Lect 9: Formal theory of scattering

§ Time-dependent method:

$$i\hbar \frac{\partial}{\partial t} \Psi(t) = (H_0 + V)\, \Psi(t), \quad \text{where } H_0 = \frac{-i\hbar \nabla^2}{2m}.$$

at $t \to -\infty$, because $V(r)$ is short ranged, $\Psi(t)$ should be a wave packet with well-defined momentum as

$$\phi_0(r,t) = \int C(p;\, P_0, \Delta P_0)\, \frac{1}{(2\pi\hbar)^{3/2}}\, e^{i(pr - E(p)t)/\hbar}\, d^3\vec{p}$$

$C(p; P_0, \Delta P_0)$ is a sharp distribution around $\vec{P}_0$ and width ΔP_0. If $\Delta P_0 \to 0$

$C(p; P_0\, \Delta P_0) \xrightarrow[\Delta P_0 \to 0]{} \delta(\vec{p} - \vec{P}_0)$. $\phi_0(r,t)$ satisfies

$$i\hbar \frac{\partial}{\partial t} \phi_0(r,t) = H_0\, \phi_0(r,t).$$

Let us suppose; at $t \le T$ — a certain negative time — distant past;

$$\boxed{\Psi(t) = \phi_0(t);\ t \le T}$$

Let us take T as the starting time, with $\Psi(T) = \phi_0(T)$ as initial condition

$$\Psi(t) = e^{-iH(t-T)/\hbar}\, \phi_0(T)$$

$$= e^{-iHt}\, \underbrace{e^{iHT/\hbar}\, \phi_0(T)} = e^{-iHt}\, \Psi(0)$$

$\Psi(0)$ is defined as $\lim\limits_{T\to-\infty} e^{iHT/\hbar}\,\phi_0(T)$

$$= \lim_{T\to-\infty} e^{iHT/\hbar}\, e^{-iH_0T/\hbar}\, \underbrace{e^{iH_0T/\hbar}\,\phi_0(T)}_{\phi_0(0)}$$

$$= \lim_{T\to-\infty} \mathcal{U}(0,T)\,\phi_0(0)$$

$T=0$ means particle moves to the center region of $V(r)$.

we can see $\Psi(0)$ is different from the free-wave state. And

$$\mathcal{U}(0,\ T) = e^{iHT/\hbar}\, e^{-iH_0T/\hbar} \Longrightarrow$$ change to the integral form

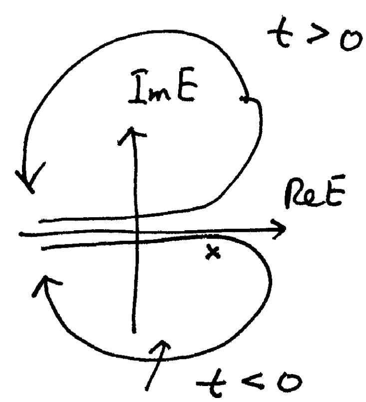
$$\frac{\partial}{\partial T}\mathcal{U}(0,T) = \frac{i}{\hbar}e^{iHT/\hbar}(H-H_0)e^{-iH_0T/\hbar} = e^{iHT/\hbar}\,V\,e^{-iH_0T/\hbar}$$

$$\mathcal{U}(0,T) = 1 + \frac{1}{i\hbar}\int_T^0 dt\; e^{iHt/\hbar}\,V\,e^{-iH_0t/\hbar}$$

$T\to-\infty$

$$\mathcal{U}(0,-\infty) = 1 + \frac{1}{i\hbar}\int_{-\infty}^0 dt\; e^{iHt/\hbar}\,V\,e^{-iH_0t/\hbar}$$

$$= 1 + \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt\; \theta(-t)\, e^{iHt/\hbar}\,V\,e^{-iH_0t/\hbar}$$



$\boxed{\mathrm{Res} = -2\pi i}$

use the identity

$$\theta(-t)\,e^{iHt/\hbar} = \frac{i}{2\pi}\lim_{\eta\to0^+}\int_{-\infty}^{+\infty} dE\, \frac{e^{iEt/\hbar}}{E-H+i\eta}$$

$$\mathcal{U}(0,-\infty) = 1 + \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} dt \ \frac{i}{2\pi} \ \frac{e^{iEt/\hbar}}{E - H + i\eta} \ V e^{-iH_0 t/\hbar}$$

perform the integral of $\int dt$

$$\int_{-\infty}^{+\infty} dt \ e^{iEt/\hbar} e^{-iH_0 t/\hbar} = 2\pi \, \delta\left(\frac{E-H_0}{\hbar}\right) = 2\pi\hbar \, \delta(E - H_0)$$

$$\mathcal{U}(0,-\infty) = 1 + \int_{-\infty}^{+\infty} dE \ \frac{1}{E - H + i\eta} \ V \, \delta(E - H_0)$$

define $\Omega_+(E) = 1 + \dfrac{1}{E - H + i\eta} V$, we have

$$\Psi(0) = \mathcal{U}(0,-\infty)\, \phi_0(0) = \int dp \ C(p; p_0 \Delta p) \ \mathcal{U}(0,-\infty) |p\rangle$$

where $|p\rangle = \dfrac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}\vec{r}}$ plane wave state

define

$$\psi_p^+ = \mathcal{U}(0,-\infty) |p\rangle$$

$$= \left[1 + \int_{-\infty}^{+\infty} dE \ \frac{1}{E - H + i\eta} \ V \, \delta(E - H_0)\right] |p\rangle$$

$$= \left[1 + \frac{1}{E(p) - H + i\eta} V\right] |p\rangle = \Omega_+(E(p)) \, |p\rangle,$$

where $E(p) = \dfrac{\hbar^2 p^2}{2m}$

thus $\Psi(0) = \int dp \ C(p; p_0 \Delta p) \ \psi_p^+$.

ψ_p^+ is the eigenstate of H. we can prove it as

$$H\psi_p^+ = \left[H + \frac{1}{E(p)-H+i\eta} HV \right] |p\rangle = \left[H_0 + V + \frac{1}{E(p)-H+i\eta}[HV] \right] |p\rangle$$

$$= \left[H_0 + V + \frac{1}{E(p)-H+i\eta}[-E(p)+H]V + \frac{1}{E(p)-H+i\eta} V E(p) \right] |p\rangle$$

$$= \left[H_0 + \frac{1}{E(p)-H+i\eta} V H_0 \right] |p\rangle = \Omega_+(E(p)) \, \underbrace{H_0 |p\rangle} = E(p) \, \psi_p^+ .$$

It can be shown later that

$$\psi_p^+(r) \xrightarrow{r\to+\infty} \frac{1}{(2\pi\hbar)^{3/2}} \left(e^{i\vec{p}\cdot\vec{r}/\hbar} + \frac{e^{ikr}}{r} f(\theta,\varphi) \right), \quad k = p/\hbar .$$

outgoing eigenstate. This is just the boundary condition that we used for the time-independent method.

Similarly, we can also formally define the initial condition from the future by setting $T \to +\infty$. Then

$$\Psi'(0) = \lim_{T\to+\infty} \mathcal{U}(0, T)\, \phi_0(0)$$

$$\mathcal{U}(0,+\infty) = 1 + \frac{1}{i\hbar} \int_0^{T\to+\infty} dt \; e^{iHt/\hbar} \, V \, e^{-iH_0 t/\hbar}$$

$$= 1 + \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt\, \underline{\theta(t)\, e^{iHt/\hbar}}\, V\, e^{-iH_0 t/\hbar}$$

$$= 1 + \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dE \int_{-\infty}^{+\infty} dt\, \frac{i}{2\pi}\, \frac{e^{iEt/\hbar}}{E-H-i\eta}\, V\, e^{-iH_0 t/\hbar} = 1 + \int_{-\infty}^{+\infty} dE\, \frac{1}{E-H+i\eta}\, V\, \delta(E-H_0)$$

define $\Omega_-(E) = 1 + \dfrac{1}{E-H-i\eta}\, V$

$$\Psi'(0) = \mathcal{U}(0,+\infty)\,\phi_0(0) = \int dp\; C(p;p_0\Delta p)\, \underbrace{\mathcal{U}(0,+\infty)\,|p\rangle}$$

define $\psi_p^- = \mathcal{U}(0,+\infty)\,|p\rangle = \left(1 + \dfrac{1}{E(p)-H-i\eta}\right)|p\rangle = \Omega_-(E(p))\,|p\rangle$

and

$$\psi_p^-(r) \xrightarrow{r\to+\infty} \frac{1}{(2\pi\hbar)^{3/2}}\left(e^{i\vec{p}\cdot\vec{r}/\hbar} + \frac{e^{-ikr}}{r} f(\theta,\varphi)\right)$$

incoming eigenstates.

$\mathcal{U}(0,\pm\infty)$ satisfy

$$\boxed{H\,\mathcal{U}(0,\pm\infty) = \mathcal{U}(0,\pm\infty)\, H_0}$$

§ properties of $\mathcal{U}(0,\pm\infty)$

Consider two incoming wavepackets $\phi_1(T)$, $\phi_2(T)$ as $T\to-\infty$

their corresponding $\Psi_1(0)$ and $\Psi_2(0)$, according to the relation

$$\Psi(0) = e^{iHT}\,\phi(T) = e^{iHT}\, e^{-iH_0 T}\,\phi(0)$$

$\Rightarrow \langle \Psi_1(0) | \Psi_2(0) \rangle = \langle \phi_1(0) | \phi_2(0) \rangle \Rightarrow$

$\langle \psi^+_{p'} | \psi^+_p \rangle = \langle p' | p \rangle = \delta(p'-p)$ or $\Rightarrow$ $\boxed{\mathcal{U}(-\infty,0)\,\mathcal{U}(0,-\infty) = 1}$

outgoing waves are orthgonal / normalized

Similarly $\boxed{\mathcal{U}(\infty,0)\,\mathcal{U}(0,\infty) = 1}$

orthagonality

where $\mathcal{U}(\pm\infty, 0) = \lim_{T \to \pm\infty} e^{iH_0T/\hbar} e^{-iHT/\hbar}$

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Outgoing / incoming eigenstates represent the <u>complete basis</u> for the <u>SCATERING states</u> of H,

$$\int dp \; |\psi^+_p\rangle\langle\psi^+_p| = \int dp \; |\psi^-_p\rangle\langle\psi^-_p| \neq 1 \text{ in general}$$

because H may have bound states.

$|\psi^+_p\rangle = \mathcal{U}(0,-\infty)|p\rangle$, $|\psi^-_p\rangle = \mathcal{U}(0,+\infty)|p\rangle$

$$\Rightarrow \mathcal{U}(0,-\infty)\int dp\,|p\rangle\langle p|\,\mathcal{U}(-\infty,0) = \mathcal{U}(0,+\infty)\int dp\,|p\rangle\langle p|\,\mathcal{U}(+\infty,0)$$

$$\Rightarrow \mathcal{U}(0,-\infty)\,\mathcal{U}(-\infty,0) = \mathcal{U}(0,+\infty)\,\mathcal{U}(+\infty,0) \neq 1$$

not complete

§ Cross-section:

We write $\Psi(t) = \phi_0(t) + u(t)$; $\phi_0(t)$ is the incoming wavepacket without scattering, $u(t)$ is the scattering wave. $\phi_0(t) \to e^{-iE_0t/\hbar} |p_0\rangle$

The scattering rate to the states $\vec{p}$

$$w_{\vec{p}}\, d\vec{p} = \frac{\partial}{\partial t} |\langle p | u(t)\rangle|^2 d\vec{p};$$ on the hand the cross-section

$$\sigma(\theta, \phi) d\Omega = \frac{(2\pi\hbar)^3}{v_0} d\Omega \int_0^{+\infty} w_p\, p^2 dp, \quad \text{where } v_0 = \frac{p_0}{m}.$$

We derive the equation of $u(t)$ as

$$\frac{d}{dt} u(t) = \frac{1}{i\hbar}[H\Psi(t) - H_0\phi_0(t)] = \frac{1}{i\hbar}[H_0\Psi(t) + V\Psi(t) - H_0\phi_0(t)]$$

$$= \frac{1}{i\hbar}[V\Psi(t) - H_0 u(t)]$$

$$\frac{\partial}{\partial t} |\langle p | u(t)\rangle|^2 = \langle u(t) | p\rangle\langle p | \dot{u}(t)\rangle + \text{c.c.}$$

$$= \frac{1}{i\hbar}\langle u(t) | p\rangle\langle p | V | \Psi(t)\rangle + \text{c.c.}$$ $\longrightarrow$ set $t=0$, and $\Delta p \to 0$,

$$|u(t)\rangle \to \psi^+_{p_0} - |p_0\rangle = [\Omega_+(E_p) - 1]|p_0\rangle$$

$$\Rightarrow \quad \frac{\partial}{\partial t} |\langle p | u(t)\rangle|^2 \longrightarrow \frac{1}{i\hbar}\langle p_0 | (\Omega_+^{\dagger}(E_p) - 1) | p\rangle\langle p | V | \psi^+_{p_0}\rangle + \text{c.c.}$$

$$\Omega_+ = 1 + \frac{1}{E-H+i\eta} V$$

$$\frac{1}{E-H+i\eta} = \qquad = \frac{1}{E-H_0+i\eta} + \frac{1}{E-H_0+i\eta} V \frac{1}{E-H_0+i\eta} + \cdots$$

$$= \frac{1}{E-H_0+i\eta} + \frac{1}{E-H_0+i\eta} V \frac{1}{E-H+i\eta}$$

$$\Rightarrow \boxed{\begin{aligned} \Omega_+ &= 1 + \frac{1}{E-H_0+i\eta} V \left[1 + \frac{1}{E-H_0+i\eta} V + \cdots \right] \\ &= 1 + \frac{1}{E-H_0+i\eta} V \Omega_+ \end{aligned}}$$

$$\lim_{\eta\to 0^+} \frac{1}{E-H_0+i\eta} = \frac{\mathbb{P}}{E-H_0} \mp i\pi\delta(E_p - H_0)$$

$$\Rightarrow \langle p_0 | (\Omega_+^\dagger - 1) | p\rangle = \langle \psi_{p_0}^+ | V | p\rangle \left\{ \frac{P}{E-H_0} + i\pi\delta\left(E - \frac{p^2}{2m}\right) \right\}$$

incoming

$$\Rightarrow \frac{\partial}{\partial t} |\langle p\nu' | u(t)\rangle|^2 = \frac{2\pi}{\hbar} |\langle p | V | \psi_{p_0}^+\rangle|^2 \delta\left(E - \frac{p^2}{2m}\right)$$

$$\Rightarrow \omega_{p_0\to p}\, d^3p = \frac{2\pi}{\hbar} |\langle p | V | \psi_{p_0}^+\rangle|^2 \delta\left(E_{p_0} - \frac{p^2}{2m}\right) \qquad p^2 dp\, d\Omega$$

$$\sigma(\theta,\varphi)\, d\Omega = \frac{(2\pi\hbar)^3}{v_0} d\Omega \int_0^\infty \omega_{p_0\to p}\, p^2 dp = \frac{(2\pi\hbar)^3}{v_0} \left(\frac{2\pi}{\hbar}\right) |\langle p | V | \psi_{p_0}^+\rangle|^2 m P$$

$$= 16\pi^4 m^2 \hbar^2 |\langle p | V | \psi_{p_0}^+\rangle|^2$$

§ another method for cross section — Lipman-Schwinger

$$\Omega_+ = 1 + \frac{1}{E - H_0 + i\eta} V \Omega_+$$

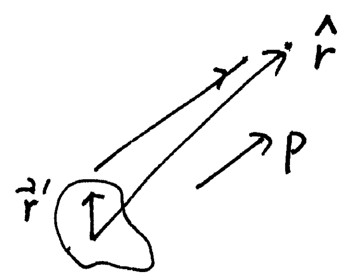
$$|\psi^+_{p_0}\rangle = |p_0\rangle + \frac{1}{E-H_0+i\eta} V |\psi^+_{p_0}\rangle \xrightarrow{\text{Coordinate Rep.}}$$

$$\psi^+_{p_0}(r) = \frac{1}{(2\pi\hbar)^{3/2}} e^{i p_0 \cdot r/\hbar} + \int \langle r| \frac{1}{E-H_0+i\eta} |r'\rangle \; V(r') \psi^+_{p_0}(r') dr'$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}_0\cdot\vec{r}/\hbar} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \; V(r') \, \psi^+_{p_0}(r') \, dr'$$

$$\psi^+_{p_0}(r) - \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{p}_0\cdot\vec{r}/\hbar} \xrightarrow{r\to+\infty} -\frac{m}{2\pi\hbar^2} \frac{e^{ik_0 r}}{r} \int e^{-i\underline{k_0}\hat{r}\cdot\vec{r}'} V(r') \psi^+_{p_0}(r') dr'$$

$$|\vec{r}-\vec{r}'| = r - \vec{r}'\cdot\hat{r}$$



$$\approx \frac{1}{(2\pi\hbar)^{3/2}} \underline{(-m4\pi^2\hbar)} \frac{e^{ik_0 r}}{r} \langle p | V | \psi^+_{p_0}\rangle$$

$$\Rightarrow \boxed{f(\theta,\varphi) = -4\pi^2\hbar m \langle p | V | \psi^+_{p_0}\rangle}$$

where $\vec{p}$ is along the direction of $\hat{r}$, $\hat{p}_0$ as $\hat{z}$ axis

$$\langle p | V | \psi^+_{p_0}\rangle = \langle p| \left(1 + \frac{1}{E-H+i\eta}\right) V | p_0\rangle$$

$$= \langle (1+\frac{1}{E-H-i\eta}) p | V | p_0\rangle = \langle \psi^-_p | V | p_0\rangle$$

check unit

$|p\rangle$ normalized to $\delta^{(3)}(p-p)$

$\Rightarrow \langle \psi_p | V | p_0\rangle \to p^{-3} \frac{p^2}{2m} = \frac{1}{mp}$

$$\Rightarrow \boxed{f(\theta,\varphi) = -4\pi^2\hbar m \langle \psi^{(-)}_p | V | p_0\rangle}$$

$[f] = \hbar/p = [\frac{1}{k}]$

§ optical theory

$$\sigma_T = \int \sigma(\theta,\varphi)\, d\Omega = \frac{\partial}{\partial t}\int d^3\vec{p}\ \langle u(t)|p\rangle\langle p|u(t)\rangle \cdot \frac{(2\pi\hbar)^3}{v_0}$$

$$\frac{\partial}{\partial t}\langle u(t)|u(t)\rangle = \langle u(t)|\dot{u}(t)\rangle + c.c.$$

real

$$= \frac{\langle u(t)|V\Psi(t)\rangle}{i\hbar} - \frac{\langle u(t)|H_0|u(t)\rangle}{i\hbar} + c.c.$$

$$= \frac{\langle u(t)|V|\Psi(t)\rangle}{i\hbar} + c.c. \qquad |u(t)\rangle = \psi(t) - \phi_0(t)$$

$$\Rightarrow \sigma_T = -\frac{\langle\phi_0(t)|V|\Psi(t)\rangle}{i\hbar} + c.c = \frac{i}{\hbar}\langle\phi_0(t)|V|\Psi(t)\rangle + c.c$$

set $\Delta p \to 0$

$$\boxed{\begin{aligned}\sigma_T &= -\frac{2}{\hbar}\, \mathrm{Im}\langle\vec{p}_0|V|\psi^+_{p_0}\rangle \cdot \frac{(2\pi\hbar)^3}{v_0} \\ &= \frac{4\pi}{k_0}\, \mathrm{Im}\, f(p_0,p_0)\end{aligned}}$$

forward scattering

§. Scattering matrix

$$S = U(+\infty, -\infty)$$

§ interaction picture

$$H = H_0 + V,$$

state vector $\quad |\psi_I(t)\rangle = e^{iH_0 t} |\psi_S(t)\rangle$

operator $\quad O_I(t) = e^{iH_0 t} O_S e^{-iH_0 t}$

time evolution operator

$$|\psi_I(t)\rangle = e^{iH_0 t} |\psi_S(t)\rangle = e^{iH_0 t} e^{-iH(t-t_0)} |\psi_S(t_0)\rangle$$

$$= e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0} |\psi_I(t_0)\rangle$$

$$\Rightarrow \boxed{\begin{array}{l} U_I(t, t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t} \\ U(t,t) = 1; \quad U(t, t_1) U(t_1, t_0) = U(t, t_0); \quad U^\dagger(t, t_0) = U(t_0, t) = U^{-1}(t, t_0) \\ i\hbar \frac{\partial}{\partial t} U(t, t_0) = V_I(t) U(t, t_0); \; -i\hbar \frac{\partial}{\partial t_0} U(t, t_0) = U(t, t_0) V_I(t_0) \end{array}}$$

§ Scattering matrix

$$S = U(\infty, -\infty) = \lim_{t \to +\infty} \lim_{t_0 \to -\infty} U(t, t_0)$$

$$= U(\infty, 0) \; U(0, -\infty)$$

← wave operator defined in last lecture

$$\Psi_S(t=0) = U(0, -\infty) \phi_0(0) = \lim_{T \to -\infty} e^{iHT} \phi_0(T) = \lim_{T \to -\infty} e^{iHT} e^{-iH_0 T} \phi_0(0)$$

$\boxed{\text{In the interaction picture}}$

$$\Psi_I(t) = \mathcal{U}_I(t, 0) \Psi_I(0)$$

$$= \mathcal{U}_I(t,0) \Psi_S(0) = \mathcal{U}_I(t,0)\, \mathcal{U}(0,-\infty) \quad \phi_0(0) = \mathcal{U}_I(t,0)\, \mathcal{U}_I(0,-\infty) \phi_0(0)$$

$$= \mathcal{U}_I(t,-\infty)\, \phi_0(0)$$

$$\boxed{\Psi_I(+\infty) = \mathcal{U}(+\infty,-\infty)\, \phi_0(0)}$$

$\mathcal{U}(+\infty,-\infty)$ Contains all the information of the scattering problem.

$\mathcal{U}(+\infty,-\infty) = S$ satisfies

① time reversal

$$T S = T\, \mathcal{U}(+\infty, 0)\, \mathcal{U}(0,-\infty) \xrightarrow{t\to -t} = \mathcal{U}(-\infty, 0)\, \mathcal{U}(0,+\infty)\, T$$

$$\Rightarrow \boxed{T S T^{-1} = S^\dagger}$$

② $$H_0 S = H_0\, \mathcal{U}(+\infty, 0)\, \mathcal{U}(0,-\infty) = \mathcal{U}(+\infty,0)\, H\, \mathcal{U}(0,-\infty)$$

$$= \mathcal{U}(+\infty,0)\,\mathcal{U}(0,-\infty)\, H_0 = S H_0$$

③ Unitarity

Completeness for the scattering states

$$S^\dagger S = \mathcal{U}(-\infty,0)\, \underline{\mathcal{U}(0\ \infty)\, \mathcal{U}(\infty, 0)}\, \mathcal{U}(0,-\infty)$$

$$= \mathcal{U}(-\infty,0)\, \underline{\mathcal{U}(0,-\infty)\, \mathcal{U}(\infty,0)}\, \mathcal{U}(0,-\infty)$$

$$= I \cdot I = I$$

Similarly $S S^\dagger = I$

③ S-matrix and scattering amplitude.

$$\langle p'|\,S\,|p\rangle = \langle p'|\,\mathcal{U}(+\infty,0)\,\mathcal{U}(0,-\infty)|p\rangle = \langle \psi^-_{p'}|\psi^+_p\rangle$$

$$|\psi^+_p\rangle - |\psi^-_p\rangle = \lim_{\eta\to 0^+}\left(\frac{1}{E(p)-H+i\eta} - \frac{1}{E(p)-H-i\eta}\right) V\,|p\rangle$$

$$= -2\pi i\,\delta(E(p)-H)\,V\,|p\rangle$$

$$\Rightarrow \langle p'|S|p\rangle = \langle\psi^+_{p'}|\psi^+_p\rangle - 2\pi i\,\langle p'|\,V\,\delta(E(p')-H)\,|\psi^+_p\rangle$$

$$= \delta(p'-p) - 2\pi i\,\langle p'|V|\psi^+_p\rangle\,\delta(E(p')-E(p))$$

$$= \delta(p'-p) - 2\pi i\,\langle \psi^{(-)}_{p'}|V|p\rangle\,\delta(E(p)-E(p'))$$

or $$\langle p'|\,S-1\,|p\rangle = -2\pi i\,\langle p'|V|\psi^{(+)}_p\rangle\,\delta(E(p')-E(p))$$

$$= -2\pi i\,\langle\psi^{(-)}_{p'}|V|p\rangle\,\delta(E(p)-E(p'))$$

$$\Rightarrow \boxed{\langle p|(S-1)|p_0\rangle = \frac{i}{2\pi\hbar m}\,f(p,p_0)\,\delta(E(p)-E(p_0)) \quad \text{where } f(p,p_0) = -4\pi^2\hbar m\,\langle p|\,V\,|\psi^{(+)}_{p_0}\rangle}$$

check unit $[\text{left}] = \delta^3(p-p_0) \sim [p]^{-3}$

$[\text{right}] = \frac{1}{\hbar m}\,\frac{1}{\hbar}\cdot\frac{1}{E} = \frac{1}{pm}\,\frac{m}{p^2} = [p]^{-3}$ ✓

④ ~~Reciprocal relation~~

$$\cancel{\langle p'|S|p\rangle = \langle p'|\,T^{-1}S^\dagger T\,|p\rangle = \langle Tp'|S^\dagger|Tp\rangle}$$

$$\cancel{= \langle Tp'|\,S^\dagger\,|Tp\rangle}$$

$$\cancel{\langle A|O|B\rangle = \langle A|TT^{-1}O|B\rangle = \langle}$$

☆ Reciprocal relation :

T: time reversal

$$\langle A|B\rangle = \langle TA|TB\rangle^* = \langle TB|TA\rangle$$

$$\langle A|\hat{O}|B\rangle = \langle TOB|TA\rangle = \langle (TOT^{-1})TB|TA\rangle$$

$$= \langle TB|(TOT^{-1})^\dagger|TA\rangle$$

$$\Rightarrow \langle p'|S|p\rangle = \langle Tp|(TST^{-1})^\dagger|Tp'\rangle = \langle Tp|(S^\dagger)^\dagger|Tp'\rangle$$

$$= \langle Tp|S|Tp'\rangle$$

← Scattering amplitudes are the same for two processes related by TR.

i.e $f(\vec{p}'; \vec{p}) = f(T\vec{p}; T\vec{p}')$