + Be^{-iax}$ is $\left|\frac{B}{A}\right|^2 \Rightarrow$

$R = \left|\frac{0.4}{1}\right|^2 = 0.16 \Rightarrow$ for every 1,000 incident electrons,

$\boxed{\text{160 electrons are reflected}}$

(b) Continuity of the wavefunction :

$\psi(x=0^-) = \psi(x=0^+) \Rightarrow \quad 1 + 0.4 = C \Rightarrow C = 1.4$

Continuity of derivative:

$\psi'(x=0^-) = \psi'(x=0^+) \Rightarrow \quad k(1-0.4) = k_2 C \Rightarrow 0.6k = a_2 C \Rightarrow$

$\Rightarrow \quad k_2 = \frac{0.6}{C} k = \frac{0.6}{1.4} k = \boxed{0.429 \text{Å}^{-1}} \quad (k = 1 \text{Å}^{-1})$

(c) $\quad k_2 = \sqrt{\frac{2m}{\hbar^2}(E - U_0)} \Rightarrow U_0 = E - \frac{\hbar^2 k_2^2}{2m_e}$

and $k = \sqrt{\frac{2m}{\hbar^2} E} \Rightarrow E = \frac{\hbar^2 a^2}{2m_e}$

$\Rightarrow \quad U_0 = \frac{\hbar^2 k^2}{2m_e} - \frac{\hbar^2 a_2^2}{2m_e} = \frac{\hbar^2}{2m_e}(a^2 - a_2^2) \Rightarrow$

$U_0 = 3.81 \text{eV Å}^2 (1 \text{Å}^{-2} - 0.429^2 \text{Å}^{-2}) \Rightarrow \boxed{U_0 = 3.11 \text{ eV}}$

Problem 7

$$R(r) = \frac{1}{(2a_0)^{3/2}} \frac{r}{\sqrt{3}a_0} e^{-r/2a_0}$$

$$R^2(r) = \frac{1}{(2a_0)^3} \frac{r^2}{3a_0^2} e^{-r/a_0} \Rightarrow P(r) = r^2 R^2(r) = \frac{r^4}{24a_0^5} e^{-r/a_0}.$$

(a) most probable r is given by $P'(r) = 0 \Rightarrow$

$$\Rightarrow 4r^3 - \frac{r^4}{a_0} = 0 \Rightarrow \boxed{r = 4a_0}$$

(b)
$$\langle r \rangle = \int_0^\infty dr\, r P(r) = \frac{1}{24a_0^5} \int_0^\infty dr\, r^5 e^{-r/a_0} =$$

$$= \frac{1}{24a_0^5} \cdot 5!\, a_0^6 = \frac{5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{4}} a_0 \Rightarrow \boxed{\langle r \rangle = 5a_0}$$

(c)
$$\langle \tfrac{1}{r} \rangle = \int_0^\infty dr\, \frac{1}{r} P(r) = \frac{1}{24a_0^5} \int_0^\infty dr\, r^3 e^{-r/a_0} =$$

$$= \frac{1}{\underset{4}{\cancel{24}}a_0^5} a_0^4 \cdot \cancel{3 \cdot 2} = \frac{1}{4a_0} \Rightarrow \boxed{\langle \tfrac{1}{r} \rangle = \frac{1}{4a_0}}$$

(d)
$$\frac{P(r_1)}{P(r_2)} = \frac{r_1^4 e^{-r_1/a_0}}{r_2^4 e^{-r_2/a_0}}$$ for $r_1 = 4a_0$, $r_2 = 5a_0 \Rightarrow$

$$\boxed{\frac{P(r_1)}{P(r_2)} = \left(\frac{4}{5}\right)^4 e^{+1} = 1.113}$$

it is slightly more likely.

Problem 8

(a) I am positive!

(b) Na: $1s^2 2s^2 2p^6 3s^1$

(c) The ionization energy of Na is relatively small (5.1 eV) compared to other atoms, which means it is relatively easy for Na to lose an electron.

For hydrogen, ionization energy is $I = 13.6\,eV$

For a hydrogen-like ion, $I = 13.6\,eV \cdot \frac{Z^2}{n^2}$.

For an atom with many electrons, Z is screened by the inner electrons, so it should be replaced by Z_{eff}. Because Na has 1 electron in the outer shell ($n=3$), the other electrons are very effective in screening the nuclear Z and Z_{eff} is relatively small.

For Na, $Z_{eff}^2 = \frac{n^2}{13.6\,eV} \times 5.1\,eV = 3.38 \Rightarrow Z_{eff} = 1.84$.

As more electrons are added to the $n=3$ shell, Z_{eff} increases because electrons in the same shell screen less effectively.

So Mg has higher ionization energy than Na (7.6 eV) and Ar much higher (15.8 eV).