Lect 14 Dirac equation - plane wave solution

§ Weyl spinors: the generator in the 4×4 Dirac representation is block diagonal as given in last lecture. Thus this representation is reducible.

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma_x\\-\sigma_x&0\end{pmatrix}\begin{pmatrix}-i\sigma_x&0\\0&-i\sigma_x\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&1\end{pmatrix}$$

each of γ_5 eigenstate with ± 1 do not mix under Lorentz transformation.
↑ chirality quantum number.

☆ Chirality is different from parity here, where parity operator is γ_0.

we write $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$ where $\gamma_5\psi_L = -\psi_L$, $\gamma_5\psi_R = \psi_R$.

Under infinitesimal Lorentz transformation θ (rotation), β (boost)

$$\psi_L \to (1 - i\vec\theta\cdot\frac{\vec\sigma}{2} - \beta\frac{\vec\sigma}{2})\,\psi_L;$$

$$\psi_R \to (1 - i\vec\theta\cdot\frac{\vec\sigma}{2} + \beta\frac{\vec\sigma}{2})\,\psi_R.$$

using the identity $\sigma^2\sigma^* = -\sigma\sigma^2$, we can show

$\sigma_2\psi_L^*$ transformat like ψ_R, i.e.

$$\sigma_2\psi_L^* \to \sigma_2(1 - i\vec\theta\cdot\frac{\vec\sigma}{2} - \beta\frac{\vec\sigma}{2})^*\,\psi_L^* = (1 - i\vec\theta\cdot\frac{\vec\sigma}{2} + \beta\frac{\vec\sigma}{2})(\sigma_2\psi_L^*)$$

The Dirac equation becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\nabla) \\ i(\partial_0 - \vec{\sigma}\cdot\nabla) & m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

The mass term mixes the ψ_L and ψ_R parts.

① $m = 0$, ψ_L and ψ_R decouples $\boxed{\swarrow \text{ Weyl equations.}}$

$$i(\partial_0 - \sigma\cdot\nabla)\psi_L = 0 \quad ; \quad i(\partial_0 + \sigma\cdot\nabla)\psi_R = 0.$$

for $\psi_L = e^{-ipx} u(p) = e^{i(\vec{p}\cdot\vec{x} - \omega t)} u_L(p) \Rightarrow (E + c\vec{\sigma}\cdot\vec{p})u_p = 0$ and $E = \pm p\cdot c$

$$\Rightarrow \frac{\vec{\sigma}\cdot\vec{p}}{|p|} u_p = \pm u_p \quad \left(\begin{array}{l} + \text{ for } E = -pc; \\ - \text{ for } E = +pc \end{array} \right).$$

helicity

for $\psi_R = e^{-ipx} u_R(p) = e^{i(\vec{p}\cdot\vec{x} - \omega t)} u_R(p) \Rightarrow (E - c\vec{\sigma}\cdot\vec{p}) u_R(p) = 0$

$$\Rightarrow \frac{\vec{\sigma}\cdot\vec{p}}{|p|} u_R = \pm u_R \quad \left(\begin{array}{ll} + & \text{for } E = pc \\ - & \text{for } E = -pc \end{array} \right).$$

For massless particle, "neutrino" in the old days, only u_L exists. Now we know neutrino actually has mass, but only u_L participates weak interaction. Parity is broken. ← Lee and Yang, 1957.

② $m \neq 0$. Let us consider in the rest frame $p = (m, 0)$, the general solution can be obtained by boost. We first consider the positive energy solution

In the rest frame, $\psi(x) = u(p) e^{-ipx} \Rightarrow (m\gamma^0 - m) u(p_0) = 0$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0) = 0 \qquad u(p_0) = m\begin{pmatrix} \xi \\ \xi \end{pmatrix},$$ where $\xi^\dagger \xi = 1$, two-component vector.

Let us choose $\xi = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, and first consider the boost along z-axis.

$$\begin{pmatrix}E\\P_3\end{pmatrix} = \left[1 + \eta\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{bmatrix}m\\0\end{bmatrix}$$ η is call rapidity

$$\to \exp\left[\eta\begin{pmatrix}0&1\\1&0\end{pmatrix}\right]\begin{bmatrix}m\\0\end{bmatrix} = \begin{bmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{bmatrix}\begin{bmatrix}m\\0\end{bmatrix} = \begin{pmatrix}\cosh\eta\cdot m\\ \sinh\eta\cdot m\end{pmatrix}$$

$$e^{\eta} = \sqrt{\frac{E+P_3}{m}}$$

$$e^{-\eta} = \sqrt{\frac{E-P_3}{m}}$$

$$u(p) = \exp\left[-\frac{1}{2}\eta\begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}\right]\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \underbrace{\cosh\frac{\eta}{2}}_{\frac{1}{2}(e^{\eta/2}+e^{-\eta/2})}\begin{pmatrix}1&0\\0&1\end{pmatrix} - \underbrace{\sinh\frac{\eta}{2}}_{\frac{1}{2}(e^{\eta/2}-e^{-\eta/2})}\begin{pmatrix}\sigma_3 & \\ & -\sigma_3\end{pmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \begin{pmatrix} e^{\eta/2}\left(\frac{1-\sigma^3}{2}\right) + e^{-\eta/2}\left(1+\frac{\sigma^3}{2}\right), & 0 \\ 0\ , & e^{\eta/2}\left(\frac{1+\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right)\end{pmatrix}\sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E+P_3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-P_3}\,\frac{1+\sigma_3}{2} \\ \sqrt{E+P_3}\,\frac{1+\sigma^3}{2} + \sqrt{E-P^3}\,\frac{1-\sigma_3}{2}\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$

$\frac{1\pm\sigma_3}{2}$ are projection operators

$$\left(\sqrt{E+P_3}\,\frac{1-\sigma^3}{2} + \sqrt{E-P_3}\,\frac{1+\sigma_3}{2}\right)^2 = E+P_3\left(\frac{1-\sigma_3}{2}\right) + (E-P_3)\frac{1+\sigma_3}{2} = E - P_3\sigma_3$$

$$\left(\sqrt{E+P_3}\,\frac{1+\sigma^3}{2} + \sqrt{E-P_3}\,\frac{1-\sigma_3}{2}\right)^2 = E + P_3\sigma_3$$

define $\sigma = (1, \vec{\sigma})$ $\quad \bar{\sigma} = (1, -\vec{\sigma})$

$u(p)$ can be write as $\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi\\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix}$ where $\sqrt{\ }$ is the square root of matrix, which take the positive eigenvalue.

if $\xi = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow u(p) = \begin{pmatrix}\sqrt{E-P_3}\begin{pmatrix}1\\0\end{pmatrix}\\ \sqrt{E+P_3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xrightarrow[\text{boost}]{\text{large}} \sqrt{2E}\begin{pmatrix}0\\ \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}$,

$\begin{pmatrix}0\\1\end{pmatrix} \qquad u(p) = \begin{pmatrix}\sqrt{E+P_3}\begin{pmatrix}1\\0\end{pmatrix}\\ \sqrt{E-P_3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xrightarrow[\text{boost}]{\text{large}} \sqrt{2E}\begin{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\\ 0\end{pmatrix}$.

For general direction of $\vec{P}$, we only need to replace the projection operator $\frac{1\pm\sigma^3}{2} \rightarrow \frac{1\pm\hat{p}\cdot\vec{\sigma}}{2}$

and $u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi\\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix} = \begin{pmatrix}\sqrt{E+pc}\,\frac{1-\hat{p}\cdot\vec{\sigma}}{2}+\sqrt{E-pc}\,\frac{1+\hat{p}\cdot\vec{\sigma}}{2}\\ \sqrt{E+pc}\,\frac{1+\hat{p}\cdot\vec{\sigma}}{2}+\sqrt{E-pc}\,\frac{1-\hat{p}\cdot\vec{\sigma}}{2}\end{pmatrix}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$

define $\frac{1\pm\hat{p}\cdot\vec{\sigma}}{2}\,\xi_\pm = \xi_\pm$ i.e $\hat{p}\cdot\vec{\sigma}\,\xi_\pm = \pm\,\xi_\pm$

$$\Rightarrow \boxed{\underbrace{u_\pm}_{\text{helicity index}}(p) = \begin{pmatrix}\sqrt{E\mp pc}\;\xi_\pm\\ \sqrt{E\pm pc}\;\xi_\pm\end{pmatrix},}$$

Similarly, we can define the negative energy states

$\psi = v(p)\,e^{ipx+Et}$, $-P_0 < 0$

for $p = (-m, 0) \Rightarrow (-m\gamma^0 - m)\,v(-P_0) = 0$

$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} v(-P_0) = 0 \Rightarrow v(-P_0) = \sqrt{m}\begin{pmatrix}\xi\\ -\xi\end{pmatrix}$.

under the Lorentz transform :

$$\begin{pmatrix} -E \\ P_3 \end{pmatrix} = \exp\left[\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{bmatrix} -m \\ 0 \end{bmatrix} = \begin{pmatrix} -\cosh\eta\ m \\ -\sinh\eta\ m \end{pmatrix} \Rightarrow \begin{array}{l} e^{\eta} = \dfrac{E - P_3}{m} \quad (\eta < 0) \\ e^{-\eta} = \dfrac{E + P_3}{m} \end{array}$$

$$v(-P_3) = \exp\left[-\frac{1}{2}\eta \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}\right]\sqrt{m}\begin{pmatrix} \xi \\ -\xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E - P_3}\ \frac{1-\sigma^3}{2} + \sqrt{E+P_3}\ \frac{1+\sigma^3}{2} \\ \sqrt{E - P_3}\ \frac{1+\sigma^3}{2} + \sqrt{E+P_3}\ \frac{1-\sigma^3}{2} \end{pmatrix}\begin{pmatrix} \xi \\ -\xi \end{pmatrix} \qquad E > 0$$

$$\Rightarrow v(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ -\sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix}, \qquad \text{i.e. } v(\vec{p}) = \begin{pmatrix} \sqrt{E - pc}\ \frac{1-\hat{p}\cdot\vec{\sigma}}{2} + \sqrt{E+pc}\ \frac{1+\hat{p}\cdot\vec{\sigma}}{2} \\ \sqrt{E+pc}\ \frac{1+\hat{p}\cdot\vec{\sigma}}{2} \quad \sqrt{E+pc}\ \frac{1-\hat{p}\cdot\vec{\sigma}}{2} \end{pmatrix}\begin{pmatrix} \xi \\ -\xi \end{pmatrix}$$

$$v_\pm(\vec{p}) = \begin{pmatrix} \sqrt{E \pm pc}\ \xi_\pm \\ -\sqrt{E \mp pc}\ \xi_\pm \end{pmatrix}$$

$$\Rightarrow \psi^p_\pm(\vec{p}) = \frac{1}{\sqrt{E^2 + (pc)^2}}\begin{pmatrix} \sqrt{E \mp pc}\ \xi_\pm \\ \sqrt{E \pm pc}\ \xi_\pm \end{pmatrix} e^{i(px - Et)}$$

$$\psi^n_\pm(\vec{p}) = \frac{1}{\sqrt{E^2 + (pc)^2}}\begin{pmatrix} \sqrt{E \pm pc}\ \xi_\pm \\ -\sqrt{E \mp pc}\ \xi_\pm \end{pmatrix} e^{i(px + Et)}$$

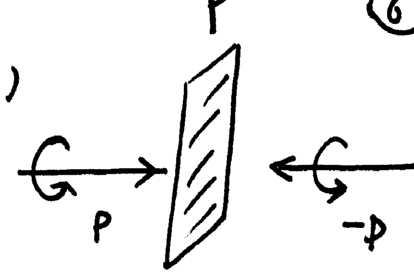
where $\xi_\pm$ are helicity eigenstate of $\vec{\sigma}\cdot\hat{p}$.

§ test symmetry operation:

a. parity: $P\,\psi^p_\pm(\vec{p}, t) = \begin{pmatrix} \sqrt{E \pm pc}\ \xi_\pm \\ \sqrt{E \mp pc}\ \xi_\pm \end{pmatrix} e^{i(-px - Et)}$

← respect to $-p$

$$= \begin{pmatrix} \sqrt{E \pm pc}\ \xi'_{\mp} \\ \sqrt{E \mp pc}\ \xi'_{\mp} \end{pmatrix} e^{i(-px - Et)} = \psi^{p}_{\mp}(-\vec{p}, t)$$



$$P\,\psi^{n}_{\pm}(\vec{p}, t) = \begin{pmatrix} -\sqrt{E \mp pc}\ \xi_{\pm} \\ \sqrt{E \pm pc}\ \xi_{\pm} \end{pmatrix} e^{-ipx + iEt}$$

$$= -\begin{pmatrix} \sqrt{E \mp pc}\ \xi'_{\mp} \\ -\sqrt{E \pm pc}\ \xi'_{\mp} \end{pmatrix} e^{-ipx + iEt} = -\psi^{n}_{\mp}(-\vec{p}, t)$$

under parity transformation $(\vec{p}, t) \to (-\vec{p}, t)$, helicity flip the sign.

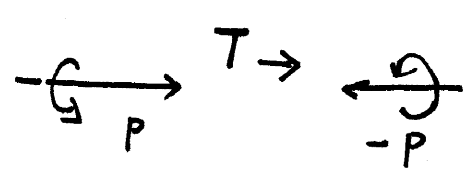
b. time reversal $\qquad \gamma^1\gamma^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}$

$$T\,\psi^{p}_{\pm}(\vec{p}, t) = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \begin{pmatrix} \sqrt{E \mp pc}\ \xi^{*}_{\pm} \\ \sqrt{E \pm pc}\ \xi^{*}_{\pm} \end{pmatrix} e^{i(-px - Et)}$$

$$\hat{p}\cdot\vec{\sigma}\ \xi_{\pm} = \pm\,\xi_{\pm} \Rightarrow \hat{p}\cdot\vec{\sigma}\,(i\sigma_2 \xi^{*}_{\pm}) = -i\sigma_2 (\hat{p}\cdot\vec{\sigma}^{*})\,\xi^{*}_{\pm} = \mp(i\sigma_2 \xi^{*}_{\pm})$$

$$-\hat{p}\cdot\vec{\sigma}\,(i\sigma_2 \xi^{*}_{\pm}) = \pm(i\sigma_2 \xi^{*}_{\pm}) \Rightarrow i\sigma_2 \xi^{*}_{\pm} = \xi'_{\pm}$$

$$\Rightarrow T\,\psi^{p}_{\pm}(\vec{p}, t) = \psi^{p}_{\pm}(-\vec{p}, t)$$



$$T\,\psi^{n}_{\pm}(\vec{p}, t) = \psi^{n}_{\pm}(-\vec{p}, t)$$

$$\boxed{\begin{aligned} i\sigma_2 \xi^{*}_{\pm} &= \xi'_{\pm} \\ i\sigma_2 \xi'^{*}_{\pm} &= -\xi_{\pm} \end{aligned}} \qquad T^2 = -1$$

C. Charge conjugation: $i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}$

$$C\psi^p_\pm(\vec{p}) = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E\mp pc}\ \xi^*_\pm \\ \sqrt{E\pm pc}\ \xi^*_\pm \end{pmatrix} e^{-ipx+iEt}$$

$$= \begin{pmatrix} \sqrt{E\pm pc}\ \xi'_\pm \\ -\sqrt{E\mp pc}\ \xi'_\pm \end{pmatrix} e^{-ipx+iEt} = \psi^n_\pm(-p)$$

$$C\psi^n_\pm(\vec{p}) = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E\pm pc}\ \xi^*_\pm \\ -\sqrt{E\mp pc}\ \xi^*_\pm \end{pmatrix} e^{-ipx-iEt}$$

$$= -\begin{pmatrix} \sqrt{E\mp pc}\ \xi'_\pm \\ \sqrt{E\pm pc}\ \xi'_\pm \end{pmatrix} e^{-ipx-iEt} = -\psi^p_\pm(-p) \qquad \boxed{C^2=1}$$

§ In the above we used the chirality basis where

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \gamma^{123} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

This is convenient for high energy states which are almost eigenstate of γ^5.

But for low energy states, they are not convenient. We often also use another basis with γ^{123} unchanged but $\gamma'_0 = \begin{pmatrix} I & \\ & -I \end{pmatrix}$ and $\gamma'^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

These two are upto a unitay transformation

$$\mathcal{U} = \exp\left[+\frac{i}{2}\sigma_2 \frac{\pi}{2}\right] = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} I & +I \\ -I & I \end{pmatrix}$$

$$\Rightarrow \gamma_0' = \mathcal{U}\gamma_0\mathcal{U}^\dagger = \frac{1}{2}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\gamma_5' = \mathcal{U}\gamma_5\mathcal{U}^\dagger = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \checkmark$$

$\Rightarrow$ wavefunction $\psi' = \mathcal{U}\psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\psi$

$$\Rightarrow \psi'^{p}_{\pm}(\vec{p}) = \frac{1}{\sqrt{2(E^2+(pc)^2)}}\begin{pmatrix} (\sqrt{E\mp pc} + \sqrt{E\pm pc})\,\xi_\pm \\ (\sqrt{E\pm pc} - \sqrt{E\mp pc})\,\xi_\pm \end{pmatrix} e^{i(px-Et)}$$

$$\frac{\sqrt{E+pc} - \sqrt{E-pc}}{\sqrt{E+pc} + \sqrt{E-pc}} = \frac{2pc}{2E^2 + 2\sqrt{E^2-(pc)^2}} = \frac{pc}{E+mc^2}$$

$$\Rightarrow \psi'^{p}_{\pm}(\vec{p}) = \sqrt{\frac{mc^2+E}{2E}}\begin{pmatrix} \xi_\pm \\ \frac{\pm pc}{E+mc^2}\,\xi_\pm \end{pmatrix} e^{i(px-Et)}$$

big component

small component.

Similarly

$$\psi'^{n}_{\pm}(\vec{p}) = \sqrt{\frac{mc^2+E}{2E}}\begin{pmatrix} \frac{\mp pc}{E+mc^2}\,\xi_\pm \\ \xi_\pm \end{pmatrix} e^{i(px+Et)}$$