Chapter 11

Shock Waves

Here we shall follow closely the pellucid discussion in chapter 2 of the book by G. Whitham, beginning with the simplest possible PDE,

$$\rho_t + c_0\,\rho_x = 0 \; . \tag{11.1}$$

The solution to this equation is an arbitrary right-moving wave (assuming $c_0 > 0$), with profile

$$\rho(x,t) = f(x - c_0 t) \; , \tag{11.2}$$

where the initial conditions on eqn. 11.1 are $\rho(x, t = 0) = f(x)$. Nothing to see here, so move along.

11.1 Nonlinear Continuity Equation

The simplest nonlinear PDE is a generalization of eqn. 11.1,

$$\rho_t + c(\rho)\,\rho_x = 0 \; . \tag{11.3}$$

This equation arises in a number of contexts. One example comes from the theory of vehicular traffic flow along a single lane roadway. Starting from the continuity equation,

$$\rho_t + j_x = 0 \; , \tag{11.4}$$

one posits a constitutive relation $j = j(\rho)$, in which case $c(\rho) = j'(\rho)$. If the individual vehicles move with a velocity $v = v(\rho)$, then

$$j(\rho) = \rho\, v(\rho) \quad \Rightarrow \quad c(\rho) = v(\rho) + \rho\, v'(\rho) \; . \tag{11.5}$$

It is natural to assume a form $v(\rho) = c_0\,(1 - a\rho)$, so that at low densities one has $v \approx c_0$, with $v(\rho)$ decreasing monotonically to $v = 0$ at a critical density $\rho = a^{-1}$, presumably corresponding to bumper-to-bumper traffic. The current $j(\rho)$ then takes the form of an inverted parabola. Note the difference between the individual vehicle velocity $v(\rho)$ and what turns out to be the group velocity of a traffic wave, $c(\rho)$. For $v(\rho) = c_0\,(1 - a\rho)$, one has $c(\rho) = c_0\,(1 - 2a\rho)$, which is *negative* for $\rho \in \big[\frac{1}{2}a^{-1}, a^{-1}\big]$. For vehicular traffic, we have $c'(\rho) = j''(\rho) < 0$ but in general $j(\rho)$ and thus $c(\rho)$ can be taken to be arbitrary.

Another example comes from the study of chromatography, which refers to the spatial separation of components in a mixture which is forced to flow through an immobile absorbing 'bed'. Let $\rho(x,t)$ denote the density of the desired component in the fluid phase and $n(x,t)$ be its density in the solid phase. Then continuity requires

$$n_t + \rho_t + V\rho_x = 0 \ , \tag{11.6}$$

where V is the velocity of the flow, which is assumed constant. The net rate at which the component is deposited from the fluid onto the solid is given by an equation of the form

$$n_t = F(n,\rho) \ . \tag{11.7}$$

In equilibrium, we then have $F(n,\rho) = 0$, which may in principle be inverted to yield $n = n_{eq}(\rho)$. If we assume that the local deposition processes run to equilibrium on fast time scales, then we may substitute $n(x,t) \approx n_{eq}\big(\rho(x,t)\big)$ into eqn. 11.6 and obtain

$$\rho_t + c(\rho)\,\rho_x = 0 \quad , \quad c(\rho) = \frac{V}{1 + n'_{eq}(\rho)} \ . \tag{11.8}$$

We solve eqn. 11.3 using the *method of characteristics*. Suppose we have the solution $\rho = \rho(x,t)$. Consider then the family of curves obeying the ODE

$$\frac{dx}{dt} = c\big(\rho(x,t)\big) \ . \tag{11.9}$$

This is a family of curves, rather than a single curve, because it is parameterized by the initial condition $x(0) \equiv \zeta$. Now along any one of these curves we must have

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\,\frac{dx}{dt} = \frac{\partial\rho}{\partial t} + c(\rho)\,\frac{\partial\rho}{\partial x} = 0 \ . \tag{11.10}$$

Thus, $\rho(x,t)$ is a constant along each of these curves, which are called *characteristics*. For eqn. 11.3, the family of characteristics is a set of straight lines[1],

$$x_\zeta(t) = \zeta + c(\rho)\,t \ . \tag{11.11}$$

The initial conditions for the function $\rho(x,t)$ are

$$\rho(x = \zeta, t = 0) = f(\zeta) \ , \tag{11.12}$$

[1]The existence of straight line characteristics is a special feature of the equation $\rho_t + c(\rho)\,\rho_x = 0$. For more general hyperbolic first order PDEs to which the method of characteristics may be applied, the characteristics are curves. See the discussion in the Appendix.

where $f(\zeta)$ is arbitrary. Thus, in the (x,t) plane, if the characteristic curve $x(t)$ intersects the line $t=0$ at $x=\zeta$, then its slope is constant and equal to $c\big(f(\zeta)\big)$. We then define

$$g(\zeta) \equiv c\big(f(\zeta)\big) \ . \tag{11.13}$$

This is a known function, computed from $c(\rho)$ and $f(\zeta) = \rho(x=\zeta, t=0)$. The equation of the characteristic $x_\zeta(t)$ is then

$$x_\zeta(t) = \zeta + g(\zeta)\, t \ . \tag{11.14}$$

Do not confuse the subscript in $x_\zeta(t)$ for a derivative!

To find $\rho(x,t)$, we follow this prescription:

(i) Given any point in the (x,t) plane, we find the characteristic $x_\zeta(t)$ on which it lies. This means we invert the equation $x = \zeta + g(\zeta)\, t$ to find $\zeta(x,t)$.

(ii) The value of $\rho(x,t)$ is then $\rho = f\big(\zeta(x,t)\big)$.

(iii) This procedure yields a unique value for $\rho(x,t)$ provided the characteristics do not cross, *i.e.* provided that there is a unique ζ such that $x = \zeta + g(\zeta)\, t$. If the characteristics do cross, then $\rho(x,t)$ is either *multi-valued*, or else the method has otherwise broken down. As we shall see, the crossing of characteristics, under the conditions of single-valuedness for $\rho(x,t)$, means that a *shock* has developed, and that $\rho(x,t)$ is *discontinuous*.

We can verify that this procedure yields a solution to the original PDE of eqn. 11.3 in the following manner. Suppose we invert

$$x = \zeta + g(\zeta)\, t \qquad \Longrightarrow \qquad \zeta = \zeta(x,t) \ . \tag{11.15}$$

We then have

$$\rho(x,t) = f\big(\zeta(x,t)\big) \qquad \Longrightarrow \qquad \begin{cases} \rho_t = f'(\zeta)\, \zeta_t \\ \\ \rho_x = f'(\zeta)\, \zeta_x \end{cases} \tag{11.16}$$

To find ζ_t and ζ_x, we invoke $x = \zeta + g(\zeta)\, t$, hence

$$0 = \frac{\partial}{\partial t}\big[\zeta + g(\zeta)\, t - x\big] = \zeta_t + \zeta_t\, g'(\zeta)\, t + g(\zeta) \tag{11.17}$$

$$0 = \frac{\partial}{\partial x}\big[\zeta + g(\zeta)\, t - x\big] = \zeta_x + \zeta_x\, g'(\zeta)\, t - 1 \ , \tag{11.18}$$

from which we conclude

$$\rho_t = -\frac{f'(\zeta)\, g(\zeta)}{1 + g'(\zeta)\, t} \tag{11.19}$$

$$\rho_x = \frac{f'(\zeta)}{1 + g'(\zeta)\, t} \ . \tag{11.20}$$

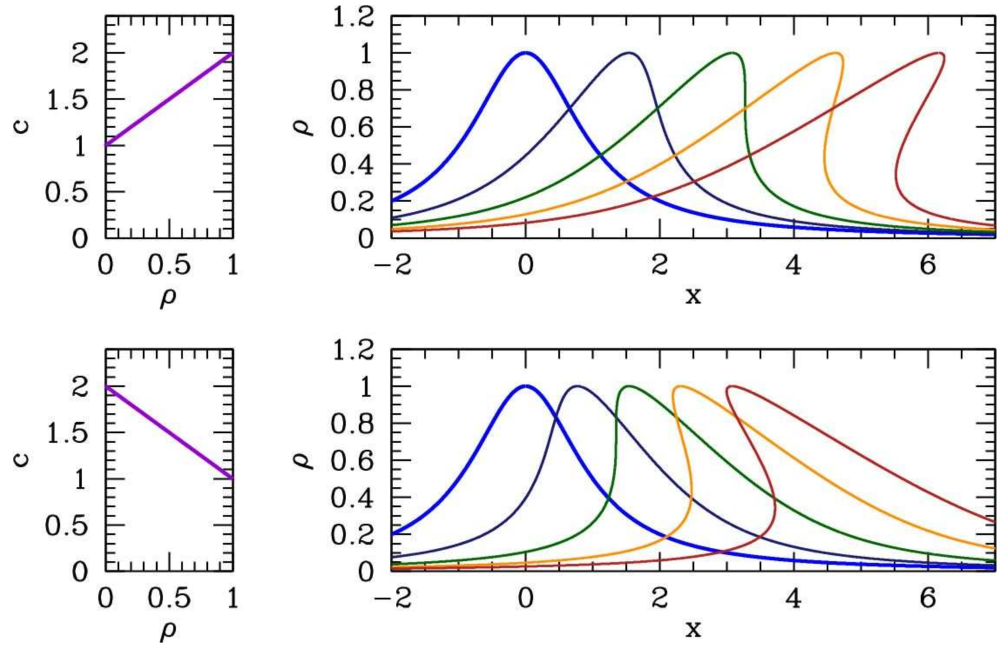

Figure 11.1: Forward and backward breaking waves for the nonlinear continuity equation $\rho_t + c(\rho)\,\rho_x = 0$, with $c(\rho) = 1 + \rho$ (top panels) and $c(\rho) = 2 - \rho$ (bottom panels). The initial conditions are $\rho(x, t = 0) = 1/(1 + x^2)$, corresponding to a break time of $t_{\rm B} = \frac{8}{3\sqrt{3}}$. Successive $\rho(x, t)$ curves are plotted for $t = 0$ (thick blue), $t = \frac{1}{2}t_{\rm B}$ (dark blue), $t = t_{\rm B}$ (dark green), $t = \frac{3}{2}t_{\rm B}$ (orange), and $t = 2t_{\rm B}$ (dark red).

Thus, $\rho_t + c(\rho)\,\rho_x = 0$, since $c(\rho) = g(\zeta)$.

As any wave disturbance propagates, different values of ρ propagate with their own velocities. Thus, the solution $\rho(x, t)$ can be constructed by splitting the curve $\rho(x, t = 0)$ into level sets of constant ρ, and then shifting each such set by a distance $c(\rho)\,t$. For $c(\rho) = c_0$, the entire curve is shifted uniformly. When $c(\rho)$ varies, different level sets are shifted by different amounts.

We see that ρ_x diverges when $1 + g'(\zeta)\,t = 0$. At this time, the wave is said to *break*. The break time $t_{\rm B}$ is defined to be the smallest value of t for which $\rho_x = \infty$ anywhere. Thus,

$$t_{\rm B} = \min_{\substack{\zeta \\ g'(\zeta)<0}} \left(-\frac{1}{g'(\zeta)} \right) \equiv -\frac{1}{g'(\zeta_{\rm B})} \ . \tag{11.21}$$

Breaking can only occur when $g'(\zeta) < 0$, and differentiating $g(\zeta) = c\big(f(\zeta)\big)$, we have that $g'(\zeta) = c'(f)\,f'(\zeta)$. We then conclude

$$\begin{aligned} c' < 0 \quad &\Longrightarrow \quad \text{need } f' > 0 \text{ to break} \\ c' > 0 \quad &\Longrightarrow \quad \text{need } f' < 0 \text{ to break} \ . \end{aligned}$$

Thus, if $\rho(x = \zeta, t = 0) = f(\zeta)$ has a hump profile, then the wave breaks forward (*i.e.* in the direction of its motion) if $c' > 0$ and backward (*i.e.* opposite to the direction of its motion)

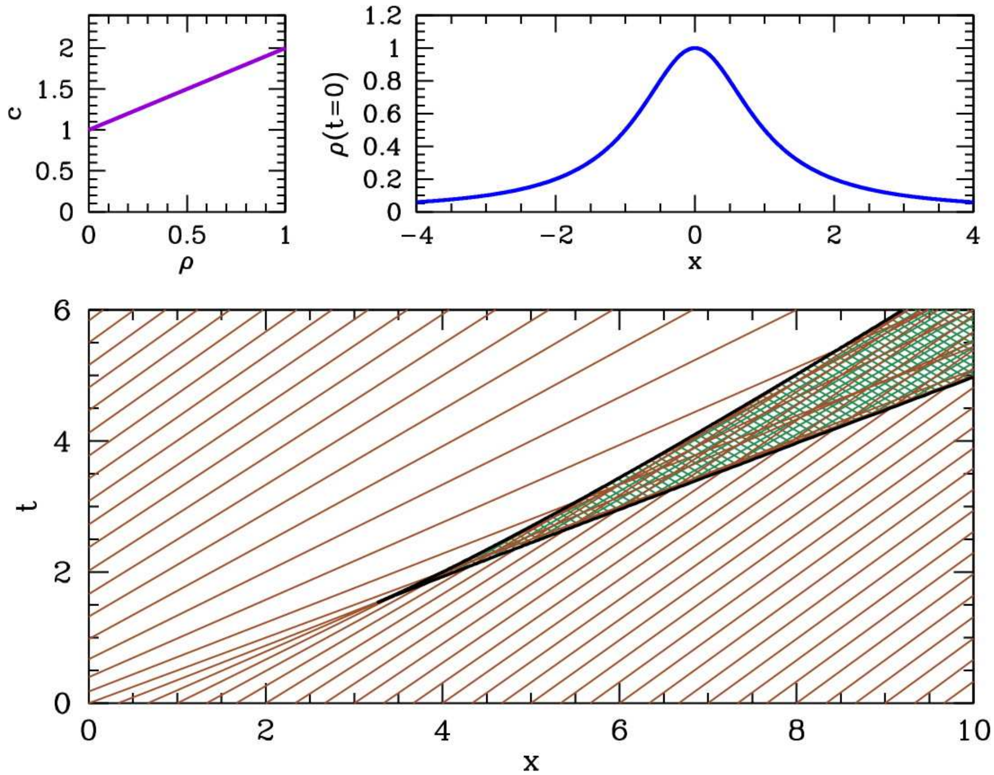

Figure 11.2: Crossing of characteristics of the nonlinear continuity equation $\rho_t + c(\rho)\,\rho_x = 0$, with $c(\rho) = 1 + \rho$ and $\rho(x, t = 0) = 1/(1 + x^2)$. Within the green hatched region of the (x, t) plane, the characteristics cross, and the function $\rho(x, t)$ is apparently multivalued.

if $c' < 0$. In fig. 11.1 we sketch the breaking of a wave with $\rho(x, t = 0) = 1/(1 + x^2)$ for the cases $c = 1 + \rho$ and $c = 2 - \rho$. Note that it is possible for different regions of a wave to break at different times, if, say, it has multiple humps.

Wave breaking occurs when neighboring characteristic cross. We can see this by comparing two neighboring characteristics,

$$x_\zeta(t) = \zeta + g(\zeta)\,t \tag{11.22}$$

$$\begin{aligned} x_{\zeta+\delta\zeta}(t) &= \zeta + \delta\zeta + g(\zeta + \delta\zeta)\,t \\ &= \zeta + g(\zeta)\,t + \big(1 + g'(\zeta)\,t\big)\,\delta\zeta + \ldots\ . \end{aligned} \tag{11.23}$$

For these characteristics to cross, we demand

$$x_\zeta(t) = x_{\zeta+\delta\zeta}(t) \quad \Longrightarrow \quad t = -\frac{1}{g'(\zeta)}\ . \tag{11.24}$$

Usually, in most physical settings, the function $\rho(x, t)$ is single-valued. In such cases, when

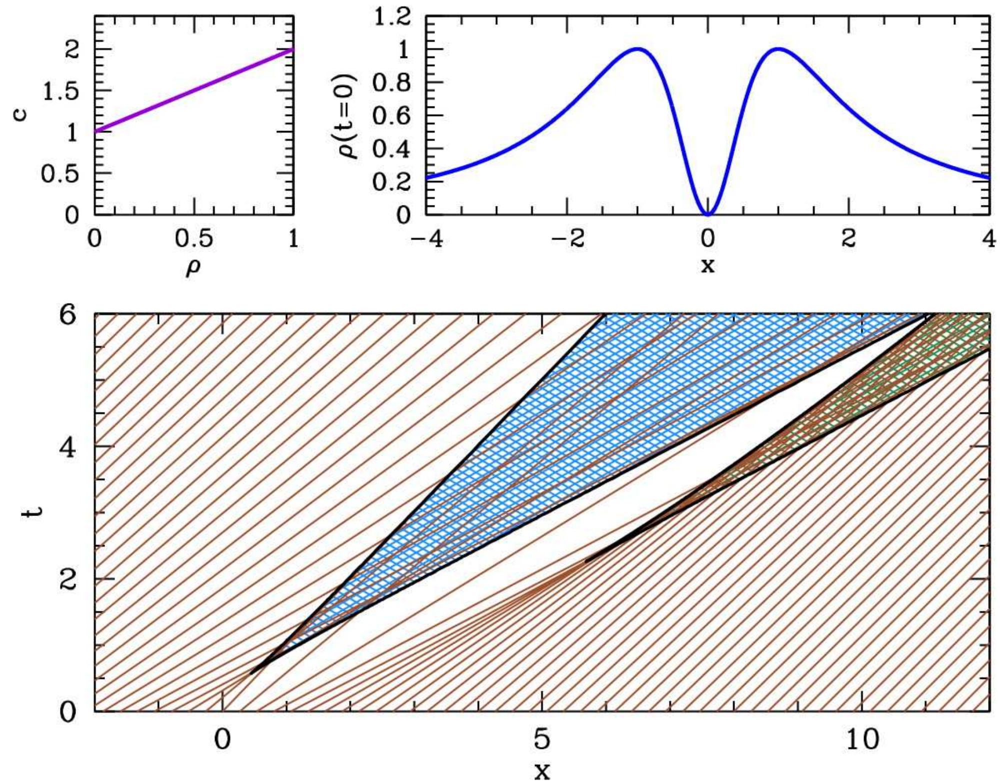

Figure 11.3: Crossing of characteristics of the nonlinear continuity equation $\rho_t + c(\rho)\,\rho_x = 0$, with $c(\rho) = 1 + \rho$ and $\rho(x, t = 0) = \big[x/(1 + x^2)\big]^2$. The wave now breaks in two places and is multivalued in both hatched regions. The left hump is the first to break.

the wave breaks, the multivalued solution ceases to be applicable[2]. Generally speaking, this means that some important physics has been left out. For example, if we neglect viscosity η and thermal conductivity κ, then the equations of gas dynamics have breaking wave solutions similar to those just discussed. When the gradients are steep – just before breaking – the effects of η and κ are no longer negligible, even if these parameters are small. This is because these parameters enter into the coefficients of higher derivative terms in the governing PDEs, and even if they are small their effect is magnified in the presence of steep gradients. In mathematical parlance, they constitute *singular perturbations*. The shock wave is then a thin region in which η and κ are crucially important, and the flow changes rapidly throughout this region. If one is not interested in this small scale physics, the shock region can be approximated as being infinitely thin, *i.e.* as a discontinuity in the inviscid limit of the theory. What remains is a set of *shock conditions* which govern the discontinuities of various quantities across the shocks.

[2]This is even true for water waves, where one might think that a multivalued height function $h(x, t)$ is physically possible.

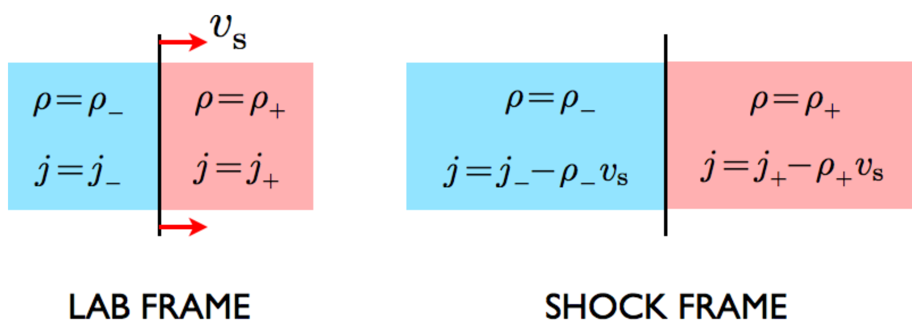

Figure 11.4: Current conservation in the shock frame yields the shock velocity, $v_s = \Delta j/\Delta\rho$.

11.2 Shocks

We now show that a solution to eqn. 11.3 exists which is single valued for almost all (x,t), *i.e.* everywhere with the exception of a set of zero measure, but which has a discontinuity along a curve $x = x_s(t)$. This discontinuity is the shock wave.

The velocity of the shock is determined by mass conservation, and is most easily obtained in the frame of the shock. The situation is as depicted in fig. 11.4. If the density and current are $(\rho_-\,,\,j_-)$ to the left of the shock and $(\rho_+\,,\,j_+)$ to the right of the shock, and if the shock moves with velocity v_s, then making a Galilean transformation to the frame of the shock, the densities do not change but the currents transform as $j \to j' = j - \rho v$. Thus, in the frame where the shock is stationary, the current on the left and right are $j_\pm = j_\pm - \rho_\pm v_s$. Current conservation then requires

$$v_s = \frac{j_+ - j_-}{\rho_+ - \rho_-} = \frac{\Delta j}{\Delta\rho}\ . \tag{11.25}$$

The special case of quadratic $j(\rho)$ bears mention. Suppose

$$j(\rho) = \alpha\rho^2 + \beta\rho + \gamma\ . \tag{11.26}$$

Then $c = 2\alpha\rho + \beta$ and

$$\begin{aligned} v_s &= \alpha(\rho_+ + \rho_-) + \beta \\ &= \tfrac{1}{2}\big(c_+ + c_-\big)\ . \end{aligned} \tag{11.27}$$

So for quadratic $j(\rho)$, the shock velocity is simply the average of the flow velocity on either side of the shock.

Consider, for example, a model with $j(\rho) = 2\rho(1-\rho)$, for which $c(\rho) = 2 - 4\rho$. Consider an initial condition $\rho(x=\zeta, t=0) = f(\zeta) = \frac{3}{16} + \frac{1}{8}\,\Theta(\zeta)$, so initially $\rho = \rho_1 = \frac{3}{16}$ for $x < 0$ and $\rho = \rho_2 = \frac{5}{16}$ for $x > 0$. The lower density part moves faster, so in order to avoid multiple-valuedness, a shock must propagate. We find $c_- = \frac{5}{4}$ and $c_+ = \frac{3}{4}$. The shock velocity is then $v_s = 1$. Ths situation is depicted in fig. 11.5.

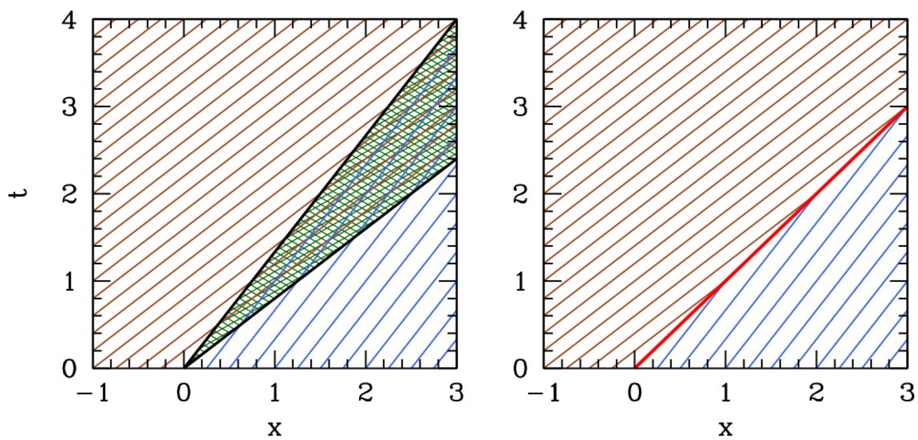

Figure 11.5: A resulting shock wave arising from $c_- = \frac{5}{4}$ and $c_+ = \frac{3}{4}$. With no shock fitting, there is a region of (x, t) where the characteristics cross, shown as the hatched region on the left. With the shock, the solution remains single valued. A quadratic behavior of $j(\rho)$ is assumed, leading to $v_s = \frac{1}{2}(c_+ + c_-) = 1$.

11.3 Internal Shock Structure

At this point, our model of a shock is a discontinuity which propagates with a finite velocity. This may be less problematic than a multivalued solution, but it is nevertheless unphysical. We should at least understand how the discontinuity is resolved in a more complete model. To this end, consider a model where

$$j = J(\rho, \rho_x) = J(\rho) - \nu \rho_x \ . \tag{11.28}$$

The $J(\rho)$ term contains a nonlinearity which leads to steepening and broadening of regions where $\frac{dc}{dx} > 0$ and $\frac{dc}{dx} < 0$, respectively. The second term, $-\nu\rho_x$, is due to diffusion, and recapitulates *Fick's law*, which says that a diffusion current flows in such a way as to reduce gradients. The continuity equation then reads

$$\rho_t + c(\rho)\, \rho_x = \nu \rho_{xx} \ , \tag{11.29}$$

with $c(\rho) = J'(\rho)$. Even if ν is small, its importance is enhanced in regions where $|\rho_x|$ is large, and indeed $-\nu\rho_x$ dominates over $J(\rho)$ in such regions. Elsewhere, if ν is small, it may be neglected, or treated perturbatively.

As we did in our study of front propagation, we seek a solution of the form

$$\rho(x, t) = \rho(\xi) \equiv \rho(x - v_s t) \qquad ; \qquad \xi = x - v_s t \ . \tag{11.30}$$

Thus, $\rho_t = -v_s\, \rho_x$ and $\rho_x = \rho_\xi$, leading to

$$-v_s\, \rho_\xi + c(\rho)\, \rho_\xi = \nu \rho_{\xi\xi} \ . \tag{11.31}$$

Integrating once, we have

$$J(\rho) - v_s\, \rho + A = \nu\, \rho_\xi \ , \tag{11.32}$$

where A is a constant. Integrating a second time, we have

$$\xi - \xi_0 = \nu \int\limits_{\rho_0}^{\rho} \frac{d\rho'}{J(\rho') - v_{\rm s}\,\rho' + A} \ . \tag{11.33}$$

Suppose ρ interpolates between the values ρ_1 and ρ_2. Then we must have

$$J(\rho_1) - v_{\rm s}\,\rho_1 + A = 0 \tag{11.34}$$

$$J(\rho_2) - v_{\rm s}\,\rho_2 + A = 0 \ , \tag{11.35}$$

which in turn requires

$$v_{\rm s} = \frac{J_2 - J_1}{\rho_2 - \rho_1} \ , \tag{11.36}$$

where $J_{1,2} = J(\rho_{1,2})$, exactly as before! We also conclude that the constant A must be

$$A = \frac{\rho_1 J_2 - \rho_2 J_1}{\rho_2 - \rho_1} \ . \tag{11.37}$$

11.3.1 Quadratic $J(\rho)$

For the special case where $J(\rho)$ is quadratic, with $J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$, we may write

$$J(\rho) - v_{\rm s}\,\rho + A = \alpha(\rho - \rho_2)(\rho - \rho_1) \ . \tag{11.38}$$

We then have $v_{\rm s} = \alpha(\rho_1 + \rho_2) + \beta$, as well as $A = \alpha\rho_1\rho_2 - \gamma$. The moving front solution then obeys

$$d\xi = \frac{\nu\, d\rho}{\alpha(\rho - \rho_2)(\rho - \rho_1)} = \frac{\nu}{\alpha(\rho_2 - \rho_1)}\, d\ln\!\left(\frac{\rho_2 - \rho}{\rho - \rho_1}\right) \ , \tag{11.39}$$

which is integrated to yield

$$\rho(x,t) = \frac{\rho_2 + \rho_1 \exp\!\Big[\alpha(\rho_2 - \rho_1)\big(x - v_{\rm s}t\big)/\nu\Big]}{1 + \exp\!\Big[\alpha(\rho_2 - \rho_1)\big(x - v_{\rm s}t\big)/\nu\Big]} \ . \tag{11.40}$$

We consider the case $\alpha > 0$ and $\rho_1 < \rho_2$. Then $\rho(\pm\infty, t) = \rho_{1,2}$. Note that

$$\rho(x,t) = \begin{cases} \rho_1 & \text{if } x - v_{\rm s}t \gg \delta \\ \rho_2 & \text{if } x - v_{\rm s}t \ll -\delta \ , \end{cases} \tag{11.41}$$

where

$$\delta = \frac{\nu}{\alpha\,(\rho_2 - \rho_1)} \tag{11.42}$$

is the thickness of the shock region. In the limit $\nu \to 0$, the shock is discontinuous. All that remains is the *shock condition*,

$$v_{\rm s} = \alpha(\rho_1 + \rho_2) + \beta = \tfrac{1}{2}\big(c_1 + c_2\big) \ . \tag{11.43}$$

We stress that we have limited our attention here to the case where $J(\rho)$ is quadratic. It is worth remarking that for *weak shocks* where $\Delta\rho = \rho_+ - \rho_-$ is small, we can expand $J(\rho)$ about the average $\frac{1}{2}(\rho_+ + \rho_-)$, in which case we find $v_{\rm s} = \frac{1}{2}(c_+ + c_-) + \mathcal{O}\big((\Delta\rho)^2\big)$.

11.4 Shock Fitting

When we neglect diffusion currents, we have $j = J$. We now consider how to fit discontinuous shocks satisfying

$$v_{\mathrm{s}} = \frac{J_+ - J_-}{\rho_+ - \rho_-} \tag{11.44}$$

into the continuous solution of eqn. 11.3, which are described by

$$x = \zeta + g(\zeta)\, t \tag{11.45}$$
$$\rho = f(\zeta) \ , \tag{11.46}$$

with $g(\zeta) = c\big(f(\zeta)\big)$, such that the multivalued parts of the continuous solution are eliminated and replaced with the shock discontinuity. The guiding principle here is number conservation:

$$\frac{d}{dt}\int\limits_{-\infty}^{\infty} dx\, \rho(x,t) = 0 \ . \tag{11.47}$$

We'll first learn how do fit shocks when $J(\rho)$ is quadratic, with $J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$. We'll assume $\alpha > 0$ for the sake of definiteness.

11.4.1 An Important Caveat

Let's multiply the continuity equation $\rho_t + c(\rho)\,\rho_x = 0$ by $c'(\rho)$. Thus results in

$$c_t + c\, c_x = 0 \ . \tag{11.48}$$

If we define $q = \frac{1}{2}c^2$, then this takes the form of a continuity equation:

$$c_t + q_x = 0 \ . \tag{11.49}$$

Now consider a shock wave. Invoking eqn. 11.25, we would find, *mutatis mutandis*, a shock velocity

$$u_{\mathrm{s}} = \frac{q_+ - q_-}{c_+ - c_-} = \tfrac{1}{2}(c_+ + c_-) \ . \tag{11.50}$$

This agrees with the velocity $v_{\mathrm{s}} = \Delta j/\Delta\rho$ only when $j(\rho)$ is quadratic. Something is wrong – there cannot be two velocities for the same shock.

The problem is that eqn. 11.48 is not valid across the shock and cannot be used to determine the shock velocity. There is no conservation law for c as there is for ρ. One way we can appreciate the difference is to add diffusion into the mix. Multiplying eqn. 11.29 by $c'(\rho)$, and invoking $c_{xx} = c'(\rho)\,\rho_{xx} + c''(\rho)\,\rho_x^2$, we obtain

$$c_t + c\, c_x = \nu c_{xx} - \nu c''(\rho)\,\rho_x^2 \ . \tag{11.51}$$

We now see explicitly how nonzero $c''(\rho)$ leads to a different term on the RHS. When $c''(\rho) = 0$, the above equation is universal, independent of the coefficients in the quadratic $J(\rho)$, and is known as *Burgers' equation*,

$$c_t + c\, c_x = \nu c_{xx} \ . \tag{11.52}$$

Later on we shall see how this nonlinear PDE may be linearized, and how we can explicitly solve for shock behavior, including the merging of shocks.

11.4.2 Recipe for shock fitting ($J'''(\rho) = 0$)

Number conservation means that when we replace the multivalued solution by the discontinuous one, the area under the curve must remain the same. If $J(\rho)$ is quadratic, then we can base our analysis on the equation $c_t + c\, c_x = 0$, since it gives the correct shock velocity $v_{\rm s} = \frac{1}{2}(c_+ + c_-)$. We then may then follow the following rules:

(i) Sketch $g(\zeta) = c\big(f(\zeta)\big)$.

(ii) Draw a straight line connecting two points on this curve at ζ_- and ζ_+ which obeys the equal area law, *i.e.*

$$\tfrac{1}{2}(\zeta_+ - \zeta_-)\Big(g(\zeta_+) + g(\zeta_-)\Big) = \int\limits_{\zeta_-}^{\zeta_+} d\zeta\, g(\zeta) \ . \tag{11.53}$$

(iii) This line evolves into the shock front after a time t such that

$$x_{\rm s}(t) = \zeta_- + g(\zeta_-)\, t = \zeta_+ + g(\zeta_+)\, t \ . \tag{11.54}$$

Thus,

$$t = -\frac{\zeta_+ - \zeta_-}{g(\zeta_+) - g(\zeta_-)} \ . \tag{11.55}$$

Alternatively, we can fix t and solve for $\zeta_\pm$. See fig. 11.6 for a graphical description.

(iv) The position of the shock at this time is $x = x_{\rm s}(t)$. The strength of the shock is $\Delta c = g(\zeta_-) - g(\zeta_+)$. Since $J(\rho) = \alpha\rho^2 + \beta\rho + \gamma$, we have $c(\rho) = 2\alpha\rho + \beta$ and hence the density discontinuity at the shock is $\Delta\rho = \Delta c/2\alpha$.

(v) The break time, when the shock first forms, is given by finding the steepest chord satisfying the equal area law. Such a chord is tangent to $g(\zeta)$ and hence corresponds to zero net area. The break time is

$$t_{\rm B} = \min_{\substack{\zeta \\ g'(\zeta)>0}} \bigg(-\frac{1}{g'(\zeta)}\bigg) \equiv -\frac{1}{g(\zeta_{\rm B})} \ . \tag{11.56}$$

(vi) If $g(\infty) = g(-\infty)$, the shock strength vanishes as $t \to \infty$. If $g(-\infty) > g(+\infty)$ then asymptotically the shock strength approaches $\Delta g = g(-\infty) - g(+\infty)$.

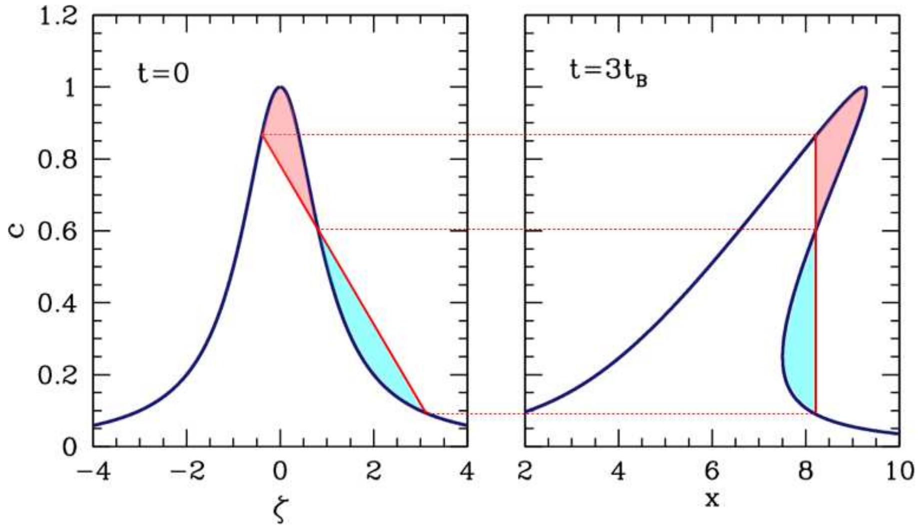

Figure 11.6: Shock fitting for quadratic $J(\rho)$.

11.4.3 Example problem

Suppose the $c(\rho)$ and $\rho(x,t=0)$ are such that the initial profile for $c(x,t=0)$ is

$$c(x,0) = c_0 \cos\left(\frac{\pi x}{2\ell}\right) \Theta\big(\ell - |x|\big) \ , \tag{11.57}$$

where $\Theta(s)$ is the step function, which vanishes identically for negative values of its argument. Thus, $c(x,0) = 0$ for $|x| \geq \ell$.

(a) Find the time $t_{\rm B}$ at which the wave breaks and a shock front develops. Find the position of the shock $x_{\rm s}(t_{\rm B})$ at the moment it forms.

Solution : Breaking first occurs at time

$$t_{\rm B} = \min_x \frac{-1}{c'(x,0)} \ . \tag{11.58}$$

Thus, we look for the maximum negative slope in $g(x) \equiv c(x,0)$, which occurs at $x = \ell$, where $c'(\ell,0) = -\pi c_0/2\ell$. Therefore,

$$t_{\rm B} = \frac{2\ell}{\pi c_0} \quad , \quad x_{\rm B} = \ell \ . \tag{11.59}$$

(b) Use the shock-fitting equations to derive $\zeta_{\pm}(t)$.

Solution : The shock fitting equations are

$$\tfrac{1}{2}\left(\zeta_+ - \zeta_-\right)\Big(g(\zeta_+) + g(\zeta_-)\Big) = \int\limits_{\zeta_-}^{\zeta_+} \! d\zeta \; g(\zeta) \tag{11.60}$$

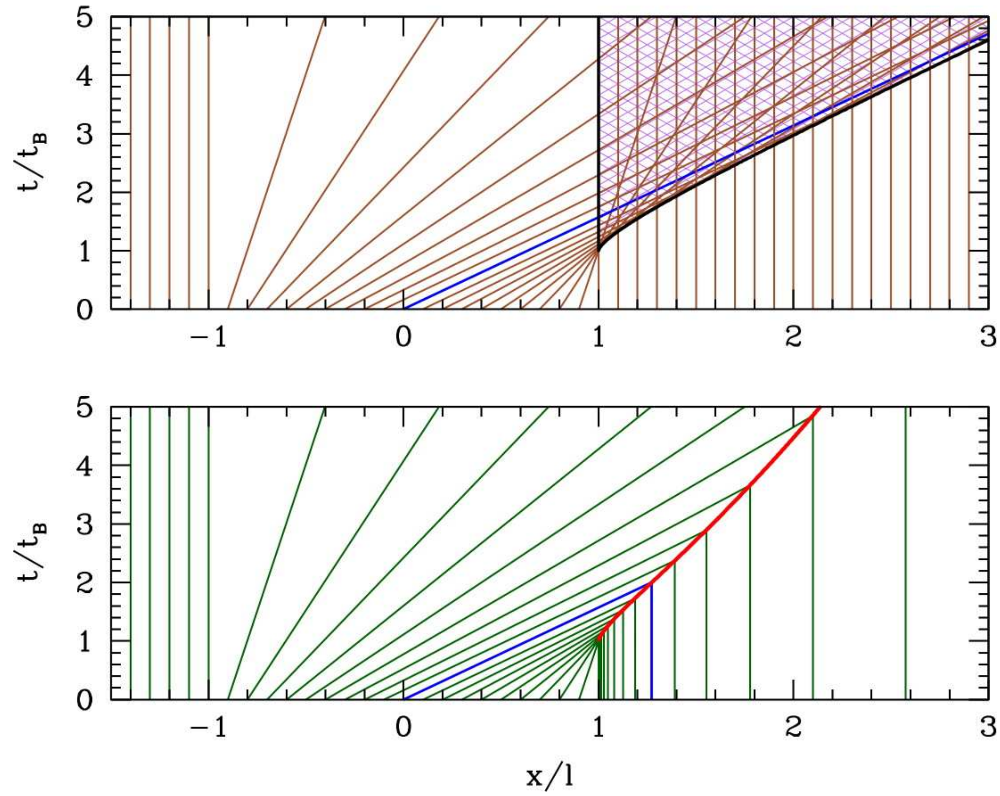

Figure 11.7: Top : crossing characteristics (purple hatched region) in the absence of shock fitting. Bottom : characteristics in the presence of the shock.

and

$$t = \frac{\zeta_+ - \zeta_-}{g(\zeta_-) - g(\zeta_+)} \ . \tag{11.61}$$

Clearly $\zeta_+ > \ell$, hence $g(\zeta_+) = 0$ and

$$\int\limits_{\zeta_-}^{\zeta_+} \! d\zeta \; g(\zeta) = c_0 \cdot \frac{2\ell}{\pi} \int\limits_{\pi\zeta_-/2\ell}^{\pi/2} \! dz \; \cos z = \frac{2\ell \, c_0}{\pi} \left\{ 1 - \sin\!\left(\frac{\pi\zeta_-}{2\ell}\right) \right\} . \tag{11.62}$$

Thus, the first shock fitting equation yields

$$\tfrac{1}{2} \, (\zeta_+ - \zeta_-) \, c_0 \cos\!\left(\frac{\pi\zeta_-}{2\ell}\right) = \frac{2\ell \, c_0}{\pi} \left\{ 1 - \sin\!\left(\frac{\pi\zeta_-}{2\ell}\right) \right\} . \tag{11.63}$$

The second shock fitting equation gives

$$t = \frac{\zeta_+ - \zeta_-}{c_0 \cos\left(\frac{\pi\zeta_-}{2\ell}\right)} \ . \tag{11.64}$$

Eliminating $\zeta_+ - \zeta_-$, we obtain the relation

$$\sin\!\left(\frac{\pi\zeta_-}{2\ell}\right) = \frac{4\ell}{\pi c_0 t} - 1 \ . \tag{11.65}$$

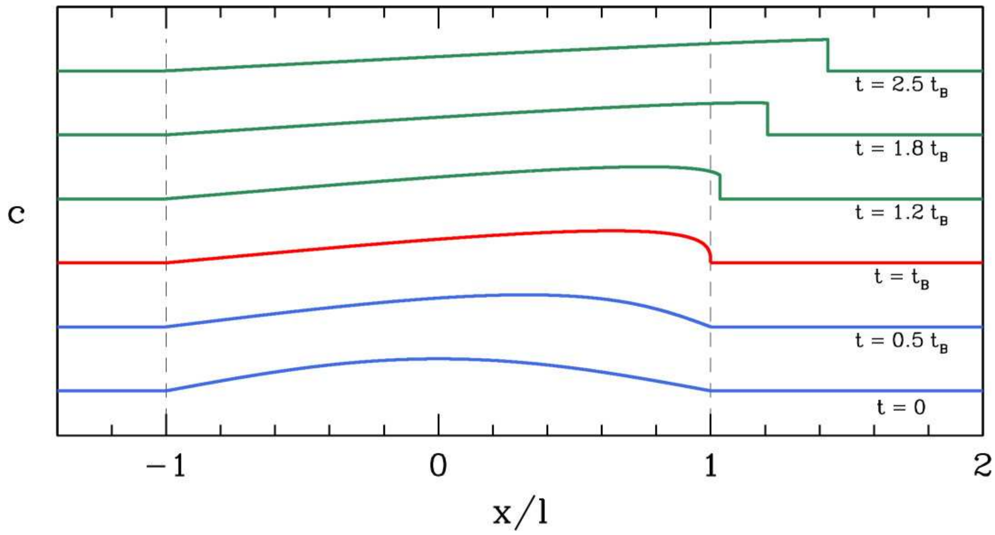

Figure 11.8: Evolution of $c(x,t)$ for a series of time values.

Thus,

$$\zeta_-(t) = \frac{2\ell}{\pi}\,\sin^{-1}\!\bigg(\frac{4\ell}{\pi c_0 t} - 1\bigg) \tag{11.66}$$

$$\begin{aligned}\zeta_+(t) &= \zeta_- + \frac{4\ell}{\pi}\cdot\frac{1-\sin(\pi\zeta_-/2\ell)}{\cos(\pi\zeta_-/2\ell)} \\ &= \frac{2\ell}{\pi}\left\{\sin^{-1}\!\bigg(\frac{4\ell}{\pi c_0 t} - 1\bigg) + 2\sqrt{\frac{\pi c_0 t}{2\ell} - 1}\,\right\} ,\end{aligned} \tag{11.67}$$

where $t \geq t_{\rm B} = 2\ell/\pi c_0$.

(c) Find the shock motion $x_s(t)$.

Solution : The shock position is

$$\begin{aligned}x_{\rm s}(t) &= \zeta_- + g(\zeta_-)\,t \\ &= \frac{2\ell}{\pi}\sin^{-1}\!\bigg(\frac{2}{\tau} - 1\bigg) + \frac{4\ell}{\pi}\,\sqrt{\tau - 1}\ ,\end{aligned} \tag{11.68}$$

where $\tau = t/t_{\rm B} = \pi c_0 t/2\ell$, and $\tau \geq 1$.

(d) Sketch the characteristics for the multivalued solution with no shock fitting, identifying the region in (x,t) where characteristics cross. Then sketch the characteristics for the discontinuous shock solution.

Solution : See fig. 11.7.

(e) Find the shock discontinuity $\Delta c(t)$.

Solution : The shock discontinuity is

$$\begin{aligned} \Delta c(t) &= g\big(\zeta_-\big) - g\big(\zeta_+\big) = c_0 \cos\!\left(\frac{\pi\zeta_-}{2\ell}\right) \\ &= \sqrt{\frac{8\ell\, c_0}{\pi t}\left(1 - \frac{2\ell}{\pi c_0 t}\right)} = 2c_0\,\frac{\sqrt{\tau - 1}}{\tau}\ . \end{aligned} \tag{11.69}$$

(f) Find the shock velocity $v_{\rm s}(t)$.

Solution : The shock wave velocity is

$$\begin{aligned} c_{\rm s}(t) &= \tfrac{1}{2}\Big[\,g\big(\zeta_-\big) + g\big(\zeta_+\big)\Big] = \tfrac{1}{2}\,\Delta c(t) \\ &= \sqrt{\frac{2\ell\, c_0}{\pi t}\left(1 - \frac{2\ell}{\pi c_0 t}\right)}\ . \end{aligned} \tag{11.70}$$

(g) Sketch the evolution of the wave, showing the breaking of the wave at $t = t_{\rm B}$ and the subsequent evolution of the shock front.

Solution : A sketch is provided in Fig. 11.8.

11.5 Long-time Behavior of Shocks

Starting with an initial profile $\rho(x,t)$, almost all the original details are lost in the $t \to \infty$ limit. What remains is a set of propagating triangular waves, where only certain gross features of the original shape, such as its area, are preserved.

11.5.1 Fate of a hump

The late time profile of $c(x,t)$ in fig. 11.8 is that of a triangular wave. This is a general result. Following Whitham, we consider the late time evolution of a hump profile $g(\zeta)$. We assume $g(\zeta) = c_0$ for $|\zeta| > L$. Shock fitting requires

$$\tfrac{1}{2}\Big[g(\zeta_+) + g(\zeta_-) - 2c_0\Big](\zeta_+ - \zeta_-) = \int\limits_{\zeta_-}^{\zeta_+}\! d\zeta\,\big(g(\zeta) - c_0\big)\ . \tag{11.71}$$

Eventually the point ζ_+ must pass $x = L$, in which case $g(\zeta_+) = c_0$. Then

$$\tfrac{1}{2}\Big[g(\zeta_+) - c_0\Big](\zeta_+ - \zeta_-) = \int\limits_{\zeta_-}^{L}\! d\zeta\,\big(g(\zeta) - c_0\big) \tag{11.72}$$

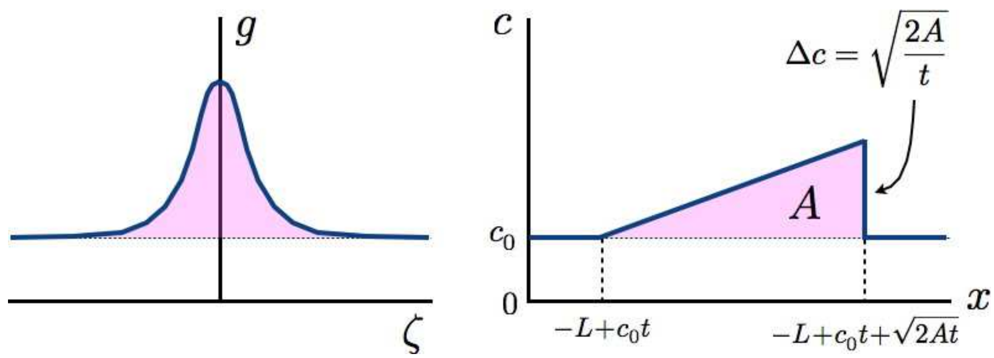

Figure 11.9: Initial and late time configurations for a hump profile. For late times, the profile is triangular, and all the details of the initial shape are lost, save for the area A.

and therefore

$$t = \frac{\zeta_+ - \zeta_-}{g(\zeta_-) - c_0} \ . \tag{11.73}$$

Using this equation to eliminate ζ_+, we have

$$\tfrac{1}{2}\big(g(\zeta_-) - c_0\big)^2\, t = \int\limits_{\zeta_-}^{L} d\zeta\, \big(g(\zeta) - c_0\big) \ . \tag{11.74}$$

As $t \to \infty$ we must have $\zeta_- \to -L$, hence

$$\tfrac{1}{2}\big(g(\zeta_-) - c_0\big)^2\, t \approx \int\limits_{-L}^{L} d\zeta\, \big(g(\zeta) - c_0\big) \equiv A \ , \tag{11.75}$$

where A is the area under the hump to the line $c = c_0$. Thus,

$$g(\zeta_-) - c_0 \approx \sqrt{\frac{2A}{t}} \ , \tag{11.76}$$

and the late time motion of the shock is given by

$$x_s(t) = -L + c_0 t + \sqrt{2At} \tag{11.77}$$

$$v_s(t) = c_0 + \sqrt{\frac{A}{2t}} \ . \tag{11.78}$$

The shock strength is $\Delta c = g(\zeta_-) - c_0 = \sqrt{2A/t}$. Behind the shock, we have $c = g(\zeta)$ and $x = \zeta + g(\zeta)\, t$, hence

$$c = \frac{x + L}{t} \qquad \text{for} \qquad -L + c_0 t < x < -L + c_0 t + \sqrt{2At} \ . \tag{11.79}$$

As $t \to \infty$, the details of the original profile $c(x, 0)$ are lost, and all that remains conserved is the area A. Both shock velocity and the shock strength decrease as $t^{-1/2}$ at long times, with $v_s(t) \to c_0$ and $\Delta c(t) \to 0$.

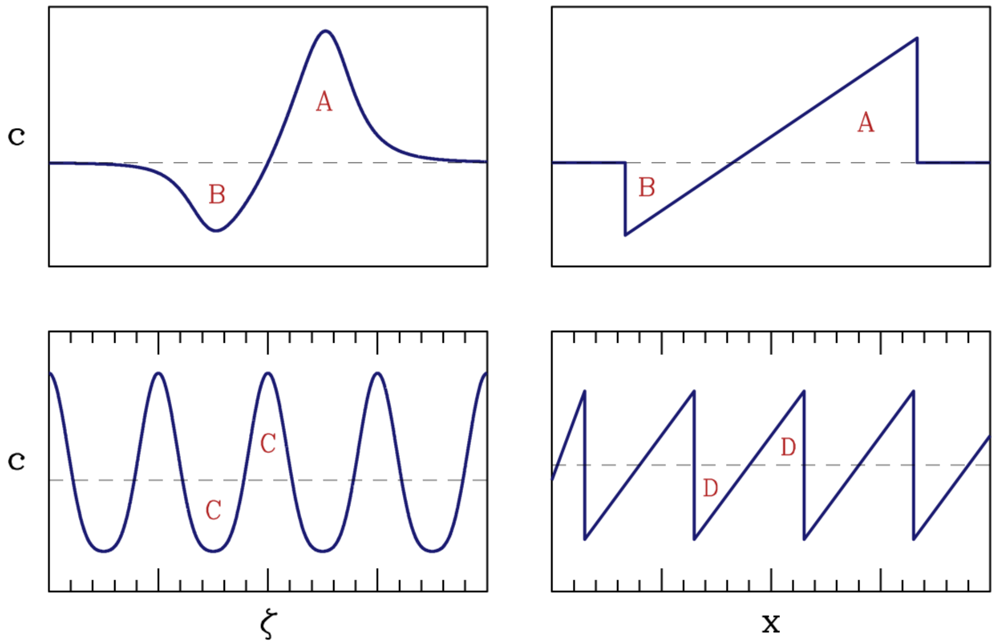

Figure 11.10: Top panels : An N-wave, showing initial (left) and late time (right) profiles. As the N-wave propagates, the areas A and B are preserved. Bottom panels : A P-wave. The area D eventually decreases to zero as the shock amplitude dissipates.

11.5.2 N-wave and P-wave

Consider the initial profile in the top left panel of fig. 11.10. Now there are two propagating shocks, since there are two compression regions where $g'(\zeta) < 0$. As $t \to \infty$, we have $(\zeta_-\,,\,\zeta_+)_{\rm A} \to (0\,,\,\infty)$ for the A shock, and $(\zeta_-\,,\,\zeta_+)_{\rm B} \to (-\infty\,,\,0)$ for the B shock. Asymptotically, the shock strength

$$\Delta c(t) \equiv c\big(x_{\rm s}^-(t),t\big) - c\big(x_{\rm s}^+(t),t\big) \tag{11.80}$$

for the two shocks is given by

$$x_{\rm s}^{\rm A}(t) \approx c_0 t + \sqrt{2At} \quad , \quad \Delta c_{\rm A} \approx +\sqrt{\frac{2A}{t}} \tag{11.81}$$

$$x_{\rm s}^{\rm B}(t) \approx c_0 t - \sqrt{2Bt} \quad , \quad \Delta c_{\rm B} \approx -\sqrt{\frac{2B}{t}}\ , \tag{11.82}$$

where A and B are the areas associated with the two features This feature is called an N-wave, for its N (or inverted N) shape.

The initial and late stages of a periodic wave, where $g(\zeta+\lambda) = g(\zeta)$, are shown in the right panels of fig. 11.10. In the $t \to \infty$ limit, we evidently have $\zeta_+ - \zeta_- = \lambda$, the wavelength. Asymptotically the shock strength is given by

$$\Delta c(t) \equiv g(\zeta_-) - g(\zeta_+) = \frac{\zeta_+ - \zeta_-}{t} = \frac{\lambda}{t}\ , \tag{11.83}$$

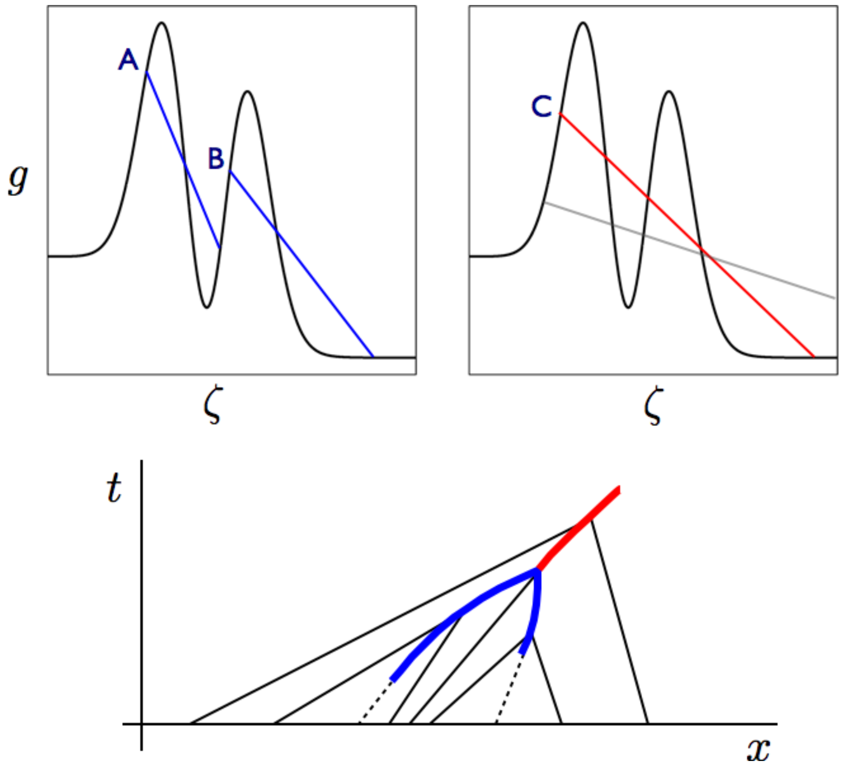

Figure 11.11: Merging of two shocks. The shocks initially propagate independently (upper left), and then merge and propagate as a single shock (upper right). Bottom : characteristics for the merging shocks.

where we have invoked eqn. 11.55. In this limit, the shock train travels with constant velocity c_0, which is the spatial average of $c(x,0)$ over one wavelength:

$$c_0 = \frac{1}{\lambda}\int\limits_0^\lambda d\zeta\, g(\zeta) \ . \tag{11.84}$$

11.6 Shock Merging

It is possible for several shock waves to develop, and in general these shocks form at different times, have different strengths, and propagate with different velocities. Under such circumstances, it is quite possible that one shock overtakes another. These two shocks then merge and propagate on as a single shock. The situation is depicted in fig. 11.11. We label the shocks by A and B when they are distinct, and the late time single shock by C. We must have

$$v_{\rm s}^{\rm A} = \tfrac{1}{2}\,g\big(\zeta_+^{\rm A}\big) + \tfrac{1}{2}\,g\big(\zeta_-^{\rm A}\big) \tag{11.85}$$

$$v_{\rm s}^{\rm B} = \tfrac{1}{2}\,g\big(\zeta_+^{\rm B}\big) + \tfrac{1}{2}\,g\big(\zeta_-^{\rm B}\big) \ . \tag{11.86}$$

The merging condition requires

$$\zeta_+^{\rm A} = \zeta_-^{\rm B} \equiv \xi \tag{11.87}$$

as well as

$$\zeta_+^{\rm C} = \zeta_+^{\rm B} \qquad , \qquad \zeta_-^{\rm C} = \zeta_-^{\rm A} \ . \tag{11.88}$$

The merge occurs at time t, where

$$t = \frac{\zeta_+ - \xi}{g(\xi) - g(\zeta_+)} = \frac{\xi - \zeta_-}{g(\zeta_-) - g(\xi)} \ . \tag{11.89}$$

Thus, the slopes of the A and B shock construction lines are equal when they merge.

11.7 Shock Fitting for General $J(\rho)$

When $J(\rho)$ is quadratic, we may analyze the equation $c_t + c\, c_x$, as it is valid across any shocks in that it yields the correct shock velocity. If $J'''(\rho) \neq 0$, this is no longer the case, and we must base our analysis on the original equation $\rho_t + c(\rho)\, \rho_x = 0$.

The coordinate transformation

$$(x, c) \longrightarrow (x + ct\, ,\, c) \tag{11.90}$$

preserves areas in the (x, c) plane and also maps lines to lines. However, while

$$(x, \rho) \longrightarrow (x + c(\rho)\, t\, ,\, \rho) \tag{11.91}$$

does preserve areas in the (x, ρ) plane, it does not map lines to lines. Thus, the 'pre-image' of the shock front in the (x, ρ) plane is not a simple straight line, and our equal area construction fails. Still, we can make progress. We once again follow Whitham, §2.9.

Let $x(\rho, t)$ be the inverse of $\rho(x, t)$, with $\zeta(\rho) \equiv x(\rho, t = 0)$. Then

$$x(\rho, t) = \zeta(\rho) + c(\rho)\, t \ . \tag{11.92}$$

Note that $\rho(x, t)$ is in general multi-valued. We still have that the shock solution covers the same area as the multivalued solution $\rho(x, t)$. Let $\rho_\pm$ denote the value of ρ just to the right $(+)$ or left $(-)$ of the shock. For purposes of illustration, we assume $c'(\rho) > 0$, which means that $\rho_x < 0$ is required for breaking, although the method works equally well for $c'(\rho) < 0$. Assuming a hump-like profile, we then have $\rho_- > \rho_+$, with the shock breaking to the right. Area conservation requires

$$\int\limits_{\rho_+}^{\rho_-} \! d\rho\, x(\rho, t) = \int\limits_{\rho_+}^{\rho_-} \! d\rho\, \big[\zeta(\rho) + c(\rho)\, t\,\big] = (\rho_- - \rho_+)\, x_{\rm s}(t) \ . \tag{11.93}$$

Since $c(\rho) = J'(\rho)$, the above equation may be written as

$$(J_+ - J_-)\,t - (\rho_+ - \rho_-) = \int\limits_{\rho_+}^{\rho_-} d\rho\;\zeta(\rho)$$

$$= \rho_-\,\zeta_- - \rho_+\,\zeta_+ - \int\limits_{\zeta_+}^{\zeta_-} d\zeta\;\rho(\zeta)\ . \tag{11.94}$$

Now the shock position $x_s(t)$ is given by

$$x_s = \zeta_- + c_-\,t = \zeta_+ + c_+ t\ , \tag{11.95}$$

hence

$$\Big[(J_+ - \rho_+ c_+) - (J_- - \rho_- c_-)\Big] = \frac{c_+ - c_-}{\zeta_+ - \zeta_-}\int\limits_{\zeta_-}^{\zeta_+} d\zeta\;\rho(\zeta)\ . \tag{11.96}$$

This is a useful result because $J_\pm$, $\rho_\pm$, and $c_\pm$ are all functions of $\zeta_\pm$, hence what we have here is a relation between ζ_+ and ζ_-. When $J(\rho)$ is quadratic, this reduces to our earlier result in eqn. 11.53. For a hump, we still have $x_s \approx c_0 t + \sqrt{2At}$ and $c - c_0 \approx \sqrt{2A/t}$ as before, with

$$A = c'(\rho_0)\int\limits_{-\infty}^{\infty} d\zeta\;\big[\rho(\zeta) - \rho_0\big]\ . \tag{11.97}$$

11.8 Sources

Consider the continuity equation in the presence of a source term,

$$\rho_t + c\,\rho_x = \sigma\ , \tag{11.98}$$

where $c = c(x, t, \rho)$ and $\sigma = \sigma(x, t, \rho)$. Note that we are allowing for more than just $c = c(\rho)$ here. According to the discussion in the Appendix, the characteristic obey the coupled ODEs[3],

$$\frac{d\rho}{dt} = \sigma(x, t, \rho) \tag{11.99}$$

$$\frac{d\sigma}{dt} = c(x, t, \rho)\ . \tag{11.100}$$

In general, the characteristics no longer are straight lines.

[3]We skip the step where we write $dt/ds = 1$ since this is immediately integrated to yield $s = t$.

11.8.1 Examples

Whitham analyzes the equation

$$c_t + c\,c_x = -\alpha\,c\ , \tag{11.101}$$

so that the characteristics obey

$$\frac{dc}{dt} = -\alpha\,c \qquad , \qquad \frac{dx}{dt} = c\ . \tag{11.102}$$

The solution is

$$c_\zeta(t) = e^{-\alpha t}\,g(\zeta) \tag{11.103}$$

$$x_\zeta(t) = \zeta + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) g(\zeta)\ , \tag{11.104}$$

where $\zeta = x_\zeta(0)$ labels the characteristics. Clearly $x_\zeta(t)$ is not a straight line. Neighboring characteristics will cross at time t if

$$\frac{\partial x_\zeta(t)}{\partial \zeta} = 1 + \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right) g'(\zeta) = 0\ . \tag{11.105}$$

Thus, the break time is

$$t_{\rm B} = \min_{\substack{\zeta \\ t_{\rm B}>0}} \left[-\frac{1}{\alpha}\,\ln\!\left(1 + \frac{\alpha}{g'(\zeta)}\right)\right] . \tag{11.106}$$

This requires $g'(\zeta) < -\alpha$ in order for wave breaking to occur.

For another example, consider

$$c_t + c\,c_x = -\alpha\,c^2\ , \tag{11.107}$$

so that the characteristics obey

$$\frac{dc}{dt} = -\alpha\,c^2 \qquad , \qquad \frac{dx}{dt} = c\ . \tag{11.108}$$

The solution is now

$$c_\zeta(t) = \frac{g(\zeta)}{1 + \alpha\,g(\zeta)\,t} \tag{11.109}$$

$$x_\zeta(t) = \zeta + \frac{1}{\alpha}\,\ln\!\big(1 + \alpha\,g(\zeta)\,t\big)\ . \tag{11.110}$$

11.8.2 Moving sources

Consider a source moving with velocity u. We then have

$$c_t + c\,c_x = \sigma(x - ut)\ , \tag{11.111}$$

where u is a constant. We seek a moving wave solution $c = c(\xi) = c(x - ut)$. This leads immediately to the ODE

$$(c - u)\, c_\xi = \sigma(\xi) \ . \tag{11.112}$$

This may be integrated to yield

$$\tfrac{1}{2}(u - c_\infty)^2 - \tfrac{1}{2}(u - c)^2 = \int\limits_\xi^\infty d\xi'\, \sigma(\xi') \ . \tag{11.113}$$

Consider the supersonic case where $u > c$. Then we have a smooth solution,

$$c(\xi) = u - \left[(u - c_\infty)^2 - 2 \int\limits_\xi^\infty d\xi'\, \sigma(\xi') \right]^{1/2} \ , \tag{11.114}$$

provided that the term inside the large rectangular brackets is positive. This is always the case for $\sigma < 0$. For $\sigma > 0$ we must require

$$u - c_\infty > \sqrt{2 \int\limits_\xi^\infty d\xi'\, \sigma(\xi')} \tag{11.115}$$

for all ξ. If $\sigma(\xi)$ is monotonic, the lower limit on the above integral may be extended to $-\infty$. Thus, if the source strength is sufficiently small, no shocks are generated. When the above equation is satisfied as an equality, a shock develops, and transients from the initial conditions overtake the wave. A complete solution of the problem then requires a detailed analysis of the transients. What is surprising here is that a supersonic source need not produce a shock wave, if the source itself is sufficiently weak.

11.9 Burgers' Equation

The simplest equation describing both nonlinear wave propagation and diffusion equation is the one-dimensional *Burgers' equation*,

$$c_t + c\, c_x = \nu\, c_{xx} \ . \tag{11.116}$$

As we've seen, this follows from the continuity equation $\rho_t + j_x$ when $j = J(\rho) - \nu \rho_x$, with $c = J'(\rho)$ and $c''(\rho) = 0$.

We have already obtained, in §11.3.1, a solution to Burgers' equation in the form of a propagating front. However, we can do much better than this; we can find *all* the solutions to the one-dimensional Burgers' equation. The trick is to employ a nonlinear transformation of the field $c(x,t)$, known as the *Cole-Hopf transformation*, which linearizes the PDE. Once again, we follow the exceptionally clear discussion in the book by Whitham (ch. 4).

The Cole-Hopf transformation is defined as follows:

$$c \equiv -2\nu \, \frac{\varphi_x}{\varphi} = \frac{\partial}{\partial x} \big(-2\nu \ln \varphi \big) \ . \tag{11.117}$$

Plugging into Burgers' equation, one finds that $\varphi(x,t)$ satisfies the *linear* diffusion equation,

$$\varphi_t = \nu \, \varphi_{xx} \ . \tag{11.118}$$

Isn't that just about the coolest thing you've ever heard?

Suppose the initial conditions on $\varphi(x,t)$ are

$$\varphi(x,0) = \Phi(x) \ . \tag{11.119}$$

We can then solve the diffusion equation 11.118 by Laplace transform. The result is

$$\varphi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int\limits_{-\infty}^{\infty} \! dx' \, e^{-(x-x')^2/4\nu t} \, \Phi(x') \ . \tag{11.120}$$

Thus, if $c(x,t=0) = g(x)$, then the solution for subsequent times is

$$c(x,t) = \frac{\int\limits_{-\infty}^{\infty} dx' \, (x - x') \, e^{-H(x,x',t)/2\nu}}{t \int\limits_{-\infty}^{\infty} dx' \, e^{-H(x,x',t)/2\nu}} \ , \tag{11.121}$$

where

$$H(x,x',t) = \int\limits_0^{x'} \! dx'' \, g(x'') + \frac{(x-x')^2}{2t} \ . \tag{11.122}$$

11.9.1 The limit $\nu \to 0$

In the limit $\nu \to 0$, the integrals in the numerator and denominator of eqn. 11.121 may be computed via the method of steepest descents. This means that extremize $H(x,x',t)$ with respect to x', which entails solving

$$\frac{\partial H}{\partial x'} = g(x') - \frac{x - x'}{t} \ . \tag{11.123}$$

Let $\zeta = \zeta(x,t)$ be a solution to this equation for x', so that

$$x = \zeta + g(\zeta) \, t \ . \tag{11.124}$$

We now expand about $x' = \zeta$, writing $x' = \zeta + s$, in which case

$$H(x') = H(\zeta) + \tfrac{1}{2} H''(\zeta) \, s^2 + \mathcal{O}(s^3) \ , \tag{11.125}$$

where the x and t dependence is here implicit. If $F(x')$ is an arbitrary function which is slowly varying on distance scales on the order of $\nu^{1/2}$, then we have

$$\int\limits_{-\infty}^{\infty} dx'\, F(x')\, e^{-H(x')/2\nu} \approx \sqrt{\frac{4\pi\nu}{H''(\zeta)}}\; e^{-H(\zeta)/2\nu}\, F(\zeta) \ . \tag{11.126}$$

Applying this result to eqn. 11.121, we find

$$c \approx \frac{x-\zeta}{t} \ , \tag{11.127}$$

which is to say

$$c = g(\zeta) \tag{11.128}$$

$$x = \zeta + g(\zeta)\, t \ . \tag{11.129}$$

This is precisely what we found for the characteristics of $c_t + c\, c_x = 0$.

What about multivaluedness? This is obviated by the presence of an additional saddle point solution. *I.e.* beyond some critical time, we have a discontinuous change of saddles as a function of x:

$$x = \zeta_\pm + g(\zeta_\pm)\, t \quad \longrightarrow \quad \zeta_\pm = \zeta_\pm(x,t) \ . \tag{11.130}$$

Then

$$c \sim \frac{1}{t} \cdot \frac{\frac{x-\zeta_-}{\sqrt{H''(\zeta_-)}}\, e^{-H(\zeta_-)/2\nu} + \frac{x-\zeta_+}{\sqrt{H''(\zeta_+)}}\, e^{-H(\zeta_+)/2\nu}}{\frac{1}{\sqrt{H''(\zeta_-)}}\, e^{-H(\zeta_-)/2\nu} + \frac{1}{\sqrt{H''(\zeta_+)}}\, e^{-H(\zeta_+)/2\nu}} \ . \tag{11.131}$$

Thus,

$$H(\zeta_+) > H(\zeta_-) \qquad \Rightarrow \qquad c \approx \frac{x-\zeta_-}{t} \tag{11.132}$$

$$H(\zeta_+) < H(\zeta_-) \qquad \Rightarrow \qquad c \approx \frac{x-\zeta_+}{t} \ . \tag{11.133}$$

At the shock, these solutions are degenerate:

$$H(\zeta_+) = H(\zeta_-) \qquad \Rightarrow \qquad \tfrac{1}{2}(\zeta_+ - \zeta_-)\big(g(\zeta_+) + g(\zeta_-)\big) = \int\limits_{\zeta_-}^{\zeta_+} d\zeta\, g(\zeta) \ , \tag{11.134}$$

which is again exactly as before. We stress that for ν small but finite the shock fronts are smoothed out on a distance scale proportional to ν.

What does it mean for ν to be small? The dimensions of ν are $[\nu] = L^2/T$, so we must find some other quantity in the problem with these dimensions. The desired quantity is the area,

$$A = \int\limits_{-\infty}^{\infty} dx\, \big[g(x) - c_0\big] \ , \tag{11.135}$$

where $c_0 = c(x = \pm\infty)$. We can now define the dimensionless ratio,

$$\mathsf{R} \equiv \frac{A}{2\nu} \ , \tag{11.136}$$

which is analogous to the Reynolds number in viscous fluid flow. R is proportional to the ratio of the nonlinear term $(c - c_0)\, c_x$ to the diffusion term νc_{xx}.

11.9.2 Examples

Whitham discusses three examples: diffusion of an initial step, a hump, and an N-wave. Here we simply reproduce the functional forms of these solutions. For details, see chapter 4 of Whitham's book.

For an initial step configuration,

$$c(x, t = 0) = \begin{cases} c_1 & \text{if } x < 0 \\ c_2 & \text{if } x > 0 \ . \end{cases} \tag{11.137}$$

We are interested in the case $c_1 > c_2$. Using the Cole-Hopf transformation and applying the appropriate initial conditions to the resulting linear diffusion equation, one obtains the complete solution,

$$c(x,t) = c_2 + \frac{c_1 - c_2}{1 + h(x,t)\, \exp\big[(c_1 - c_2)(x - v_{\mathrm{s}} t)/2\nu\big]} \ , \tag{11.138}$$

where

$$v_{\mathrm{s}} = \tfrac{1}{2}(c_1 + c_2) \tag{11.139}$$

and

$$h(x,t) = \frac{\mathrm{erfc}\Big(-\frac{x - c_2 t}{\sqrt{4\nu t}}\Big)}{\mathrm{erfc}\Big(+\frac{x - c_1 t}{\sqrt{4\nu t}}\Big)} \ . \tag{11.140}$$

Recall that $\mathrm{erfc}(z)$ is the complimentary error function:

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int\limits_0^z du\, e^{-u^2} \tag{11.141}$$

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int\limits_z^\infty du\, e^{-u^2} = 1 - \mathrm{erf}(z) \ . \tag{11.142}$$

Note the limiting values $\mathrm{erfc}(-\infty) = 2$, $\mathrm{erfc}(0) = 1$ and $\mathrm{erfc}(\infty) = 0$. If $c_2 < x/t < c_1$, then $h(x,t) \to 1$ as $t \to \infty$, in which case the solution resembles a propagating front. It is convenient to adimensionalize $(x,t) \to (y,\tau)$ by writing

$$x = \frac{\nu\, y}{\sqrt{c_1 c_2}} \quad , \quad t = \frac{\nu\, \tau}{c_1 c_2} \quad , \quad r \equiv \sqrt{\frac{c_1}{c_2}} \ . \tag{11.143}$$

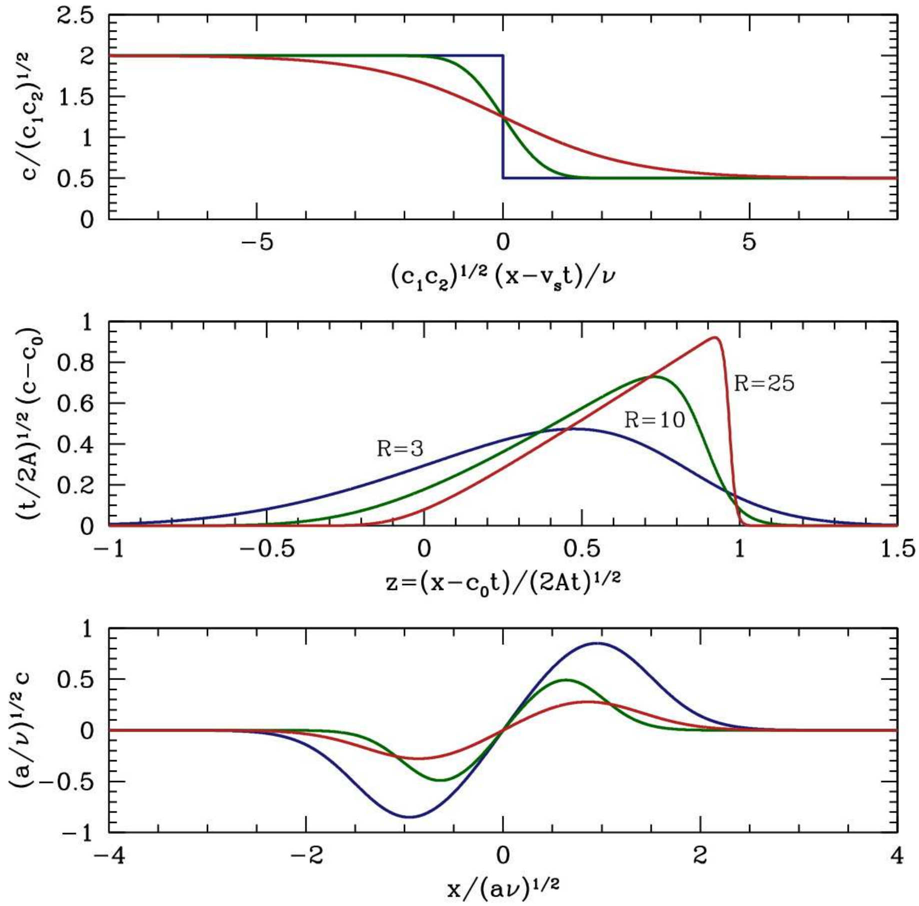

Figure 11.12: Evolution of profiles for Burgers' equation. Top : a step discontinuity evolving into a front at times $\tau = 0$ (blue), $\tau = \frac{1}{5}$ (green), and $\tau = 5$ (red).. Middle : a narrow hump $c_0 + A\delta(x)$ evolves into a triangular wave. Bottom : dissipation of an N-wave at times $\tau = \frac{1}{4}$ (blue), $\tau = \frac{1}{2}$ (green), and $\tau = 1$ (red).

We then have

$$\frac{c(z,\tau)}{\sqrt{c_1 c_2}} = r^{-1} + \frac{2\alpha}{1 + h(z,\tau)\, \exp(\alpha z)} \ , \tag{11.144}$$

where

$$h(z,\tau) = \text{erfc}\bigg(-\,\frac{z+\alpha\tau}{2\sqrt{\tau}}\bigg)\bigg/\text{erfc}\bigg(+\,\frac{z-\alpha\tau}{2\sqrt{\tau}}\bigg) \tag{11.145}$$

and

$$\alpha \equiv \tfrac{1}{2}\big(r - r^{-1}\big) \qquad , \qquad z \equiv y - \tfrac{1}{2}\big(r + r^{-1}\big)\,\tau \ . \tag{11.146}$$

The second example involves the evolution of an infinitely thin hump, where

$$c(x, t=0) = c_0 + A\,\delta(x) \ . \tag{11.147}$$

The solution for subsequent times is

$$c(x,t) = c_0 + \sqrt{\frac{\nu}{\pi t}} \cdot \frac{(e^{\mathsf{R}} - 1)\,\exp\!\left(-\frac{x-c_0 t}{4\nu t}\right)}{1 + \frac{1}{2}(e^{\mathsf{R}} - 1)\,\mathrm{erfc}\!\left(\frac{x-c_0 t}{\sqrt{4\nu t}}\right)} \ , \tag{11.148}$$

where $\mathsf{R} = A/2\nu$. Defining

$$z \equiv \frac{x - c_0 t}{\sqrt{2At}} \ , \tag{11.149}$$

we have the solution

$$c = c_0 + \left(\frac{2A}{t}\right)^{1/2} \frac{1}{\sqrt{4\pi\mathsf{R}}} \cdot \frac{(e^{\mathsf{R}} - 1)\, e^{-\mathsf{R}z^2}}{1 + \frac{1}{2}(e^{\mathsf{R}} - 1)\,\mathrm{erfc}(\sqrt{\mathsf{R}}\, z)} \ . \tag{11.150}$$

Asymptotically, for $t \to \infty$ with x/t fixed, we have

$$c(x,t) = \begin{cases} x/t & \text{if } 0 < x < \sqrt{2At} \\ 0 & \text{otherwise .} \end{cases} \tag{11.151}$$

This recapitulates the triangular wave solution with the two counterpropagating shock fronts and dissipating shock strengths.

Finally, there is the N-wave. If we take the following solution to the linear diffusion equation,

$$\varphi(x,t) = 1 + \sqrt{\frac{a}{t}}\, e^{-x^2/4\nu t} \ , \tag{11.152}$$

then we obtain

$$c(x,t) = \frac{x}{t} \cdot \frac{e^{-x^2/4\nu t}}{\sqrt{\frac{t}{a}} + e^{-x^2/4\nu t}} \ . \tag{11.153}$$

In terms of dimensionless variables (y, τ), where

$$x = \sqrt{a\nu}\; y \qquad , \qquad t = a\tau \ , \tag{11.154}$$

we have

$$c = \sqrt{\frac{\nu}{a}}\, \frac{y}{\tau} \cdot \frac{e^{-y^2/4\tau}}{\sqrt{\tau} + e^{-y^2/4\tau}} \ . \tag{11.155}$$

The evolving profiles for these three cases are plotted in fig. 11.12.

11.9.3 Confluence of shocks

The fact that the diffusion equation 11.118 is linear means that we can superpose solutions:

$$\varphi(x,t) = \varphi_1(x,t) + \varphi_2(x,t) + \ldots + \varphi_N(x,t) \ , \tag{11.156}$$

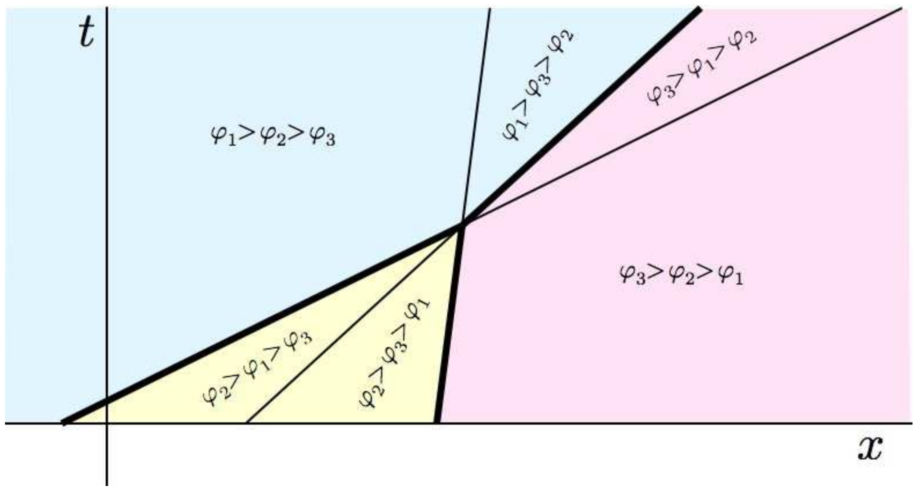

Figure 11.13: Merging of two shocks for piecewise constant initial data. The (x,t) plane is broken up into regions labeled by the local value of $c(x,t)$. For the shocks to form, we require $c_1 > c_2 > c_3$. When the function $\varphi_j(x,t)$ dominates over the others, then $c(x,t) \approx c_j$.

where

$$\varphi_j(x,t) = e^{-c_j(x-b_j)/2\nu}\, e^{+c_j^2 t/4\nu} \ . \tag{11.157}$$

We then have

$$c(x,t) = -\frac{2\nu\varphi_x}{\varphi} = \frac{\sum_i c_i\, \varphi_i(x,t)}{\sum_i \varphi_i(x,t)} \ . \tag{11.158}$$

Consider the case $N = 2$, which describes a single shock. If $c_1 > c_2$, then at a fixed time t we have that φ_1 dominates as $x \to -\infty$ and φ_2 as $x \to +\infty$. Therefore $c(-\infty,t) = c_1$ and $c(+\infty) = c_2$. The shock center is defined by $\varphi_1 = \varphi_2$, where $x = \frac{1}{2}(c_1 + c_2)\, t$.

Next consider $N = 3$, where there are two shocks. We assume $c_1 > c_2 > c_3$. We identify regions in the (x,t) plane where φ_1, φ_2, and φ_3 are dominant. One finds

$$\varphi_1 > \varphi_2 \quad : \quad x < \tfrac{1}{2}(c_1 + c_2)\, t + \frac{b_1 c_1 - b_2 c_2}{c_1 - c_2} \tag{11.159}$$

$$\varphi_1 > \varphi_3 \quad : \quad x < \tfrac{1}{2}(c_1 + c_3)\, t + \frac{b_1 c_1 - b_3 c_3}{c_1 - c_3} \tag{11.160}$$

$$\varphi_2 > \varphi_3 \quad : \quad x < \tfrac{1}{2}(c_2 + c_3)\, t + \frac{b_2 c_2 - b_3 c_3}{c_2 - c_3} \ . \tag{11.161}$$

These curves all meet in a single point at $(x_{\rm m}, t_{\rm m})$, as shown in fig. 11.13. The shocks are the locus of points along which two of the φ_j are equally dominant. We assume that the intercepts of these lines with the x-axis are ordered as in the figure, with $x^*_{12} < x^*_{13} < x^*_{23}$, where

$$x^*_{ij} \equiv \frac{b_i c_i - b_j c_j}{c_i - c_j} \ . \tag{11.162}$$

When a given $\varphi_i(x,t)$ dominates over the others, we have from eqn. 11.158 that $c \approx c_i$. We see that for $t < t^*$ one has that φ_1 is dominant for $x < x^*_{12}$, and φ_3 is dominant for $x > x^*_{23}$, while φ_2 dominates in the intermediate regime $x^*_{12} < x < x^*_{23}$. The boundaries between these different regions are the two propagating shocks. After the merge, for $t > t_{\rm m}$, however, φ_2 *never dominates*, and hence there is only one shock.

11.10 Appendix I : The Method of Characteristics

Consider the quasilinear PDE

$$a_1(\boldsymbol{x},\phi)\,\frac{\partial\phi}{\partial x_1} + a_2(\boldsymbol{x},\phi)\,\frac{\partial\phi}{\partial x_2} + \ldots + a_N(\boldsymbol{x},\phi)\,\frac{\partial\phi}{\partial x_N} = b(\boldsymbol{x},\phi) \ . \tag{11.163}$$

This PDE is called 'quasilinear' because it is linear in the derivatives $\partial\phi/\partial x_j$. The N independent variables are the elements of the vector $\boldsymbol{x} = (x_1,\ldots,x_N)$. A solution is a function $\phi(\boldsymbol{x})$ which satisfies the PDE.

Now consider a curve $\boldsymbol{x}(s)$ parameterized by a single real variable s satisfying

$$\frac{dx_j}{ds} = a_j\big(\boldsymbol{x},\phi(\boldsymbol{x})\big) \ , \tag{11.164}$$

where $\phi(\boldsymbol{x})$ is a solution of eqn. 11.163. Along such a curve, which is called a *characteristic*, the variation of ϕ is

$$\frac{d\phi}{ds} = \sum_{j=1}^N \frac{\partial\phi}{\partial x_j}\,\frac{dz_j}{ds} = b\big(\boldsymbol{x}(s),\phi\big) \ . \tag{11.165}$$

Thus, we have converted our PDE into a set of $(N+1)$ ODEs. To integrate, we must supply some initial conditions of the form

$$g\big(\boldsymbol{x},\phi\big)\Big|_{s=0} = 0 \ . \tag{11.166}$$

This defines an $(N-1)$-dimensional hypersurface, parameterized by $\{\zeta_1,\ldots,\zeta_{N-1}\}$:

$$x_j(s=0) = h_j(\zeta_1,\ldots,\zeta_{N-1}) \qquad , \qquad j=1,\ldots,N \tag{11.167}$$

$$\phi(s=0) = f(\zeta_1,\ldots,\zeta_{N-1}) \ . \tag{11.168}$$

If we can solve for all the characteristic curves, then the solution of the PDE follows. For every $\boldsymbol{x}$, we identify the characteristic curve upon which $\boldsymbol{x}$ lies. The characteristics are identified by their parameters $(\zeta_1,\ldots,\zeta_{N-1})$. The value of $\phi(\boldsymbol{x})$ is then $\phi(\boldsymbol{x}) = f(\zeta_1,\ldots,\zeta_{N-1})$. If two or more characteristics cross, the solution is multi-valued, or a shock has occurred.

11.10.1 Example

Consider the PDE

$$\phi_t + t^2\,\phi_x = -x\,\phi \ . \tag{11.169}$$

We identify $a_1(t,x,\phi)=1$ and $a_2(t,x,\phi)=t^2$, as well as $b(t,x,\phi)=-x\,\phi$. The characteristics are curves $\big(t(s),x(s)\big)$ satisfing

$$\frac{dt}{ds}=1 \qquad , \qquad \frac{dx}{ds}=t^2 \ . \tag{11.170}$$

The variation of ϕ along each characteristics is given by

$$\frac{d\phi}{ds}=-x\,\phi \ . \tag{11.171}$$

The initial data are expressed parametrically as

$$t(s=0)=0 \tag{11.172}$$
$$x(s=0)=\zeta \tag{11.173}$$
$$\phi(s=0)=f(\zeta) \ . \tag{11.174}$$

We now solve for the characteristics. We have

$$\frac{dt}{ds}=1 \quad \Rightarrow \quad t(s,\zeta)=s \ . \tag{11.175}$$

It then follows that

$$\frac{dx}{ds}=t^2=s^2 \quad \Rightarrow \quad x(s,\zeta)=\zeta+\tfrac{1}{3}s^3 \ . \tag{11.176}$$

Finally, we have

$$\frac{d\phi}{ds}=-x\,\phi=-\big(\zeta+\tfrac{1}{3}s^3\big)\,\phi \quad \Rightarrow \quad \phi(s,\zeta)=f(\zeta)\,\exp\big(-\tfrac{1}{12}s^4-s\zeta\big) \ . \tag{11.177}$$

We may now eliminate (ζ,s) in favor of (x,t), writing $s=t$ and $\zeta=x-\frac{1}{3}t^3$, yielding the solution

$$\phi(x,t)=f\big(x-\tfrac{1}{3}t^3\big)\,\exp\big(\tfrac{1}{4}t^4-xt\big) \ . \tag{11.178}$$

11.11 Appendix II : Shock Fitting an Inverted Parabola

Consider the shock fitting problem for the initial condition

$$c(x,t=0)=c_0\bigg(1-\frac{x^2}{a^2}\bigg)\,\Theta(a^2-x^2) \ , \tag{11.179}$$

which is to say a truncated inverted parabola. We assume $j'''(\rho)=0$. Clearly $-c_x(x,0)$ is maximized at $x=a$, where $-c_x(a,0)=2c_0/a$, hence breaking first occurs at

$$\big(x_{\rm B}\,,\,t_{\rm B}\big)=\bigg(a\,,\,\frac{a}{2c_0}\bigg) \ . \tag{11.180}$$

Clearly $\zeta_+ > a$, hence $c_+ = 0$. Shock fitting then requires

$$\tfrac{1}{2}(\zeta_+ - \zeta_-)(c_+ + c_-) = \int\limits_{\zeta_-}^{\zeta_+} d\zeta \, c(\zeta) \tag{11.181}$$

$$= \frac{c_0}{3a^2}\,(2a + \zeta_-)(a - \zeta_-)^2 \ . \tag{11.182}$$

Since

$$c_+ + c_- = \frac{c_0}{a^2}\,(a^2 - \zeta_-^2) \ , \tag{11.183}$$

we have

$$\zeta_+ - \zeta_- = \tfrac{2}{3}\,(2a + \zeta_-)\bigg(\frac{a - \zeta_-}{a + \zeta_-}\bigg) \ . \tag{11.184}$$

The second shock-fitting equation is

$$\zeta_+ - \zeta_- = (c_- - c_+)\, t \ . \tag{11.185}$$

Eliminating ζ_+ from the two shock-fitting equations, we have

$$t = \frac{2a^2}{3c_0} \cdot \frac{2a + \zeta_-}{(a + \zeta_-)^2} \ . \tag{11.186}$$

Inverting to find $\zeta_-(t)$, we obtain

$$\frac{\zeta_-(t)}{a} = \frac{a}{3c_0 t} - 1 + \frac{a}{3c_0 t}\,\sqrt{1 + \frac{6c_0 t}{a}} \ . \tag{11.187}$$

The shock position is then $x_{\rm s}(t) = \zeta_-(t) + c_-\big(\zeta_-(t)\big)\, t$.

It is convenient to rescale lengths by a and times by $t_{\rm B} = a/2c_0$, defining q and τ from $x \equiv aq$ and $t \equiv a\tau/2c_0$. Then

$$q_-(\tau) = \frac{\zeta_-}{a} = \frac{2}{3\tau} - 1 + \frac{2}{3\tau}\,\sqrt{1 + 3\tau} \ . \tag{11.188}$$

and

$$q_{\rm s}(\tau) = \frac{x_{\rm s}}{a} = -1 + \frac{2}{9\tau}\,\Big[(1 + 3\tau)^{3/2} + 1\Big] \ . \tag{11.189}$$

The shock velocity is

$$\dot q_{\rm s} = -\frac{2}{9\tau^2}\,\Big[1 + (1 + 3\tau)^{3/2}\Big] + \frac{1}{\tau}\,(1 + 3\tau)^{1/2} \tag{11.190}$$

$$= \tfrac{3}{4}\,(\tau - 1) + \tfrac{81}{64}\,(\tau - 1)^2 + \ldots \ ,$$

with $v_{\rm s} = 2c_0\,\dot q_{\rm s} = \frac{1}{2}c_-$ if we restore units. Note that $\dot q_{\rm s}(\tau = 1) = 0$, so the shock curve initially rises vertically in the (x, t) plane. Interestingly, $v_{\rm s} \propto (\tau - 1)$ here, while for the example in §11.4.3, where $c(x, 0)$ had a similar profile, we found $v_{\rm s} \propto (\tau - 1)^{1/2}$ in eqn. 11.69.