Lect 13 Relativistic Quantum mechanics

Relativistic dispersion: $E(k) = \sqrt{\hbar^2 k^2 + m^2}$. So

$H = \sqrt{(-i\hbar\nabla)^2 + m^2}$??? When Bohr asked Dirac what he was doing, Dirac said that he was working hard on the sqare root of an operator. Bohr's response, "Young man, you have better things to do".

one solution: $H^2 = (-i\hbar\nabla)^2 + m^2 \Rightarrow$

$$\left\{ \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2 \right\} \psi(x,t) = 0 \leftarrow \text{Klein-Gordon equation}$$

for scalor particles. The draw back, we need ~~know~~ $\psi(t=0)$ and $\psi'(t=0)$ for it's time evolution.

Can we maintain the first order time-derivative ? and have $E^2 = \hbar^2 k^2 + m^2$

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi.$$

Yes, but ψ needs to be multi-component. Let us consider the spin-1/2 particle in $\vec{B}$ field: $B_x\sigma_x + B_y\sigma_y + B_z\sigma_y$

$$\Rightarrow E^2 = B_x^2 + B_y^2 + B_z^2.$$

But now we have $E^2 = \hbar^2 k_x^2 + \hbar^2 k_y^2 + \hbar^2 k_z^2 + m^2$

$\Rightarrow$ Pauli matrices are not enough.

Clifford algebra: a next generation of Pauli matrices

— is the 4x4 extention called Γ-matrix. There are five of them, γ^0, $\gamma^{1,2,3}$; γ^5. They anti-commute with each other.

$$\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \quad = \mathrm{diag}\{1, -1, -1, -1\} \quad \text{for } \mu, \nu = 1, 2, 3, 4.$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \Rightarrow \gamma^5 = \mathbb{1}$$

To maintain the Lorentz invairance, we write
$$\boxed{\begin{aligned} \partial_\mu &= \left(\frac{\partial_t}{c}, +\partial_x, +\partial_y, +\partial_z\right) \\ \partial^\mu &= \left(\frac{\partial_t}{c}, -\partial_x, -\partial_y, -\partial_z\right) \end{aligned}}$$

Dirac equation $\rightarrow$
$$\left[i\gamma^\mu\partial_\mu - \frac{mc}{\hbar}\right]\psi(x) = 0,$$
$\psi(x)$ is a 4-component vector

or
$$\left[i\gamma^0\frac{\partial_t}{c} + i\gamma^1\partial_x + i\gamma^2\partial_y + i\gamma^3\partial_z\right]\psi(x) = \frac{mc}{\hbar}\psi(x)$$

$$i\frac{\partial_t}{c}\psi = \underbrace{\left[-i\,\partial_x\,\gamma^0\gamma^1 - i\,\partial_y\,\gamma^0\gamma^2 - i\,\partial_z\,\gamma^0\gamma^3 + \frac{mc}{\hbar}\gamma^0\right]}_{H}\psi$$

$$\boxed{\frac{1}{c\hbar}H = \alpha_x P_x + \alpha_y P_y + \alpha_z P_z + \frac{mc}{\hbar}\beta} \leftarrow \text{time-independent Dirac}$$

where
$$\beta = \gamma^0$$
$$\alpha_x = \gamma^0\gamma^1, \quad \alpha_y = \gamma^0\gamma^2, \quad \alpha_z = \gamma^0\gamma^3.$$

$\gamma^{1,2,3}$ are anti-hemitian $\quad \gamma^1\gamma^1 = -1$. Define $(\gamma^{1,2,3})^\dagger = \gamma^{1,2,3}$

γ^0 is hermtian $\quad (\gamma^0)^\dagger = \gamma^0$

$$(\alpha_x)^\dagger = \gamma^{1\dagger}\gamma^{0\dagger} = \gamma^0\gamma^1 = \alpha_x \Rightarrow \alpha\text{'s}, \beta \text{ are Hermitian.}$$

In the following, we chose the representation as.

$$\vec{\alpha} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \gamma^0$$

then $\gamma^{1,2,3} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$. $\quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & +I \end{pmatrix}$

we will explain the importance of γ^5 later.

Another form of Dirac equation:

$$\left[i\gamma^\mu \partial_\mu - \frac{mc}{\hbar}\right]\psi(x) = 0 \xrightarrow{\text{Hermitian conjugate}}$$

$$\left[\psi^\dagger (\gamma^\mu)^\dagger (-i)\partial_\mu - \psi^\dagger(x)\frac{mc}{\hbar}\right] = 0$$

$$-i\psi^\dagger\left[\gamma^0 \overleftarrow{\partial}_t \left(\tfrac{1}{c}\right) - \gamma^1 \overleftarrow{\partial}_x - \gamma^2 \partial_y - \gamma^3 \overleftarrow{\partial}_z\right] - \psi^\dagger \frac{mc}{\hbar} = 0 \qquad \boxed{\bar{\psi} = \psi^\dagger \gamma_0}$$

$$-i\bar{\psi}\left[\gamma^0 \frac{1}{c}\overleftarrow{\partial}_t + \gamma^0\gamma^1\partial_x + \gamma^0\gamma^2\partial_y + \gamma^0\gamma^3\partial_z\right] - \bar{\psi}\frac{mc}{\hbar} = 0$$

$$\Rightarrow \boxed{i\partial_\mu \bar{\psi}(x,t)\gamma^\mu + \frac{mc}{\hbar}\bar{\psi}(x,t) = 0}$$

Dirac equation with E-M field $\quad \partial_\mu \to \partial_\mu + \frac{ie}{\hbar c}A_\mu(x)$

$A_\mu = (\phi, -A)$

$$\Rightarrow \left\{ i\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu(x)\right) - \frac{mc}{\hbar}\right\}\psi = 0$$

or $\quad i\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu(x)\right)\bar{\psi}(x)\gamma^\mu + \frac{mc}{\hbar}\bar{\psi}(x) = 0$

Continuity equation

$$\bar{\psi}\left(i\gamma^\mu D_\mu - \frac{mc}{\hbar}\right)\psi = 0$$
$$i\left(D^*_\mu \bar{\psi}\right)\gamma^\mu \psi + \frac{mc}{\hbar}\bar{\psi}\psi = 0$$
$$\Rightarrow \bar{\psi} i\gamma^\mu D_\mu \psi + i\left(D^*_\mu \bar{\psi}\right)\gamma^\mu \psi = 0$$

$$\bar{\psi}\gamma^0\left(\partial_0 + \frac{ie}{\hbar c}\phi\right)\psi + \left(\partial_0 - \frac{ie}{\hbar c}\phi\right)\bar{\psi}\gamma^0\psi$$
$$+ \bar{\psi}\gamma^i\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\psi + \left(\partial_i + \frac{ie}{\hbar c}A_i\right)\bar{\psi}\gamma^i\psi = 0$$

$\leftarrow \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$ is real

$\bar{\psi}\gamma^i\psi$ is also real

$$\bar{\psi}\gamma^0\frac{\partial_t}{c}\psi + \left(\frac{\partial_t}{c}\bar{\psi}\right)\gamma^0\psi + \bar{\psi}\gamma^i\partial_i\psi + \left(\partial_i\bar{\psi}\right)\gamma^i\psi = 0$$

$$\Rightarrow \frac{1}{c}\partial_t\left[\bar{\psi}\gamma^0\psi\right] + \partial_i\left[\bar{\psi}\gamma^i\psi\right] = 0$$

$$\Rightarrow \partial_t \rho + \partial_i j_i = \partial_\mu j^\mu = 0$$

where $j^0 = \rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi$

$$j^i = c\bar{\psi}\gamma^i\psi = c\psi^\dagger\gamma^0\gamma^i\psi = c\psi^\dagger\vec{\alpha}\psi$$

velocity operator.

§ Lorentz group / transform

$$x^\mu = (ct, x, y, z) \Rightarrow c^2\Delta t^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \text{ invairant}$$

consider transformation Λ: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$\Rightarrow x'^\mu x'_\mu = x'^\mu x'^{\mu'} g_{\mu\mu'} = x^\nu x^{\nu'} \underbrace{\Lambda^\mu{}_\nu \Lambda^{\mu'}{}_{\nu'} g_{\mu\mu'}}_{g_{\nu\nu'}}$$

i.e $\Lambda^\mu{}_\nu \Lambda^{\mu'}{}_{\nu'} g_{\mu\mu'} = g_{\nu\nu'}$

$$\Rightarrow \Lambda^T g \Lambda = g \Rightarrow (\det\Lambda)^2 = 1 \Rightarrow \det\Lambda = \pm 1$$

further more: $g_{00} = g_{\mu\mu'} \Lambda^\mu{}_0 \Lambda^{\mu'}_0 = (\Lambda^0_0)^2 - \sum_{i=1}^3 (\Lambda^i{}_0)^2 = 1$

$$(\Lambda^0{}_0)^2 = 1 + \left(\sum_{i=1}^3 \Lambda^i_0\right)^2 \geq 1. \Rightarrow \Lambda^0_0 \geq 1 \text{ or } \Lambda^0_0 \leq -1.$$

We can classifly Lorentz group into four different disconneted parts.

① $\det\Lambda = 1$, $\Lambda^0_0 \geq 1$; ② $\det\Lambda = 1$, $\Lambda^0_0 \leq -1$; ③ $\det\Lambda = -1$, $\Lambda^0_0 \geq 1$; ④ $\det\Lambda = -1$ $\Lambda^0_0 \leq -1$.

The first class is called regular Lorentz transformation.

The others can be decomposed into a product between the regular Lorentz transformation, and

discrete transformations

A) Reflection $\Lambda = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

B) time reversal $\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

C) Reflection × time reversal

we first consider the regular Lorentz transformation: Under the transformation $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, the wavefunction transform as

$$\psi'(x') = S\,\psi(x). \quad \text{or} \quad \psi(x) = S^{-1}\psi'(x').$$

$$\Rightarrow \left\{ i\gamma^{\mu}\left(\partial_{\mu} + \frac{ie}{\hbar c}A_{\mu}\right) - \frac{mc}{\hbar} \right\} S^{-1}\psi'(x') = 0$$

$$i S\gamma^{\nu} S^{-1}\left(\partial_{\nu} + \frac{ie}{\hbar c}A_{\nu}(x)\right) - \frac{mc}{\hbar}\psi'(x') = 0$$

For any vector $\boxed{V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}}$

$$g^{\mu\mu'} V'_{\mu'} = \underbrace{\Lambda^{\mu}{}_{\nu}\, g^{\nu\nu'}} V_{\nu'}$$

$$\Lambda^{\mu}{}_{\nu}\, g^{\nu\nu'} \Lambda^{\mu'}{}_{\nu'} = g^{\mu\mu'} \Rightarrow \Lambda^{\mu}{}_{\nu}\, g^{\nu\nu'} = g^{\mu\mu'} (\Lambda^{-1})^{\nu'}{}_{\mu'}$$

$$g^{\mu\mu'} V'_{\mu'} = g^{\mu\mu'} (\Lambda^{-1})^{\nu'}{}_{\mu'} V_{\nu'} \Rightarrow V'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} V_{\nu}$$

$$\text{or} \quad V_{\mu'} = \Lambda^{\mu}{}_{\mu'} V'_{\mu}$$

$$\text{or} \quad \boxed{V_{\mu} = \Lambda^{\nu}{}_{\mu} V'_{\nu}}$$

$$\Rightarrow \quad \partial_{\nu} = \Lambda^{\mu}{}_{\nu}\,\partial'_{\mu}$$

$$A_{\nu}(x) = \Lambda^{\mu}{}_{\nu} A'_{\mu}(x')$$

$$\Rightarrow \left[\underbrace{i S\gamma^{\nu} S^{-1} \Lambda^{\mu}{}_{\nu}} \left(\partial'_{\mu} + \frac{ie}{\hbar c}A'_{\mu}(x')\right) - \frac{mc}{\hbar} \right] \psi'(x') = 0$$

we need $\Lambda^{\mu}{}_{\nu} S\gamma^{\nu} S^{-1} = \gamma^{\mu}$ or $\boxed{S^{-1}\gamma^{\mu} S = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}}$

define $\gamma'^{\mu} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$

$$\Rightarrow \{\gamma'^{\mu}, \gamma'^{\nu}\} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \{\gamma^{\mu'}, \gamma^{\nu'}\} = 2\Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} g^{\mu'\nu'} = 2g^{\mu\nu}$$

$\Rightarrow \gamma'^{\mu} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu}$ is another representation of γ^{μ}, which should differ ~~a~~ by S.

According to $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$, for a given Lorentz transformation Λ. S is at most defined to an arbitary coefficient. Now we put some extra constriant:

$$(S^{-1}\gamma^\mu S)^\dagger = \Lambda^\mu{}_\nu (\gamma^\nu)^\dagger \Rightarrow S^\dagger \gamma^0 \gamma^\mu \gamma^0 (S^\dagger)^{-1} = \Lambda^\mu{}_\nu \gamma^0 \gamma^\nu \gamma^0$$

$$\Rightarrow (\gamma^0 S^\dagger \gamma^0)\, \gamma^\mu\, (\gamma^0 S^\dagger \gamma^0)^{-1} = \Lambda^\mu{}_\nu \gamma^\nu \quad \Rightarrow$$

a coefficient

$$\gamma^0 S^\dagger \gamma^0 = b S^{-1} \rightarrow S^\dagger = b\,\gamma^0 S^{-1} \gamma^0$$

$$\Rightarrow \quad S^\dagger S = b\,\gamma^0 \underbrace{S^{-1}\gamma^0 S} = b\gamma^0 \Lambda^0{}_\nu \gamma^\nu = b\Lambda^0{}_0 + b\gamma^0 \sum_{k=1}^{3} \Lambda^0{}_k \gamma^k$$

$tr[S^\dagger S] = b\Lambda^0{}_0 > 0$. If $\Lambda^0{}_k = 0$, we can choose S to be unitary. For the general Lorentz transformation, S is not unitary.

b is positive number

we can redefine $\frac{S}{\sqrt{b}}$ as new S, thus $\boxed{S^\dagger = \gamma^0 S^{-1} \gamma^0}$

This constraint can determin S to a phase factor.

Futher more,

$$(S^{-1}\gamma^\mu S)^* = \Lambda^\mu{}_\nu (\gamma^\nu)^*$$

$(\gamma^\mu)^*$ is also a representation of Γ-matrix $\Rightarrow (\gamma^\mu)^* = B\gamma^\mu B^{-1}$

$\Rightarrow \gamma^0$ is Hermitian, γ^i is anti-Hermitian $\Rightarrow (\gamma^0)^* = (B^\dagger)^{-1}\gamma^0 B^\dagger$

$(\gamma^i)^* = (B^\dagger)^{-1}\gamma^i B^\dagger$

$\Rightarrow$ B can be choosen as unitary.

we can choose $B = \lambda \gamma^0 \gamma^1 \gamma^3$ [because $(\gamma^0)^* = \gamma^0$, $(\gamma^1)^* = \gamma^1$
$\underbrace{(\gamma^2)^* = -\gamma^2}_{\text{spactial}}$, $(\gamma^3)^* = \gamma^3$]

$|\lambda| = 1$, phase factor

$$\Rightarrow (S^*)^{-1} B \gamma^\mu B^{-1} S^* = \Lambda^\mu{}_\nu \, B \gamma^\nu B^{-1}$$

$$\Rightarrow (B^{-1} S^* B)^{-1} \gamma^\mu (B^{-1} S^* B) = \Lambda^\mu{}_\nu \gamma^\nu$$

$$\Rightarrow B^{-1} S^* B = e^{i\varphi} S$$

differ a phase fator, we can set $\varphi = 0$

i.e $B^{-1} S^* B = S$ i.e. $\boxed{S^* = B S B^{-1} = \gamma^0 \gamma^1 \gamma^3 S \gamma^3 \gamma^1 \gamma^0}$

Thus with the result:

$$\boxed{\begin{aligned} S^{-1} \gamma^\mu S &= \Lambda^\mu{}_\nu \gamma^\nu \\ S^\dagger &= \gamma^0 S^{-1} \gamma^0 \\ S^* &= \gamma^0 \gamma^1 \gamma^3 S \gamma^3 \gamma^1 \gamma^0 \end{aligned}}$$

and two extra constraints

S forms a 4×4 spinor Representation to the Lorentz group.

Then how about the transformation to $\bar{\psi}(x)$

$$\bar{\psi}'(x') = \psi^{\dagger\prime}(x') \gamma^0 = \psi^\dagger(x) S^\dagger \gamma^0 = \bar{\psi}(x) \gamma^0 S^\dagger \gamma^0 = \bar{\psi}(x) S^{-1}$$

§. Lorentz group & its spiner Rep

$$\Lambda^{\alpha}{}_{\beta} = \exp\left[-\tfrac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}{}_{\beta}\right], \quad J^{\mu\nu} \text{ are generators of } SO(3,1) \text{ group, satisfying}$$

$$\underline{[J^{\mu\nu}, J^{\rho\sigma}] = i\,[g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho}]}$$

$$(J^{ij})^{\alpha}{}_{\beta} = -i\,[\delta^{i}_{\alpha}\delta^{j}_{\beta} - \delta^{j}_{\alpha}\delta^{i}_{\beta}], \quad [J^{0i}]^{\alpha}{}_{\beta} = i\,[\delta^{0}_{\alpha}\delta^{i}_{\beta} + \delta^{0}_{\beta}\delta^{i}_{\alpha}]$$

i.e.

$$\underbrace{J^{12}}_{L_3} = \begin{bmatrix} 0&0&0&0\\ 0&0&-i&0\\ 0&i&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \underbrace{J^{23}}_{L_1} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{bmatrix}, \quad \underbrace{J^{31}}_{L_2} = \begin{bmatrix} 0&0&0&0\\ 0&0&0&i\\ 0&0&0&0\\ 0&-i&0&0 \end{bmatrix}$$

angular momentum

$$\underbrace{J^{01}}_{K_1} = \begin{bmatrix} 0&i&0&0\\ i&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \underbrace{J^{02}}_{K_2} = \begin{bmatrix} 0&0&i&0\\ 0&0&0&0\\ i&0&0&0\\ 0&0&0&0 \end{bmatrix}, \quad \underbrace{J^{03}}_{K_3} = \begin{bmatrix} 0&0&0&i\\ 0&0&0&0\\ 0&0&0&0\\ i&0&0&0 \end{bmatrix}$$

boost

For example $\omega_{12} = -\omega_{21} = \underset{\to 0}{\theta} \Rightarrow \Lambda = \begin{pmatrix} 1&0&0&0\\ 0&1&-\theta&0\\ 0&\theta&1&0\\ 0&0&0&1 \end{pmatrix}$

$\omega_{01} = -\omega_{10} = \beta \to 0 \Rightarrow \Lambda = \begin{pmatrix} 1&\beta&0&0\\ \beta&1&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}$

how about applying to spinor ?

Let us consider an infitisimall Lorentz transformation

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}{}_{\nu}x^{\nu} \Rightarrow (\delta^{\mu}_{\nu} + \varepsilon^{\mu}{}_{\nu})(\delta^{\mu'}_{\nu'} + \varepsilon^{\mu'}{}_{\nu'})\,g_{\mu\mu'} = g_{\nu\nu'}$$

$$\Rightarrow g_{\mu\lambda}\varepsilon^{\lambda}{}_{\nu} + g_{\nu\lambda}\varepsilon^{\lambda}{}_{\mu} = 0$$

define $\varepsilon_{\mu\nu} \equiv g_{\nu\lambda} \varepsilon^{\lambda}{}_{\nu} \Rightarrow \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0$.

define $S(\Lambda) = 1 + T(\varepsilon)$, $S^{-1}(\Lambda) = 1 - T(\varepsilon)$

$$\Rightarrow (1 - T(\varepsilon))\, \gamma^{\mu} (1 + T(\varepsilon)) = \gamma^{\mu} + \varepsilon^{\mu}{}_{\nu} \gamma^{\nu}$$

$$\gamma^{\mu} T(\varepsilon) - T(\varepsilon) \gamma^{\mu} = \varepsilon^{\mu}{}_{\nu} \gamma^{\nu}$$

$$\Rightarrow T(\varepsilon) = \frac{1}{8} \varepsilon_{\mu\nu} (\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}) + \text{cnst}$$

cnst can be determined to 0 due to other two constraints

$$\Rightarrow S(\Lambda) \approx 1 + \frac{1}{8} \varepsilon_{\mu\nu} (\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})$$

$$\boxed{S(\Lambda) = \exp\left[-\frac{i}{2} \varepsilon_{\mu\nu} S^{\mu\nu}\right] \Leftarrow S^{\mu\nu} = \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}]}$$

in our representation

$$S^{ij} = \frac{i}{4} [\gamma^i \gamma^j] = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2} \varepsilon^{ijk} \Sigma^k \leftarrow \text{Hermitian}$$

spin

$$S^{0i} = \frac{i}{4} [\gamma^0, \gamma^i] = -\frac{i}{2} \begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix} \leftarrow \text{anti-Hermitian}$$

boost

§ Rotation and spin of Dirac particle

$$\Lambda = \begin{pmatrix} 1 & 0\,0\,0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & R \end{pmatrix} \qquad R = \begin{pmatrix} 1 & \varepsilon^1{}_2 & \varepsilon^1{}_3 \\ \varepsilon^2{}_1 & 1 & \varepsilon^2{}_3 \\ \varepsilon^3{}_1 & \varepsilon^3{}_2 & 1 \end{pmatrix}$$

$$\varepsilon^1{}_2 = -\varepsilon^2{}_1 = -\Delta\theta\, n_z$$
$$\varepsilon^2{}_3 = -\varepsilon^3{}_2 = -\Delta\theta\, n_x$$
$$\varepsilon^3{}_1 = -\varepsilon^1{}_3 = -\Delta\theta\, n_y$$

$$S \approx 1 + \frac{1}{4} \varepsilon_{kj} \gamma^k \gamma^j = 1 - \frac{i}{2} \Delta\theta\, \hat{n} \cdot \vec{\Sigma}$$

$$\Rightarrow \quad \psi'(x',t) \approx (1 - \tfrac{i}{2}\Delta\theta\, n\cdot\vec{\Sigma})\,\psi(R^{-1}x',t)$$

$$= \exp[-\tfrac{i}{2}\theta\,\hat{n}\cdot\vec{\Sigma}]\;\psi(R^{-1}x',t)$$

Proof of electron has spin-½.

if $\theta = 2\pi \Rightarrow \psi(x',t) = -\psi(x,t)$

§ Space reflection (P)

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \psi'(x') = S_P\,\psi(x)$$

Similar to the regular Lorentz transform $\quad S_P^{-1}\gamma^\mu S_P = \Lambda^\mu{}_\nu\,\gamma^\nu$

i.e. $\quad S_P^{-1}\,\gamma^0\,S_P = \gamma^0$

$\quad\;\; S_P^{-1}\,\gamma^i\,S_P = \gamma^i$

$\Rightarrow$ $\boxed{\text{we can choose } S_P = \gamma^0}$

again $\quad \bar{\psi}(x') = \bar{\psi}(x)\,S_P^{-1}$

§ time-reversal transformation (T)

$$\Lambda = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad \psi'(x') = T\,\psi(x) \quad (\text{i.e. } \psi'(x_i,t) = T\psi(x,-t)$$

T is antiunitary transformation: $\quad T = \Theta\,\mathcal{U}$ ← unitary transf

↑ complex conjugate

$$\Rightarrow \psi(x) = T^{-1}\psi'(x') = \mathcal{U}^{-1}\Theta\,\psi'(x')$$

$$\left\{ i\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu(x)\right) - \frac{mc}{\hbar} \right\} \mathcal{U}^{-1}\Theta \ \psi'(x') = 0$$

$$\left\{ \Theta\mathcal{U}\left[i\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu\right) \mathcal{U}^{-1}\Theta \right] - \frac{mc}{\hbar} \right\} \psi'(x') = 0$$

we want

$$-i\,\Theta\,\mathcal{U}\,\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right)\mathcal{U}^{-1}\Theta = i\gamma^\mu\left(\partial_\mu' + \frac{ie}{\hbar c}A_\mu'(x')\right)$$

$$\partial_0 = -\partial_{0'} \qquad \partial_i = \partial_{i'}$$

$$A_0(x) = A_0'(x') \qquad A_i(x) = -A_i'(x')$$

$$\Rightarrow \quad \Theta\,\mathcal{U}\,\gamma^0\,\mathcal{U}^{-1}\Theta = \gamma^0 \quad \Rightarrow \quad \mathcal{U}\gamma^0\mathcal{U}^{-1} = (\gamma^0)^*$$

$$\Theta\,\mathcal{U}\,\gamma^i\,\mathcal{U}^{-1}\Theta = -\gamma^i \qquad \mathcal{U}\gamma^i\mathcal{U}^{-1} = -(\gamma^i)^*$$

$$T = \Theta\,\mathcal{U} \quad \Rightarrow \quad T^2 = \Theta\,\mathcal{U}[\Theta\,\mathcal{U}] = \mathcal{U}^*\mathcal{U}$$

in our representation $\mathcal{U}\,\gamma^{1,3}\,\mathcal{U}^{-1} = -\gamma^{1,3}$, $\mathcal{U}\,\gamma^{0,2}\,\mathcal{U}^{-1} = +\gamma^{0,2}$

$$\Rightarrow \quad \mathcal{U} = \gamma_1\gamma_3 \quad \Rightarrow \quad T^2 = (\gamma_1\gamma_3)^*(\gamma_1\gamma_3) = \gamma_1\gamma_3\gamma_1\gamma_3 = -1.$$

$$\boxed{T = \Theta\,\gamma_1\gamma_3 = \gamma_1\gamma_3\,\Theta}$$

$$\bar{\psi}'(x') = \psi^{\dagger\prime}(x')\,\gamma^0 = \left[\gamma_1\gamma_3\,\psi^*(x)\right]^\dagger\gamma^0 = \psi^T(x)\,\gamma_3^\dagger\gamma_1^\dagger\,\gamma^0$$

$$= \psi^T(x)\,(\gamma_1\gamma_3)^T\,\gamma^0$$

§ Charge conjugation (C)

$$\left\{ i\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu(x)\right) - \frac{mc}{\hbar} \right\} \psi(x) = 0$$

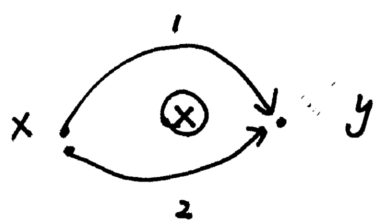
under charge conjugation, $\psi(x) \to \varphi(x)$, $\quad e \to -e$

$$\left\{ i\gamma^\mu \left(\partial_\mu - \frac{ie}{\hbar c} A_\mu(x)\right) - \frac{mc}{\hbar} \right\} \psi'(x) = 0$$

$\uparrow$ anti-particle wavefunction

This has to be an anti-unitary transformation because if

$$\psi(y) = \psi(x)\left[e^{i\int_1 d\vec{x}\, e\vec{A}} + e^{ie\int_2 d\vec{x}\, \vec{A}} \right]$$



$$\to \psi'(y) = \psi'(x)\left(e^{-i\int_1 d\vec{x}\, e\vec{A}} + e^{-i\int d\vec{x}\, e\vec{A}} \right)$$

define $\quad \psi'(x) = \mathcal{U}\psi^*(x) \quad$ or $\quad \psi(x) = (\mathcal{U}^{-1})^* \psi'^*(x)$

$$\Rightarrow \left[i\gamma^\mu\left(\partial_\mu + \frac{ie}{\hbar} A_\mu\right) - \frac{mc}{\hbar} \right] (\mathcal{U}^{-1})^* \psi'^*(x) = 0 \qquad \boxed{C^2 = 1}$$

$$\mathcal{U}\left[-i\gamma^{*\mu}\left(\partial_\mu - \frac{ie}{\hbar} A_\mu\right) - \frac{mc}{\hbar} \right] \mathcal{U}^{-1} \psi'(x) = 0$$

$$\Rightarrow \left[-i\,\mathcal{U}\gamma^{*\mu}\mathcal{U}^{-1}\left(\partial_\mu - \frac{ie}{\hbar} A_\mu\right) - \frac{mc}{\hbar} \right] \psi'(x) = 0$$

$$\mathcal{U}\gamma^{*\mu}\mathcal{U}^{-1} = -\gamma^\mu \quad \text{or} \quad \boxed{\mathcal{U}^{-1}\gamma^\mu\,\mathcal{U} = -\gamma^{\mu *}}$$

i.e. $\quad \mathcal{U}^{-1}\gamma^{0,1,3}\,\mathcal{U} = -\gamma^{0,1,3}, \qquad \mathcal{U}^{-1}\gamma^2\,\mathcal{U} = +\gamma^2$

$\Rightarrow \mathcal{U} = i\gamma^2$, $\quad$ it is often write $\quad \boxed{\psi'(x) = i\gamma^2\gamma^0\gamma^0\psi^*(x) = \underbrace{i\gamma^2\gamma^0}_{C}\bar{\psi}^T}$

§ Fermion bilinears. under Lorentz transform

Scalar	$\bar{\psi}(x)\psi(x)$	1
vector	$\bar{\psi}(x)\gamma^\mu\psi(x)$	4
tensor	$\bar{\psi}(x)\gamma^\mu\gamma^\nu\psi(x) \quad (\mu\neq\nu)$	6
pseudo-scalar	$\bar{\psi}(x)\gamma^5\psi(x)$	4
pseudo-vector	$\bar{\psi}(x)\gamma^5\gamma^\mu\psi(x)$	1

$16 = 4\times 4$

under Lortentz transformation
$$\boxed{\begin{aligned}\psi(x) &\to \psi'(x') = S\psi(x)\\ \bar{\psi}(x) &\to \bar{\psi}'(x') = \bar{\psi}(x)S^{-1}\\ S^{-1}\gamma^\mu S &= \Lambda^\mu{}_\nu\gamma^\nu\end{aligned}}$$

we easily see scalar, vector, tensor transforms as regular Lorentz ~~ten~~ scalar tensor, · ·

$$\bar{\psi}\gamma^5\psi(x) \to \bar{\psi}'\gamma^5\psi' = \bar{\psi}S^{-1}\gamma^5 S\psi$$

$$S^{-1}\gamma^5 S = S^{-1}i\gamma^0\gamma^1\gamma^2\gamma^3 S = i\Lambda^0{}_{\nu_1}\Lambda^1{}_{\nu_2}\Lambda^2{}_{\nu_2}\Lambda^3{}_{\nu_3}\gamma^{\nu_1}\gamma^{\nu_2}\gamma^{\nu_3}\gamma^{\nu_4}$$

$$= \det(\Lambda)\gamma^5 \to \text{pseudo Scalar}$$

Similarly $S^{-1}\gamma^5\gamma^\mu S = \det(\Lambda)\gamma^5\gamma^\mu \to$ vector.

For example: under parity transformation $\boxed{\psi'(x') = \gamma^0\psi(x)}$

	$\bar{\psi}(x)\psi(x)$	$\bar{\psi}(x)\gamma^5\psi(x)$	$\bar{\psi}(x)\gamma^\mu\psi(x)$	$\bar{\psi}(x)\gamma^5\gamma^\mu$	$\bar{\psi}(x)\gamma^\mu\gamma^\nu\psi(x)$
P	+	−	+ $\mu=0$ − $\mu=1,2,3$	− $\mu=0$ + $\mu=1,2,3$	+ $\mu,\nu=1,2,3$ − $\mu=0,\nu=1,2,3$

under time-reversal: $\psi'(x') = \gamma_1\gamma_3\Theta\,\psi(x) = \gamma_1\gamma_3\,\psi^*(x)$

$$\bar\psi'(x')\psi'(x') = \psi^T(x)(\gamma_1\gamma_3)^T\gamma^0\gamma_1\gamma_3\psi^*(x) = \psi^T\gamma^0\psi^*(x) = (\psi^T\gamma^0\psi^*)^T$$

$$= \psi^\dagger\gamma^0\psi = \bar\psi\psi$$

$$\bar\psi'(x')\gamma^\mu\psi'(x') = \psi^T(x)(\gamma^1\gamma^3)^T\gamma^0\gamma^\mu\gamma_1\gamma^3\psi^*(x)$$

$$= \psi^\dagger(x)\,(\gamma^1\gamma^3)^T(\gamma^\mu)^T\gamma^0(\gamma^1\gamma^3)\,\psi(x)$$

$$= \bar\psi(x)\,\gamma^0\underbrace{(\gamma^1\gamma^3)^T(\gamma^\mu)^T(\gamma^1\gamma^3)}_{\gamma_\mu}\gamma^0\psi(x) = \bar\psi(x)\gamma^0\gamma^\mu\gamma^0\psi(x)$$

$$= \gamma^3\gamma^1(\gamma^\mu)^T\gamma^1\gamma^3$$

$\Rightarrow$ $\bar\psi\gamma^0\psi$ even $\quad\bar\psi\gamma^i\psi$ odd $(i=1,2,3)$

similarly $\bar\psi\gamma^0\gamma^i\psi$ even, $\bar\psi\gamma^i\gamma^j\psi$ odd

$\bar\psi\gamma^5\psi$ odd $\quad\bar\psi\gamma^5\gamma^0\psi$ even, $\quad\bar\psi\gamma^5\gamma^i\psi$ odd