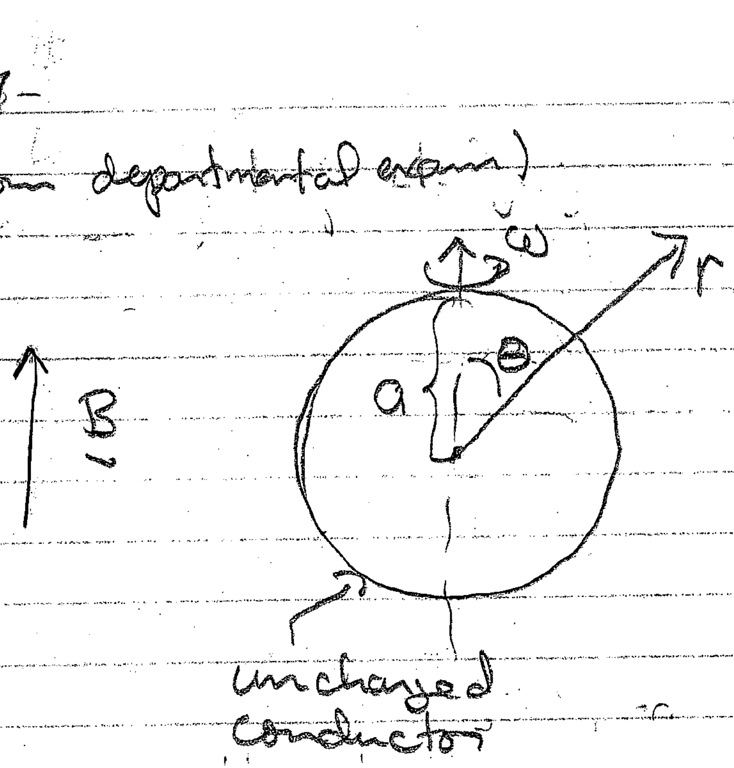
4/22/03 start

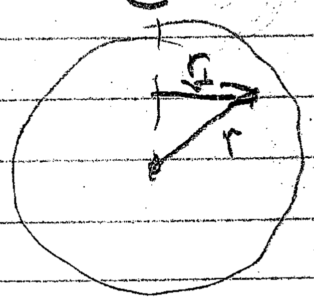
example (from departmental exam)



find steady state E-field and charge density

inside sphere

$$0 = \underset{\sim}{E} + \frac{\underset{\sim}{v}}{c} \times \underset{\sim}{B} = -\nabla \Phi_{in} + \frac{(\underset{\sim}{\omega} \times \underset{\sim}{r}) \times \underset{\sim}{B}}{c}$$



$$0 = -\underset{\sim}{\nabla} \Phi_{in} + \frac{\omega B}{c} \underset{\sim}{r_\perp}$$

$$\Phi_{in} = \frac{\omega B}{c} \frac{r_\perp^2}{2} + C$$

use cylindrical coordinates to evaluate $\nabla^2 \Phi_{in}$

$$-4\pi \rho = \frac{1}{r_\perp} \frac{\partial}{\partial r_\perp} r_\perp \frac{\partial \Phi_{in}}{\partial r_\perp} = \frac{2\omega B}{c}$$

$B_0 = 0$ for $Q_{total} = 0$

returning to spherical coordinates

$$r_\perp^2 = r^2 \sin^2\theta = r^2(1-\cos^2\theta)$$

$$\Phi_{in} = \frac{\omega B}{2c} r^2 (1-\cos^2\theta) + C$$

$$\Phi_{out} = \sum_l \left(A_l r^l + B_l/r^{l+1}\right) P_l(\cos\theta)$$

$\Phi(r) \to 0$ as $r \to \infty$

implies that $A_l = 0$ for all l

$$\Phi_{in}(a,\theta) = \Phi_{out}(a,\theta) \qquad E_t \text{ cont.}$$

$$C + \frac{\omega B}{2c} a^2 (1-\cos^2\theta) = \cancel{\frac{B_0}{a} P_0(\cos\theta)} + \frac{B_1}{a^2} \underbrace{P_1(\cos\theta)}_{\cos\theta}$$

$$+ \frac{B_2}{a^3} \underbrace{P_2(\cos\theta)}_{\frac{1}{2}[3\cos^2\theta - 1]} + \cdots$$

$$1 - \cos^2\theta = -\frac{2}{3} P_2(\cos\theta) \underbrace{-\frac{1}{3} + 1}_{2/3}$$

$$e + \frac{\omega B}{2c} a^2 \left[-\frac{2}{3} P_2(\cos\theta) + \frac{2}{3} \right] = \frac{B_0}{a} \underbrace{P_0(\cos\theta)}_{1} + \frac{B_1}{a} \underbrace{P_1(\cos\theta)}_{\cos\theta}$$

$$+ \frac{B_2}{a^3} P_2(\cos\theta) + -$$

$$\therefore B_2 = -\frac{2}{3} \frac{\omega B}{2c} a^5$$

$$B_1 = 0$$

$$e + \frac{\omega B}{2c} a^2 \frac{2}{3} = 0$$

$$\therefore \Phi_{out}(r,\theta) = -\frac{\omega B}{3c} \frac{a^5}{r^3} P_2[\cos\theta]$$

quadrupole field outside

surface charge

$$4\pi\sigma = \left.\frac{\partial \Phi_{in}}{\partial r}\right)_{r=a} - \left.\frac{\partial \Phi_{out}}{\partial r}\right)_{r=a}$$

$$= + \frac{\omega B}{c} a \left\{ \frac{-2}{3} P_2[\cos\theta] + \frac{2}{3} \right\} + \frac{\omega B a}{c} P_2[\cos\theta]$$

$$= \frac{\omega B a}{3c} P_2[\cos\theta] + \frac{2}{3}\frac{\omega B a}{c}$$

Have neglected change in B due to rotating charge (current)

Homework:

(12) A superconducting sphere excludes magnetic field from its interior (type I superconductor). Calculate the magnetic field near the sphere if it is placed in a uniform magnetic field $B_0 \hat{z}$.

(18) Everywhere inside a solid sphere of radius R the magnetization is $\underset{\sim}{M}$ = const. Calculate $\underset{\sim}{B}$ in all of space [Hint: $\underset{\sim}{\nabla} \times \underset{\sim}{H} = \frac{4\pi}{c} \underset{\sim}{J} = 0$

$\therefore \underset{\sim}{H} = -\underset{\sim}{\nabla} \varphi$

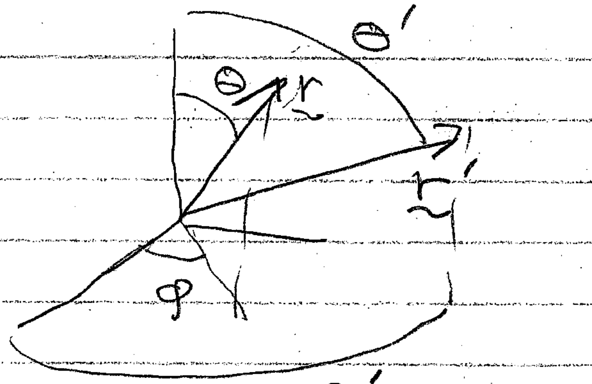
$0 = \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = \underset{\sim}{\nabla} \cdot (-\underset{\sim}{\nabla} \varphi + 4\pi \underset{\sim}{M})$]

varies only at surface

~~start~~

Multipole expansions re-examined

Jackson Chapter 4



~~J. page 102~~

$$\frac{1}{|\underset{\sim}{r} - \underset{\sim}{r}'|} = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

$r_<$ is smaller of $|\underset{\sim}{r}|, |\underset{\sim}{r}'|$

$r_>$ is larger of $|\underset{\sim}{r}|, |\underset{\sim}{r}'|$

potential at large distance from localized charge distribution

$$\varphi(\underline{r}) = \iiint d^3\underline{r}' \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}$$

$$\varphi(\underline{r}) = \sum_{l,m} \frac{4\pi}{2l+1} \Big[\underbrace{\iiint d^3\underline{r}' \, Y^*_{l,m}(\theta',\varphi') |\underline{r}'|^l \rho(\underline{r}')}_{\equiv q_{lm}} \Big] \frac{Y_{l,m}(\theta,\varphi)}{|\underline{r}|^{l+1}}$$

$$q_{0,0} = \frac{1}{\sqrt{4\pi}} \iiint \rho(\underline{r}') d^3\underline{r}' = \frac{q}{\sqrt{4\pi}}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \iiint \underbrace{r'\cos\theta'}_{z'} \rho(\underline{r}') d^3\underline{r}' = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \iiint r'\sin\theta' e^{-i\varphi'} \rho(\underline{r}') d^3\underline{r} = -\sqrt{\frac{3}{8\pi}}(p_x - ip_y)$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \iiint \underbrace{P_2(\cos\theta') r'^2}_{\frac{3}{2}z'^2 - \frac{1}{2}r'^2} \rho(\underline{r}') = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}$$

etc.

$$Q_{ij} = \iiint [3x_i'x_j' - r'^2\delta_{ij}] \rho(\underline{r}')$$

Electromagnetic waves   J. Chapter 7
L+L Vol 2 chapter 5
L+L Vol 8 chapter 9

In vacuum

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}, \quad \underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{1}{c}\frac{\partial \underset{\sim}{E}}{\partial t}$$

$$\underbrace{\nabla \times \nabla \times \underset{\sim}{E}}_{\underset{\sim}{\nabla}\,\underset{\sim}{\nabla}\cdot\underset{\sim}{E} - \nabla^2 \underset{\sim}{E}} = -\frac{1}{c}\frac{\partial}{\partial t}\underset{\sim}{\nabla}\times\underset{\sim}{B} = -\frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2}$$

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{E} = 0$$

$$\nabla^2 \underset{\sim}{E} - \frac{1}{c^2}\frac{\partial^2 \underset{\sim}{E}}{\partial t^2} = 0$$

monochromatic plane wave

$$\underset{\sim}{E} = \mathrm{Re}\, \underset{\sim}{E}_0 e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t}, \quad \omega^2 = c^2 k^2$$

$$0 = \underset{\sim}{\nabla}\cdot\underset{\sim}{E} = \mathrm{Re}\, i\underset{\sim}{k}\cdot\underset{\sim}{E}_0 e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t}$$

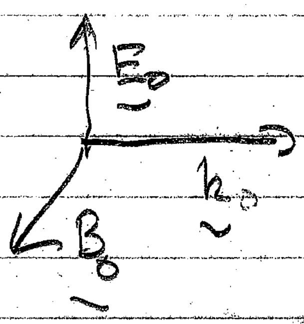
$\therefore \underset{\sim}{k}\cdot\underset{\sim}{E}_0 = 0$ transverse wave

Let $\underset{\sim}{B} = \mathrm{Re}\, \underset{\sim}{B}_0\, e^{i\underline{k}\cdot\underline{r} - i\omega t}$

↑ this functional form is necessary

$$\mathrm{Re}\; i\underset{\sim}{k}\times\underset{\sim}{E}_0\, e^{i\underline{k}\cdot\underline{r}-i\omega t} = \mathrm{Re}\; \frac{i\omega}{c}\underset{\sim}{B}_0\, e^{i\underline{k}\cdot\underline{r}-i\omega t}$$

$$\therefore\quad \underset{\sim}{B}_0 = \frac{c\underline{k}}{\omega}\times\underset{\sim}{E}_0 = \hat{k}\times\underset{\sim}{E}_0$$



note that $|\underset{\sim}{E}_0| = |\underset{\sim}{B}_0|$

↑ necessary, since electric and magnetic field energy are traded back and forth start

Polarization

let $\underset{\sim}{k} = k\hat{z}$

$$\underset{\sim}{E} = \mathrm{Re}\, \underset{\sim}{E}_0\, e^{ikz - i\omega t} \quad , \quad \hat{z}\cdot\underset{\sim}{E}_0 = 0$$

↑ complex vector

$$\underset{\sim}{E}_0\cdot\underset{\sim}{E}_0 = E_0^2 = |E_0^2|\, e^{-2i\alpha}$$

let $\underset{\sim}{b} \equiv \underset{\sim}{E}_0\, e^{+i\alpha}$

$$\therefore\ b^2 = E_0^2\, e^{+2i\alpha} = |E_0^2|$$

↑ real

let $\underset{\sim}{b} = \underset{\sim}{b}_1 + i\,\underset{\sim}{b}_2$

real

$$b^2 = b_1^2 - b_2^2 + 2i\,\underset{\sim}{b}_1\cdot\underset{\sim}{b}_2$$

$$\therefore\ \underset{\sim}{b}_1\cdot\underset{\sim}{b}_2 = 0$$

let $\hat{x} = \frac{\underset{\sim}{b_1}}{|\underset{\sim}{b_1}|}$, $\hat{y} = \frac{\underset{\sim}{b_2}}{|\underset{\sim}{b_2}|}$, $\hat{x} \otimes \hat{y} = \hat{z}$

$$\therefore \underset{\sim}{E_0} = (b_1 \hat{x} - i b_2 \hat{y}) e^{-i\alpha}$$

positive ← can be negative

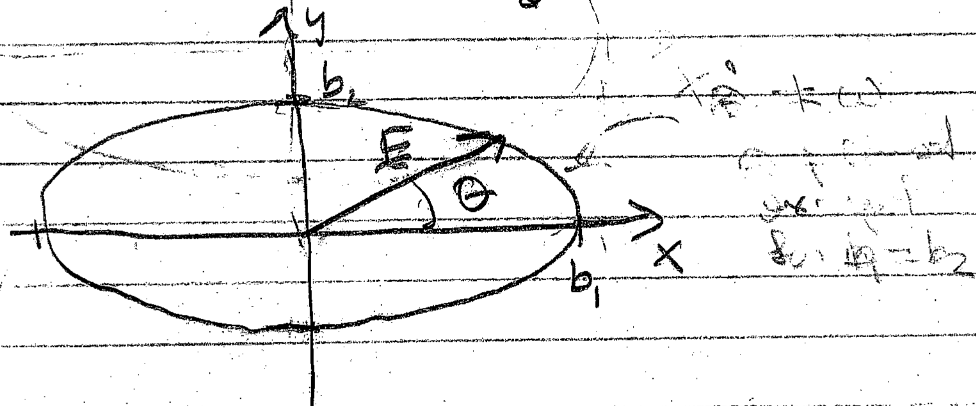
$$E_x = b_1 \cos(kz - \omega t - \alpha)$$

$$E_y = b_2 \sin(kz - \omega t - \alpha)$$

$$\therefore \frac{E_x^2}{b_1^2} + \frac{E_y^2}{b_2^2} = \cos^2(\;) + \sin^2(\;) = 1$$

↑ equation for ellipse

∴ elliptically polarized



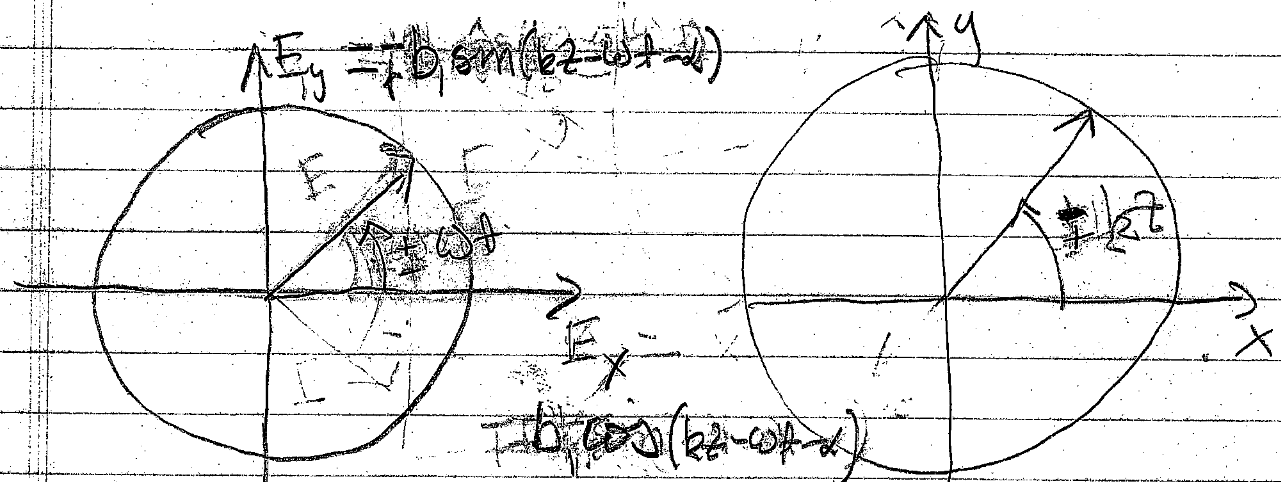
Special cases

(1) circularly polarized for $|b_1| = |b_2|$

$$E_0 = b_1 e^{-i\alpha}(\hat{x} \pm i\hat{y})$$

+ sign left hand circularly polarized (positive helicity)

− sign right hand circularly polarized (negative helicity)



fixed point in space

fixed point in time

Homework

(14) J. 7.28 (plane wave of finite cross section)

(15) J. 7.29 $\frac{k_z}{W} = \pm \frac{1}{\omega}$

(2) linearly polarized for

b_1 or $b_2 = 0$

for example $\rightarrow$ $\underset{\sim}{E}_0 = b_1 e^{-i\alpha}\hat{x}$

$$\underset{\sim}{E} = \hat{x} b_1 \cos(kz - \omega t - \alpha)$$

note that superposition of left hand circularly polarized wave and right hand circularly polarized wave of equal amplitude make a linearly polarized wave.

time average quantities

$$g(t) = \mathrm{Re}\, g\, e^{-i\omega t}$$

$$h(t) = \mathrm{Re}\, h\, e^{-i\omega t}$$

$$\langle g(t) h(t) \rangle = \frac{1}{\frac{2\pi}{\omega}} \int_0^{2\pi/\omega} dt \left[g_R \cos\omega t + g_I \sin\omega t \right] \cdot \left[h_R \cos\omega t + h_I \sin\omega t \right]$$

$$= \frac{1}{2} \left[g_R h_R + g_I h_I \right] = \frac{1}{2} \mathrm{Re}\, g\, h^*$$

Time average Poynting flux

$$\underset{\sim}{E} = \mathrm{Re}\, \underset{\sim}{E}_0\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t}$$

$$\underset{\sim}{B} = \mathrm{Re}\, \underset{\sim}{B}_0\, e^{i \underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t}$$

$$\underset{\sim}{B}_0 = \hat{k} \times \underset{\sim}{E}_0$$

$$\langle \underset{\sim}{S} \rangle = \frac{1}{2} \mathrm{Re} \frac{c}{4\pi} \underset{\sim}{E}_0 \times \underset{\sim}{B}_0^*$$

$$\langle \underset{\sim}{S} \rangle = \frac{c}{8\pi} \underbrace{\underset{\sim}{E}_0 \cdot \underset{\sim}{E}_0^*}_{=|\underset{\sim}{E}_0|^2 \equiv |E_x|^2 + |E_y|^2} \hat{k}$$

time average energy density

$$\langle W \rangle = \frac{1}{2} \mathrm{Re} \left[\frac{\underset{\sim}{E}_0 \cdot \underset{\sim}{E}_0^*}{8\pi} + \frac{\underset{\sim}{B}_0 \cdot \underset{\sim}{B}_0^*}{8\pi} \right] = \frac{|\underset{\sim}{E}_0|^2}{8\pi}$$

note that $\underset{\sim}{S} = W c \hat{k}$

time average momentum density

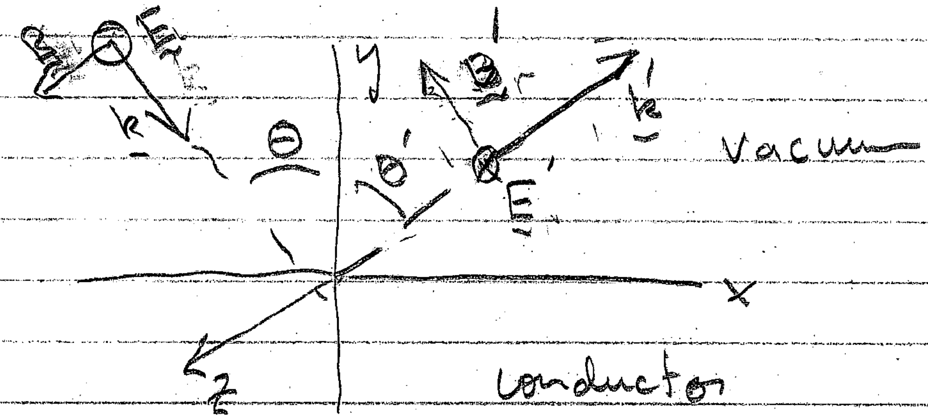
$$\langle \underset{\sim}{g} \rangle = \frac{1}{2} \mathrm{Re} \frac{\underset{\sim}{E}_0 \times B_0^*}{4\pi c} = \frac{|E_0|^2}{8\pi} \frac{\hat{k}}{c}$$

note that

$\langle W \rangle = |\langle g \rangle| c$ (i.e. $\mathcal{E} = cp$ for photons)

Plane wave incident on conducting surface

case 1 polarization $\perp$ to plane of incidence



incident wave

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \mathrm{Re}\, \underset{\sim}{E}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t}$$

$$\underset{\sim}{B} = \hat{k} \times \underset{\sim}{E}$$

reflected wave

$$\underset{\sim}{E}'(\underset{\sim}{r},t) = \mathrm{Re}\, \underset{\sim}{E}'\, e^{i\underset{\sim}{k}'\cdot\underset{\sim}{r} - i\omega t}$$

$$\underset{\sim}{B}' = \hat{k}' \times \underset{\sim}{E}'$$

$\underset{\sim}{E}, \underset{\sim}{B} = 0$ inside a perfect conductor; however, there can be a surface charge and a surface current.

$\underset{\sim}{\nabla} \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \Longrightarrow E_t$ is cont. $\therefore E_t = 0$

$\nabla \cdot B = 0 \Longrightarrow B_n$ is cont. $\therefore B_n = 0$

note that

B_t and E_n are not continuous

matching requires

$$\underset{\sim}{k} \cdot \underset{\sim}{r})_{y=0} - \omega t = \underset{\sim}{k}' \cdot \underset{\sim}{r})_{y=0} - \omega' t + \text{const.}$$

$$\therefore \omega = \omega' \Rightarrow |\underset{\sim}{k}| = |\underset{\sim}{k}'|$$

$$k_x = k_x' \quad , \quad \underset{\substack{\parallel \\ 0}}{k_z} = k_z'$$

$\leftarrow$ in this case

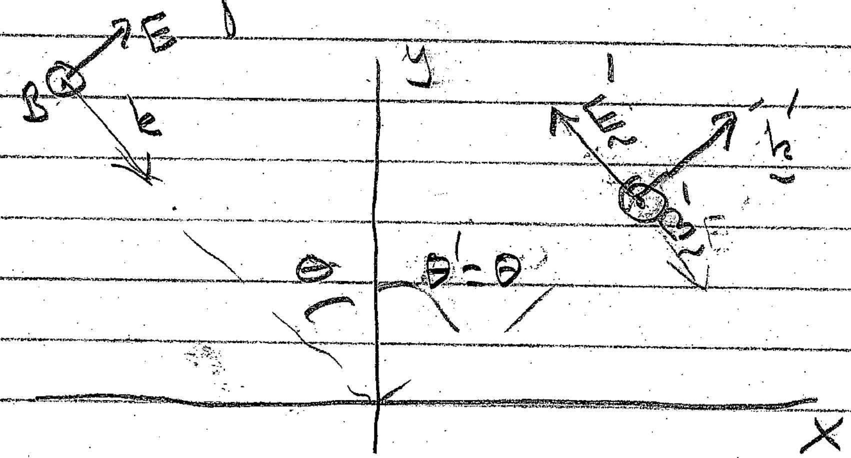
$$\therefore \theta = \theta'$$

at the surface $(\underset{\sim}{E} + \underset{\sim}{E}')_t = 0$

$$(\underset{\sim}{B} + \underset{\sim}{B}')_n = 0$$

satisfied if $\underset{\sim}{E}' = -\underline{E}$

<u>case 2</u> polarization parallel to plane of incidence



$$(\underset{\sim}{E} + \underset{\sim}{E}')_t = 0$$

$$(\underline{B} + \underline{B}')_n = 0$$

satisfied if $|\underline{E}| = |\underline{E}'|$

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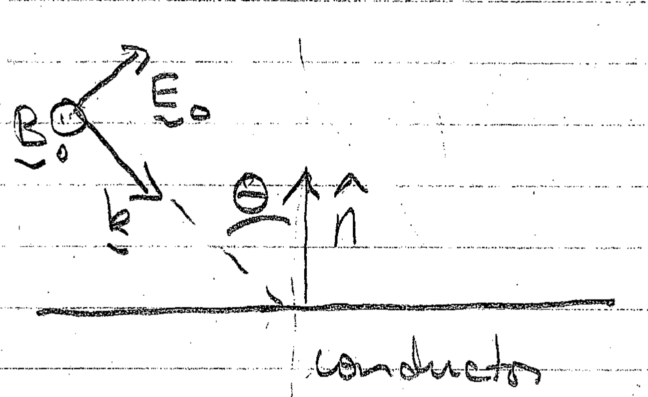
Homework

(16)



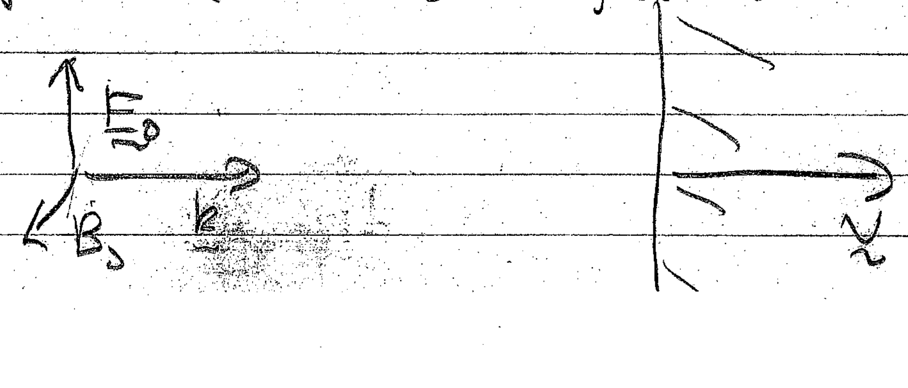
A plane wave $\underset{\sim}{E} = \mathrm{Re}\, \underset{\sim}{E_0} e^{i \underset{\sim}{k} \cdot \underset{\sim}{r} - i\omega t}$ is incident at angle θ on conducting surface

a. Using the stress tensor, calculate time average force/area (radiation pressure) on the surface.

b. Also calculate this quantity using conservation of momentum.

(17) A plane wave is normally incident on a conducting surface that moves with velocity v. Calculate the radiation pressure as measured by observer on mirror

Hint: transform fields to moving frame



Dispersion

In general ϵ and μ depend on frequency

$$\underset{\sim}{D}(\omega) = \epsilon(\omega)\underset{\sim}{E}(\omega) \quad , \quad \underset{\sim}{B}(\omega) = \mu(\omega)\underset{\sim}{H}(\omega)$$

monochromatic plane wave

$$\underset{\sim}{E} = \mathrm{Re}\,\underset{\sim}{E_0}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t} \quad , \quad \underset{\sim}{B} = \mathrm{Re}\,\underset{\sim}{B_0}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r} - i\omega t}$$

$$i\underset{\sim}{k} \times \underset{\sim}{E_0} = \frac{i\omega}{c}\underset{\sim}{B_0} \quad , \quad i\underset{\sim}{k}\cdot\epsilon(\omega)\underset{\sim}{E_0} = 0$$

$$i\underset{\sim}{k} \times \underset{\sim}{B_0}/\mu(\omega) = -\frac{i\omega}{c}\epsilon(\omega)\underset{\sim}{E_0} \quad , \quad \underset{\sim}{k}\cdot\underset{\sim}{B_0} = 0$$

$$\therefore \quad \underbrace{\underset{\sim}{k} \times \underset{\sim}{k} \times \underset{\sim}{E_0}}_{-k^2 \underset{\sim}{E_0}} = -\frac{\omega^2}{c^2}\epsilon(\omega)\mu(\omega)\underset{\sim}{E_0}$$

$$\therefore \quad k^2 = \frac{\omega^2}{c^2}\epsilon(\omega)\mu(\omega)$$

ω and k not linearly related (dispersion)

We say that there is <u>dispersion</u> when ω and k are <u>not</u> linearly related; wave packet distorts as it propagates

<u>phase velocity</u>

$$\underset{\sim}{v}_{ph} \equiv \frac{\omega}{k}\hat{k}$$

for $\underset{\sim}{r} = \underset{\sim}{r}_0 + \underset{\sim}{v}_{ph}\, t$

$$\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t = \underset{\sim}{k}\cdot\underset{\sim}{r}_0 + \cancel{\underset{\sim}{k}\cdot\underset{\sim}{v}_{ph}\, t} - \cancel{\omega t} = \text{const},$$

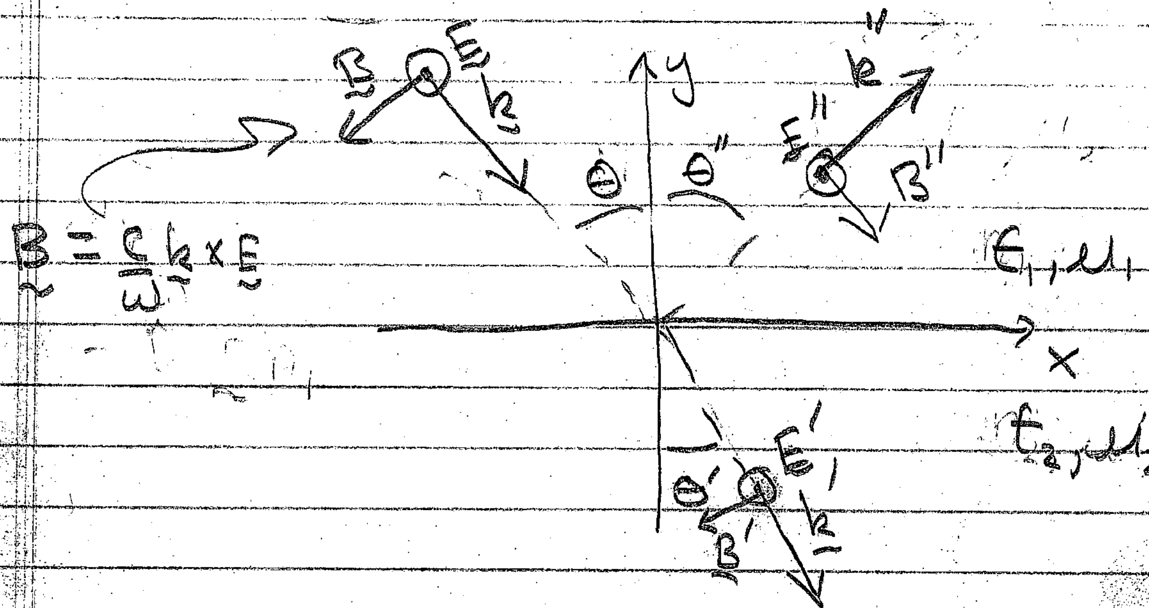
<u>index of refraction</u> for transparent medium (i.e., $\sqrt{\epsilon(\omega)\mu(\omega)}$ is real at freq. of interest)

$$n(\omega) \equiv \frac{c}{|\underset{\sim}{v}_{ph}|} = \sqrt{\epsilon\mu}$$

reflection and refraction at interface between two transparent media

case 1

polarization $\perp$ to plane of incidence



matching requires

$k_x = k'_x = k''_x \quad , \quad k_z = k'_z = k''_z = 0$

$k \sin\theta = k' \sin\theta' = k'' \sin\theta''$

$$\omega = \omega' = \omega''$$
$$\frac{ck}{n_1} \quad \frac{ck'}{n_2} \quad , \quad \frac{ck''}{n_1}$$

Snell's laws $\begin{cases} \therefore \sin\theta = \sin\theta'' \text{ or } \theta = \theta'' \\ n_1 \sin\theta = n_2 \sin\theta' \end{cases}$

Start

b.c.

(1) D_n is cont. (satisfied)

(2) E_t is cont.

$$E + E'' = E'$$

(3) B_n is cont.

$$\underset{\frac{ck}{\omega}E}{\underset{\|}{B}} \sin\theta + \underset{\frac{ck''}{\omega''}E''}{\underset{\|}{B''}} \sin\theta'' = \underset{\frac{ck'}{\omega'}E'}{\underset{\|}{B'}} \sin\theta'$$

$$k \sin\theta = k'' \sin\theta'' = k' \sin\theta' \quad , \quad \omega = \omega' = \omega''$$

$$\therefore E + E'' = E' \quad \text{same as (2)}$$

3 [illegible]

(4) H_t is cont. $n_1 \sin\theta = n_2 \sin\theta'$

$$\frac{1}{\mu_1}\left[\frac{ck}{\omega}E\cos\theta + \frac{ck''}{\omega''}E''\cos\theta''\right] = \frac{-1}{\mu_2}\frac{ck'}{\omega}E'\cos\theta'$$

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\,(E - E'')\cos\theta = \sqrt{\frac{\epsilon_2}{\mu_2}}\,E'\cos\theta'$$

$$E - E'' = \sqrt{\frac{\epsilon_2}{\mu_2}}\sqrt{\frac{\mu_1}{\epsilon_1}}\, E' \frac{\cos\theta'}{\cos\theta} = \frac{n_2}{n_1}\frac{\mu_1}{\mu_2}\frac{\cos\theta'}{\cos\theta} E'$$

$$E + E'' = E'$$

$$2E = E'\left[1 + \frac{n_2}{n_1}\frac{\mu_1}{\mu_2}\frac{\sqrt{1-(n_1/n_2)^2\sin^2\theta}}{\cos\theta}\right]$$

Fresnel's formulas

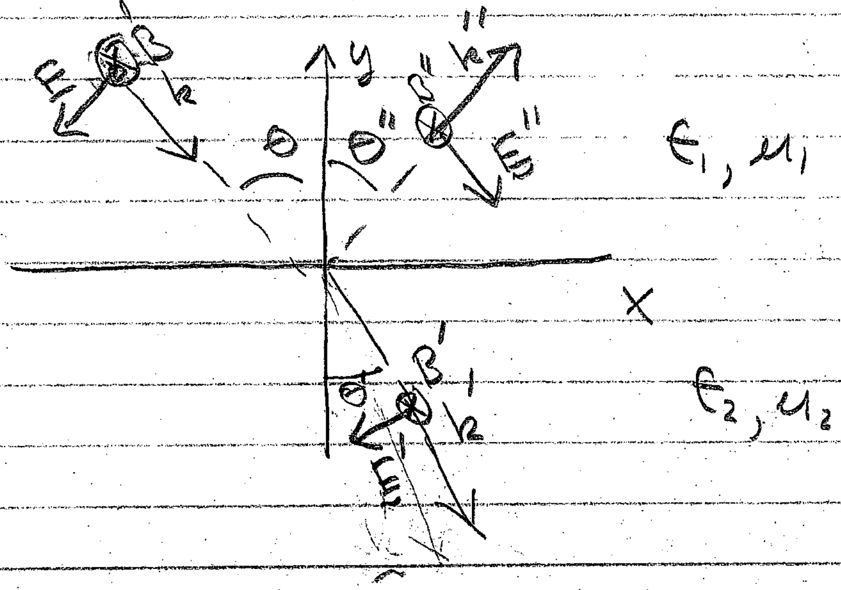
$$\frac{E'}{E} = \frac{2 n_1 \cos\theta}{n_1\cos\theta + \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin^2\theta}}$$

$$\frac{E''}{E} = \frac{n_1\cos\theta - \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin^2\theta}}{n_1\cos\theta + \frac{\mu_1}{\mu_2}\sqrt{n_2^2 - n_1^2\sin^2\theta}}$$

note that $\frac{\mu_1}{\mu_2}$ is typically near 1

5/5 start

<u>case 2</u> polarization ∥ to plane of incidence



Fresnel's formulas

$$\frac{E'}{E} = \frac{2 n_1 n_2 \cos\theta}{\frac{\mu_1}{\mu_2} n_2^2 \cos\theta + n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}}$$

$$\frac{E''}{E} = \frac{\frac{\mu_1}{\mu_2} n_2^2 \cos\theta - n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}}{\frac{\mu_1}{\mu_2} n_2^2 \cos\theta + n_1\sqrt{n_2^2 - n_1^2 \sin^2\theta}}$$

typically $\frac{\mu_1}{\mu_2} \simeq 1$