PHYSICS 221A : NONLINEAR DYNAMICS
HW ASSIGNMENT #2

(1) In an ODE, the functions and their derivatives are all evaluated at the same time t. In a *delay differential equation*, this is no longer the case, and different terms may be evaluated at different values of the time t. Consider the delay equation

$$y'(t) + a\,y(t-1) = 0\ ,$$

subject to the boundary conditions

$$y(t) = y_0 \qquad \text{when} \quad -1 \le t \le 0\ .$$

(a) Solve using the method of the Laplace transform:

$$Y(s) = \int_0^\infty \!dt\; y(t)\, e^{-st}$$

$$y(t) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, Y(s)\, e^{st}\ .$$

Recall that the contour for the inverse transform lies to the right of all singularities of the integrand. Find an analytic expression for $Y(s)$. Show that your result can be Taylor expanded to yield

$$Y(s) = \frac{y_0}{s} + y_0 \sum_{n=1}^\infty (-a)^n\, e^{-(n-1)s}\, s^{-(n+1)}\ .$$

Performing the inverse transform, show that

$$y(t) = y_0 \sum_{n=0}^{[t]+1} \frac{(-a)^n}{n!}\,(t-n+1)^n\ ,$$

where $[t]$ is the greatest integer less than or equal to t. *Hint: You will close in the LHP or RHP depending on the value of* $[t]$ *in relation to* $n-1$.

(b) Solve by the following alternate method. For $t \in [0,1]$ we have

$$y'(t) + a\,y_0 = 0\ ,$$

with $y(0) = y_0$. The solution is then

$$y(t) = y_0\,(1-at) \qquad t \in [0,1]\ .$$

On the interval $t \in [1,2]$, then, we have

$$y'(t) + a y_0\,\big[1 - a(t-1)\big]\ ,$$

with $y(1) = y_0\,(1-a)$. The solution on this interval is then

$$y(t) = y_0\Big[1 - at + \tfrac{1}{2}a^2(t-1)^2\Big] \qquad t \in [1,2]\ .$$

Proceeding thusly, show that you obtain the same solution as in part (a).

(c) Dispensing with the boundary condition, and assuming $t \in \mathbb{R}$, find at least one periodic solution. Show that periodicity requires that a be fine tuned to a specific value or set of values.

(2) Numerically integrate the system

$$\begin{aligned}\dot{r} &= r(1-r^2) + \lambda r \cos\theta \\ \dot{\theta} &= 1\end{aligned}$$

with $0 < \lambda < 1$, and show that any initial condition lying between the concentric circles of radii $\sqrt{1 \pm \lambda}$ approaches a closed limit cycle in the long time limit. Choose whatever value of λ suits your taste.

(3) For the Lotka-Volterra system

$$\begin{aligned}\dot{x} &= x\,(r - x - ky) \\ \dot{y} &= y\,(1 - y - k'x)\ ,\end{aligned}$$

discussed in §3.4.2 of the notes, sketch the phase diagram in the upper right quadrant of the (k, k') plane for fixed r. That is, label the regions of the plane by their fixed point structure.

(4) In §4.1.2 of the notes, the Poincaré-Lindstedt method was applied to ODEs of the form

$$\ddot{x} + \Omega_0^2\, x = \epsilon\, h(x)\ .$$

Consider the more general equation,

$$\ddot{x} + \Omega_0^2\, x = \epsilon\, h(x, \dot{x})\ .$$

Generalize the discussion in §4.1.2 for such equations, and exhibit explicitly the equations of the resulting hierarchy through order ϵ^3. Apply this method to lowest nontrivial order to the van der Pol oscillator,

$$\ddot{x} + \epsilon(x^2 - 1)\,\dot{x} + x = 0\ ,$$

by taking $h(x, \dot{x}) = (1 - x^2)\,\dot{x}$.