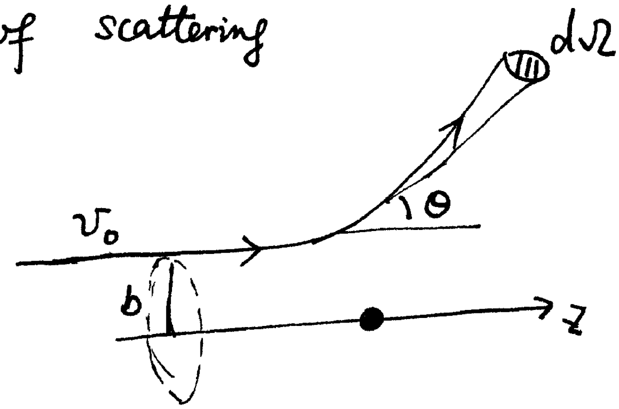
Lect 4: Description of Scattering theory

§ Cross section; the classical description of scattering



$$dn = j_i \, \sigma \, d\Omega \quad \text{or} \quad \sigma = \frac{1}{j_i} \frac{dn}{d\Omega}$$

σ ↗ differential cross section

$$\sigma_t = \int d\Omega \, \sigma(\theta,\varphi) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi \; \sigma(\theta,\varphi)$$

The deflection angle θ depends on the distance of b.

let's set $b \to b + db$, $\theta \to \theta + d\theta$

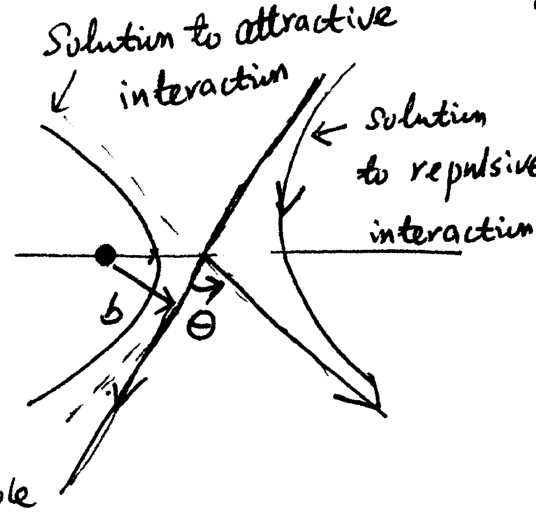
$$dn = j_i \, b db \, d\varphi = j_i \sigma \sin\theta d\theta \, d\varphi \Rightarrow \sigma(\theta,\varphi) = \frac{b db}{\sin\theta d\theta}$$

★ example: Coulumb scattering $V(r) = \frac{\kappa}{r}$, where $\kappa > 0$.

Forom classic physics, we know the solution of the ~~trajectory~~ of a particle in the polar coordinate, where the force center is the focus.

$$r = \frac{p}{1 + e\cos\theta}, \quad \text{where } e = \sqrt{1 + \frac{2EL^2}{\kappa^2 m}} > 1$$

is the eccentricity, L is the angular momentum, $p = \frac{L^2}{\kappa m}$ is the distance from the focus to line of directrix.



The direction of the asyptotes

$$1 + e\cos\theta' = 0 \Rightarrow \theta' = \pi \pm \cos^{-1}\frac{1}{e}$$

The deflection angle $\underline{\theta = \pi - 2\cos^{-1}\frac{1}{e}}$

The distance from b to the incoming asymptole

is $\underline{m v_0 b = L}$.

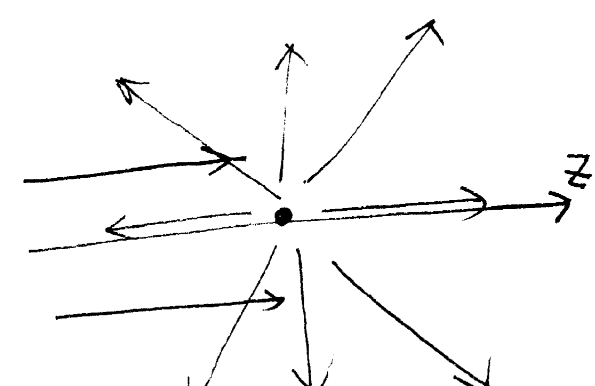
we have $\sin\frac{\theta}{2} = \frac{1}{e} \Rightarrow ctg\frac{\theta}{2} = \frac{\sqrt{1-1/e^2}}{1/e} = \sqrt{e^2-1} = \sqrt{\frac{2E}{\varkappa^2 m}}\cdot L$

$$= \frac{v_0}{\varkappa} m v_0 b$$

$\Rightarrow b = \frac{\varkappa}{E} ctg\frac{\theta}{2}$ where $E = \frac{1}{2} m v_0^2$

$$\boxed{\sigma = \frac{b\,db}{\sin\theta\, d\theta} = \frac{\varkappa^2}{16E^2}\frac{1}{\sin^4\frac{\theta}{2}}}$$ Rutherford formula.

§2. Quantum mechanics description



incoming wave $\psi_i = e^{ikz}$

scattering wave $\frac{f(\theta)}{r} e^{ikr}$

no dependence on the azimuthal angle on φ, due to the cylindric symmetry.

Let us assume short range scattering (Coulumb scattering actually does not fit into this category). As $r \to +\infty$, we have

$$\boxed{\psi \xrightarrow{r\to\infty} e^{ikz} + f(\theta)\frac{1}{r}e^{ikr}}$$ boundary condition

$f(\theta)$: scattering amplitude to be determined

Next let us justify.

$$H = \frac{p^2}{2m} + V(r)$$

The incoming wave $e^{ikz} = e^{ikr\cos\theta}$ whose $l_z = 0$, (i.e. symmetric around z-axis).

If we take (H, l^2, l_z) as the complete set of good quantum number, the scattering wave can be expanded as

$\sum_l C_l R_{kl}(r) Y_{l0}(\theta)$, where $R_{kl}(r)$ is ~~rd~~ radius function satisfying

$$\left.\begin{aligned} &\frac{d^2\chi_{kl}(r)}{dr^2} + \left(k^2 - \frac{l(l+1)}{r^2} - \frac{2mV(r)}{\hbar^2}\right)\chi_{kl}(r) = 0 \\ &r R_{kl} = \chi_{kl}(r) \end{aligned}\right\}$$ where.

As $r \to +\infty$, $V(r) \to 0$. we have $\chi_{kl} \approx e^{ikr}$ for scattering wave, thus

the scattering wave as $r \to +\infty$

$$\frac{1}{r} e^{ikr} \underbrace{\sum_l C_l P_l(\cos\theta)}$$

denoted by $f(\theta)$

The scattering problem is to solve $f(\theta)$.

Suppose we have already had the information of $f(\theta)$

then $\quad j_{in} = \psi_{in}^* \frac{i\hbar\nabla}{2m} \psi_{in} - c.c = \frac{\hbar k}{m}$

$$j_s = \frac{i\hbar}{2m}\left[f(\theta) \frac{e^{ikr}}{r} \frac{\partial}{\partial r}\left(f^*(\theta) \frac{e^{-ikr}}{r} \right) - c.c.\right] = \frac{\hbar k}{m} |f(\theta)|^2 \frac{1}{r^2}$$

$$dn = j_s r^2 d\Omega = j_i \sigma \, d\Omega$$

$$\Rightarrow \frac{\hbar k}{m} |f(\theta)|^2 = \frac{\hbar k}{m} \sigma \quad \Rightarrow \boxed{\sigma(\theta) = |f(\theta)|^2}$$

Question needs to be addressed: in deriving $\sigma(\theta) = |f(\theta)|^2$, we neglect the interference between the incoming and scattering waves. Now let us justify this: plug in $\psi(r) = e^{ikr\cos\theta} + \frac{e^{ikr}}{r} f(\theta)$ into the

current density $j = \frac{\hbar}{2im}[\psi^* \nabla \psi - c.c.]$. In the spher. coordinate,

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\vec{j}_\varphi = 0$$

$$\vec{j}_r = \frac{\hbar}{2im}\left\{\left(e^{-ikr\cos\theta} + \frac{e^{-ikr}}{r} f(\theta)\right)\frac{\partial}{\partial r}\left(e^{ikr\cos\theta} + \frac{e^{ikr}}{r} f(\theta)\right) - c.c.\right)$$

$$= \frac{\hbar k}{m}\left[\cos\theta + \frac{1}{r^2}|f|^2\right] + \frac{\hbar}{2m}\left\{ f(\theta)\left(kr(1+\cos\theta)+i\right) \underline{\frac{e^{ik(r-z)}}{r^2}} + c.c.\right\}$$

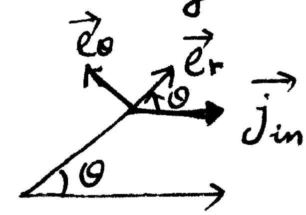
$$\vec{j}_\theta = \frac{\hbar}{2imr}\left[\left(e^{-ikr\cos\theta} + \frac{e^{-ikr}}{r} f(\theta)\right)\frac{\partial}{\partial \theta}\left(e^{+ikr\cos\theta} + \frac{e^{+ikr}}{r} f(\theta)\right) - c.c.\right]$$

$$= -\frac{\hbar k}{m}\sin\theta + \frac{\hbar}{2im}\left[\left(\frac{df}{d\theta} - ikrf\sin\theta\right)\frac{1}{r^2}\underline{e^{ik(r-z)}} - c.c\right\}$$

$$\underline{+ \frac{\hbar}{2im}\left(\frac{df}{d\theta} f^* - f\frac{df^*}{d\theta}\right)/r^3}$$

① The interfence term has the facter $e^{ik(r-z)} = \boxed{e^{ikr(1-\cos\theta)}}$

unless $\theta \to 0°$, at $kr \to +\infty$, this phase is fast oscillating within the solid angle $\Delta\Omega$.



② $1/r^3$ term can be neglected.

$$\Rightarrow \quad j_\varphi = 0, \quad j_r = \frac{\hbar k}{m}\Big(\underbrace{\cos\theta}_{\text{incoming wave}} + \underbrace{\frac{|f|^2}{r^2}}_{\text{scattering wave}}\Big), \quad j_\theta = \frac{\hbar k}{m}\underbrace{(-\sin\theta)}_{\text{incoming wave}}$$

Optical theorem:

let us consider a sphere with $r \to +\infty$, the net particle flux is 0,

$\oint j_r r^2 d\Omega = 0$. Plug in $j_r = \frac{\hbar k}{m}(\cos\theta + \frac{1}{r^2}|f|^2)$

$$+ \frac{\hbar}{2m}\left(f(\theta)(kr(1+\cos\theta)+i)\frac{e^{ik(r-z)}}{r^2} + c.c.\right)$$

$$\Rightarrow \oint |f|^2 d\Omega + \oint \frac{\hbar}{2m} f(\theta)\{[kr(1+\cos\theta)+i]e^{ikr(1-\cos\theta)} + c.c.\} = 0$$

$$\lim_{kr\to+\infty} e^{ikr(1-\cos\theta)} = \frac{2i}{kr}\delta(1-\cos\theta) \quad \text{under the integral } \int d\Omega$$

$$\oint \frac{\hbar}{2m} f(\theta)[(kr(1+\cos\theta) + \underbrace{i}_{\text{this term is negligible}}]\frac{2i}{kr}\delta(1-\cos\theta) + c.c.$$

$$\boxed{\int_{-1}^{1} d\cos\theta\, \delta(1-\cos\theta) = \frac{1}{2}}$$

$$= \oint \frac{\hbar}{2m}\left[\frac{2i}{k^2}f(\theta)(1+\cos\theta) + c.c.\right]\delta(1-\cos\theta)$$

$$\Rightarrow \oint |f|^2 d\Omega = \frac{-1}{k^2}\int_{-1}^{1} d\cos\theta \int d\varphi\, [i2f(\theta) + c.c]\delta(1-\cos\theta) = \frac{2\pi \cdot 2\, \mathrm{Im} f(0)}{k^2} = \frac{4\pi}{k^2}\mathrm{Im} f(0)$$