LINEAR AND NONLINEAR WAVES

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CHAPTER 2

Waves and First Order Equations

We start the detailed discussion of hyperbolic waves with a study of first order equations. As noted in Chapter 1, the simplest wave equation is

$$\rho_t + c_0 \rho_x = 0, \qquad c_0 = \text{constant}. \tag{2.1}$$

When this equation arises, the dependent variable is usually the density of something so we now use the symbol ρ rather than the all-purpose symbol φ of the introduction. The general solution of (2.1) is $\rho = f(x - c_0 t)$, where $f(x)$ is an arbitrary function, and the solution of any particular problem consists merely of matching the function f to initial or boundary values. It clearly describes a wave motion since an initial profile $f(x)$ would be translated unchanged in shape a distance $c_0 t$ to the right at time t. At two observation points a distance s apart, exactly the same disturbance would be recorded with a time delay of s/c_0.

Although this linear case is almost trivial, the nonlinear counterpart

$$\rho_t + c(\rho)\rho_x = 0, \tag{2.2}$$

where $c(\rho)$ is a given function of ρ, is certainly not and a study of it leads to most of the essential ideas for nonlinear hyperbolic waves. As remarked earlier, many of the classical examples of wave propagation are described by second or higher order equations such as the wave equation $c_0^2 \nabla^2 \varphi = \varphi_{tt}$, but a surprising number of physical problems do lead directly to (2.2) or extensions of it. Examples will be given after a preliminary discussion of the solution. Even in higher order problems, one often searches for special solutions or approximations that involve (2.2).

2.1 Continuous Solutions

One approach to the solution of (2.2) is to consider the function $\rho(x,t)$ at each point of the (x,t) plane and to note that $\rho_t + c(\rho)\rho_x$ is the total derivative of ρ along a curve which has slope

$$\frac{dx}{dt} = c(\rho) \tag{2.3}$$

at every point of it. For along any curve in the (x,t) plane, we may consider x and ρ to be functions of t, and the total derivative of ρ is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt}\frac{\partial \rho}{\partial x}.$$

The total derivative notation should be sufficient to indicate when x and ρ are being treated as functions of t on a certain curve; the introduction of new symbols each time this is done eventually becomes confusing. We now consider a curve $\mathcal{C}$ in the (x,t) plane which satisfies (2.3). Of course such a curve cannot be determined explicitly in advance since the defining equation (2.3) involves the unknown values of ρ on the curve. However, its consideration will lead us to a simultaneous determination of a possible curve $\mathcal{C}$ and the solution ρ on it. On $\mathcal{C}$ we deduce from the total derivative relation and from (2.2) that

$$\frac{d\rho}{dt} = 0, \qquad \frac{dx}{dt} = c(\rho). \tag{2.4}$$

We first observe that ρ remains constant on $\mathcal{C}$. It then follows that $c(\rho)$ remains constant on $\mathcal{C}$, and therefore that the curve $\mathcal{C}$ must be a straight line in the (x,t) plane with slope $c(\rho)$. Thus the general solution of (2.2) depends on the construction of a family of straight lines in the (x,t) plane, each line with slope $c(\rho)$ corresponding to the value of ρ on it. This is easily done in any specific problem.

Let us take for example the initial value problem

$$\rho = f(x), \quad t = 0, \quad -\infty < x < \infty,$$

and refer to the (x,t) diagram in Fig. 2.1. If one of the curves $\mathcal{C}$ intersects $t=0$ at $x=\xi$ then $\rho = f(\xi)$ on the whole of that curve. The corresponding slope of the curve is $c(f(\xi))$, which we will denote by $F(\xi)$; it is a known function of ξ calculated from the function $c(\rho)$ in the equation and the given initial function $f(\xi)$. The equation of the curve then is

$$x = \xi + F(\xi)t.$$

This determines one typical curve and the value of ρ on it is $f(\xi)$. Allowing ξ to vary, we obtain the whole family:

$$\rho = f(\xi), \qquad c = F(\xi) = c(f(\xi)) \tag{2.5}$$

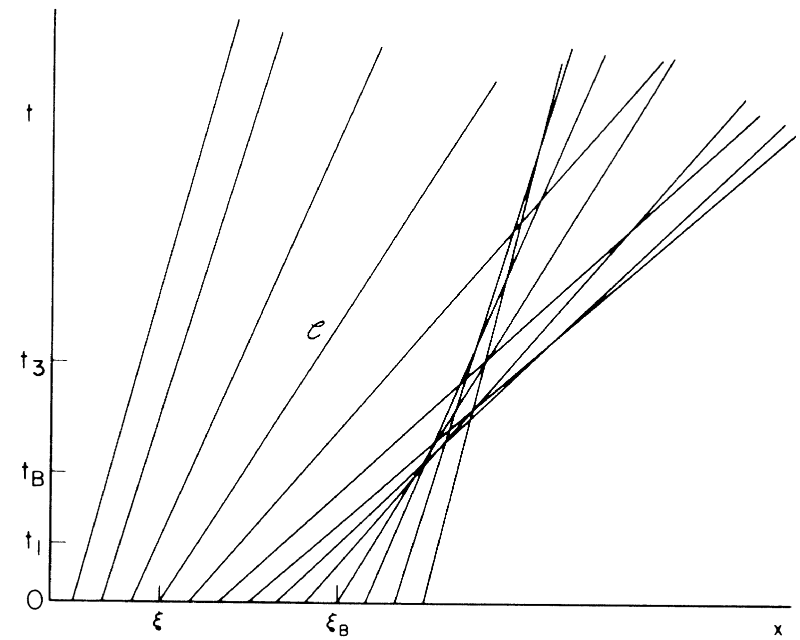

Fig. 2.1. Characteristic diagram for nonlinear waves.

on

$$x = \xi + F(\xi)t. \tag{2.6}$$

We may now change the emphasis and use (2.5) and (2.6) as an analytic expression for the solution, free of the particular construction. That is, ρ is given by (2.5) where $\xi(x,t)$ is defined implicitly by (2.6). Let us check that this gives the solution. From (2.5),

$$\rho_t = f'(\xi)\xi_t, \qquad \rho_x = f'(\xi)\xi_x,$$

and from the t and x derivatives of (2.6),

$$0 = F(\xi) + \{1 + F'(\xi)t\}\xi_t,$$

$$1 = \{1 + F'(\xi)t\}\xi_x.$$

Therefore

$$\rho_t = -\frac{F(\xi)f'(\xi)}{1 + F'(\xi)t}, \qquad \rho_x = \frac{f'(\xi)}{1 + F'(\xi)t}, \tag{2.7}$$

and we see that

$$\rho_t + c(\rho)\rho_x = 0$$

since $c(\rho)=F(\xi)$. The initial condition $\rho=f(x)$ is satisfied because $\xi=x$ when $t=0$.

The curves used in the construction of the solution are the *characteristic curves* for this special problem. Similar characteristics play an important role in all problems involving hyperbolic differential equations. In general, characteristic curves do not have the property that the solution remains constant along them. This happens to be true in the special case of (2.2); it is not the defining property of characteristics. The general definitions will be considered later, but it will be convenient now to refer to the curves defined by (2.3) as characteristics.

The basic idea of wave propagation is that some recognizable feature of the disturbance moves with a finite velocity. For hyperbolic equations, the characteristics correspond to this idea. Each characteristic curve in (x,t) space represents a moving wavelet in x space, and the behavior of the solution on a characteristic curve corresponds to the idea that information is carried by that wavelet. The mathematical statement in (2.4) may be given this type of emphasis by saying that different values of ρ "propagate" with velocity $c(\rho)$. Indeed, the solution at time t can be constructed by moving each point on the initial curve $\rho=f(x)$ a distance $c(\rho)t$ to the right; the distance moved is different for the different values of ρ. This is shown in Fig. 2.2 for the case $c'(\rho)>0$; the corresponding time levels are indicated in Fig. 2.1. The dependence of c on ρ produces the typical nonlinear distortion of the wave as it propagates. When $c'(\rho)>0$, higher values of ρ propagate faster than lower ones. When $c'(\rho)<0$, higher values of ρ propagate slower and the distortion has the opposite tendency to that shown in Fig. 2.2. For the linear case, c is constant and the profile is translated through a distance ct without any change of shape.

It is immediately apparent from Fig. 2.2 that the discussion is far from complete. Any compressive part of the wave, where the propagation velocity is a decreasing function of x, ultimately "breaks" to give a

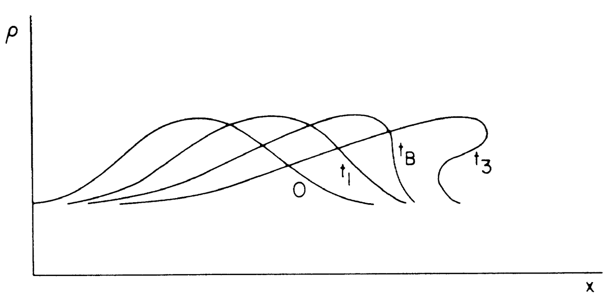

Fig. 2.2. Breaking wave: successive profiles corresponding to the times 0, t_1, t_B, t_3 in Fig. 2.1.

triple-valued solution for $\rho(x,t)$. The breaking starts at the time indicated by $t=t_B$ in Fig. 2.2, when the profile of ρ first develops an infinite slope. The analytic solution (2.7) confirms this and allows us to determine the breaking time t_B. On any characteristic for which $F'(\xi)<0$, ρ_x and ρ_t become infinite when

$$t=-\frac{1}{F'(\xi)}.$$

Therefore breaking first occurs on the characteristic $\xi=\xi_B$ for which $F'(\xi)<0$ and $|F'(\xi)|$ is a maximum; the time of first breaking is

$$t_B=-\frac{1}{F'(\xi_B)}. \tag{2.8}$$

This development can also be followed in the (x,t) plane. A compressive part of the wave with $F'(\xi)<0$ has converging characteristics; since the characteristics are straight lines, they must eventually overlap to give a region where the solution is multivalued, as in Fig. 2.1. This region may be considered as a fold in the (x,t) plane made up of three sheets, with different values of ρ on each sheet. The boundary of the region is an envelope of characteristics. The family of characteristics is given by (2.6) with ξ as parameter. The condition that two neighboring characteristics ξ, $\xi+\delta\xi$ intersect at a point (x,t) is that

$$x=\xi+F(\xi)t$$

and

$$x=\xi+\delta\xi+F(\xi+\delta\xi)t$$

hold simultaneously. In the limit $\delta\xi\to 0$, these give

$$x=\xi+F(\xi)t \qquad \text{and} \qquad 0=1+F'(\xi)t$$

for the implicit equations of an envelope. The second of these relations shows that an envelope is formed in $t>0$ by those characteristics for which $F'(\xi)<0$. The minimum value of t on the envelope occurs for the value of ξ for which $-F'(\xi)$ is maximum. This is the first time of breaking in agreement with (2.8). If $F''(\xi)$ is continuous, the envelope has a cusp at $t=t_B$, $\xi=\xi_B$, as shown in Fig. 2.1.

An extreme case of breaking arises when the initial distribution has a discontinuous step with the value of $c(\rho)$ behind the discontinuity greater than that ahead. If we have the initial functions

$$f(x)=\begin{cases}\rho_1, & x>0\\ \rho_2, & x<0\end{cases}$$

and

$$F(x)=\begin{cases} c_1=c(\rho_1), & x>0\\ c_2=c(\rho_2), & x<0\end{cases}$$

with $c_2>c_1$, then breaking occurs immediately. This is shown in Fig. 2.3 for the case $c'(\rho)>0$, $\rho_2>\rho_1$. The multivalued region starts right at the origin and is bounded by the characteristics $x=c_1t$ and $x=c_2t$; the boundary is no longer a cusped envelope since F and its derivatives are not continuous. Nevertheless, the result may be considered as the limit of a series of smoothed-out steps, and the breaking point moves closer to the origin as the initial profile approaches the discontinuous step.

On the other hand, if the initial step function is expansive with $c_2<c_1$, there is a perfectly good continuous solution. It may be obtained as the limit of (2.5) and (2.6) in which all the values of F between c_2 and c_1 are taken on characteristics through the origin $\xi=0$. This corresponds to a fan of characteristics in the (x,t) plane as in Fig. 2.4. Each member of the fan has a different slope F but the same ξ. The function F is a step function but we use all the values of F between c_2 and c_1 on the face of the step and take them all to correspond to $\xi=0$. In the fan, the solution (2.5), (2.6) then reads

$$c=F, \qquad x=Ft, \qquad \text{for } c_2<F<c_1,$$

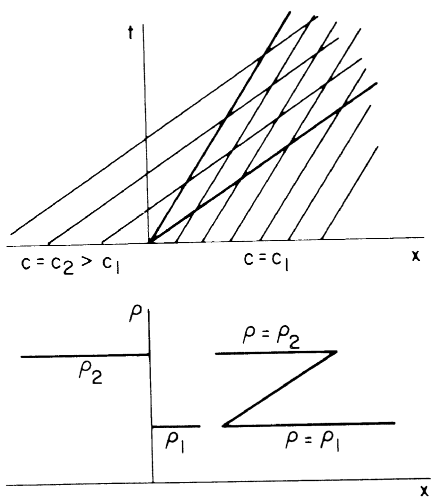

Fig. 2.3. Centered compression wave with overlap.

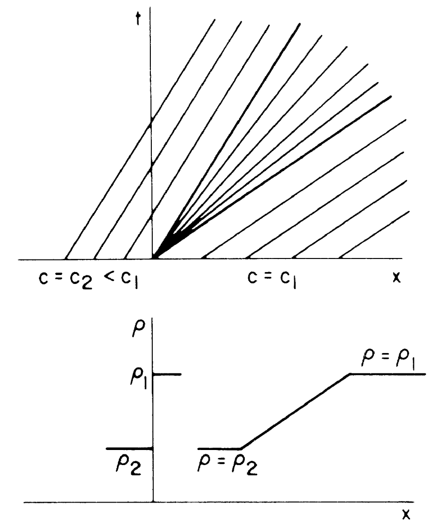

Fig. 2.4. Centered expansion wave.

and by elimination of F we have the simple explicit solution for c:

$$c=\frac{x}{t}, \qquad c_2<\frac{x}{t}<c_1.$$

The complete solution for c is

$$c=\begin{cases} c_1, & c_1<\dfrac{x}{t},\\ \dfrac{x}{t}, & c_2<\dfrac{x}{t}<c_1,\\ c_2, & \dfrac{x}{t}<c_2.\end{cases} \tag{2.9}$$

The relation $c=c(\rho)$ can be solved to determine ρ. For the compressive step, $c_2>c_1$, the fan in the (x,t) plane is reversed to produce the overlap shown in Fig. 2.3.

In most physical problems where this theory arises, $\rho(x,t)$ is just the density of some medium and is inherently single-valued. Therefore when breaking occurs (2.2) must cease to be valid as a description of the physical problem. Even in cases such as water waves where a multivalued solution for the height of the surface could at least be interpreted, it is still found that (2.2) is inadequate to describe the process. Thus the situation is that some assumption or approximate relation in the formulation leading to (2.2) is no longer valid. In principle one must return to the physics of the problem, see what went wrong, and formulate an improved theory. However, it turns out, as we shall see, that the foregoing solution can be saved by allowing discontinuities into the solution; there is then a single-valued solution with a simple jump discontinuity to replace the multivalued continuous solution. This requires some mathematical extension of what we mean by a "solution" to (2.2), since strictly speaking the derivatives of ρ will not exist at a discontinuity. It can be done through the concept of a "weak solution." But it is important to appreciate that the real issue is not just a mathematical question of extending the solution of (2.2). The breakdown of the continuous solution is associated with the breakdown of some approximate relation in the physics, and the two aspects must be considered together. It is found, for example, that there are several possible families of discontinuous solutions, all satisfactory mathematically; the nonuniqueness can be resolved only by appeal to the physics.

Clearly then, we cannot proceed further without discussion of some physical problems. The prototype is the nonlinear theory of waves in a gas and the formation of shock waves. When viscosity and heat conduction are ignored, the equations of gas dynamics have breaking solutions similar to the preceding ones. As the gradients become steep, just before breaking, the effects of viscosity and heat conduction are no longer negligible. These effects can be included to give an improved theory and waves no longer break in that theory. There is a thin region, a shock wave, in which viscosity and heat conduction are crucially important; outside the shock wave, viscosity and heat conduction may still be neglected. The flow variables change rapidly in the shock. This shock region is idealized into a discontinuity in the "extended" inviscid theory, and only shock conditions relating the jumps of the flow variables across the discontinuity need to be added to the inviscid theory.

We will study all these various aspects in detail. However, gas dynamics is not the simplest example, since it involves higher order equations, and we shall discuss the essential ideas first in the context of the simpler first order problems. It should be remembered, though, that these ideas were developed for gas dynamics, and we are reversing the chronological order. The basic ideas were elucidated by Poisson (1807), Stokes (1848), Riemann (1858), Earnshaw (1858), Rankine (1870), Hugoniot (1889), Rayleigh (1910), Taylor (1910)—a most impressive list. The time required indicates that putting the different aspects together was quite a complicated affair.

2.2 Kinematic Waves

In many problems of wave propagation there is a continuous distribution of either material or some state of the medium, and (for a one

dimensional problem) we can define a density $\rho(x,t)$ per unit length and a flux $q(x,t)$ per unit time. We can then define a flow velocity $v(x,t)$ by

$$v=\frac{q}{\rho}.$$

Assuming that the material (or state) is conserved, we can stipulate that the rate of change of the total amount of it in any section $x_1>x>x_2$ must be balanced by the net inflow across x_1 and x_2. That is,

$$\frac{d}{dt}\int_{x_2}^{x_1}\rho(x,t)\,dx+q(x_1,t)-q(x_2,t)=0. \tag{2.10}$$

If $\rho(x,t)$ has continuous derivatives, we may take the limit as $x_1\to x_2$ and obtain the conservation equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial x}=0. \tag{2.11}$$

The simplest wave problems arise when it is reasonable, on either theoretical or empirical grounds, to postulate (in a first approximation!) a functional relation between q and ρ. If this is written as

$$q=Q(\rho), \tag{2.12}$$

(2.11) and (2.12) form a complete system. On substitution we have

$$\rho_t+c(\rho)\rho_x=0 \tag{2.13}$$

where

$$c(\rho)=Q'(\rho). \tag{2.14}$$

This leads to our (2.2) and a typical solution is given by (2.5) to (2.6). The breaking requires us to reconsider both the mathematical assumption that ρ and q have derivatives and the physical assumption that $q=Q(\rho)$ is a good approximation. To fix ideas for the further development of the theory some specific examples are noted briefly here. We shall return to them in Chapter 3 for a more detailed discussion after the theoretical ideas are complete.

An amusing case (which is also important) concerns traffic flow. It is reasonable to suppose that some essential features of fairly heavy traffic flow may be obtained by treating a stream of traffic as a continuum with an observable density $\rho(x,t)$, equal to the number of cars per unit length, and a flow $q(x,t)$, equal to the number of cars crossing the position x per unit time. For a stretch of highway with no entries or exits, cars are conserved! So we stipulate (2.10). For traffic it also seems reasonable to argue that the traffic flow q is determined primarily by the local density ρ and to propose (2.12) as a first approximation. Such functional relations have been studied and documented to some extent by traffic engineers. We can then apply the theory. But it is clear in this case that when breaking occurs there is no lack of possible explanations for some breakdown in the formulation. Certainly the assumption $q=Q(\rho)$ is a very simplified view of a very complicated phenomenon. For example, if the density is changing rapidly (as it is near breaking), one expects the drivers to react to more than the local density and one also expects that there will be a time lag before they respond adequately to the changing conditions. One might also question the continuum assumption itself.

Another example is flood waves in long rivers. Here ρ is replaced by the cross-sectional area of the channel, A, and this varies with x and t as the level of the river rises. If q is the volume flux across the section, then (2.10) between A and q expresses the conservation of water. Although the fluid flow is extremely complicated, it seems reasonable to start with a functional relation $q=Q(A)$ as a first approximation to express the increase in flow as the level rises. Such relations have been plotted from empirical observations on various rivers. But it is again clear that this assumption is an oversimplification which may well have to be corrected if troubles arise in the theory.

A similar example, proposed and studied extensively by Nye (1960), is the example of glacier flow. The flow velocity is expected to increase with the thickness of the ice, and it seems reasonable to assume a functional dependence between the two.

In chromatography and in similar exchange processes studied in problems of chemical engineering, the same theory arises. The formulation is a little more complicated. The situation is that a fluid carrying dissolved substances or particles or ions flows through a fixed bed and the material being carried is partially adsorbed on the fixed solid material in the bed. The fluid flow is idealized to have a constant velocity V. Then if ρ_f is the density of the material carried in the fluid, and ρ_s is the density deposited on the solid,

$$\rho=\rho_f+\rho_s, \qquad q=V\rho_f.$$

Hence the conservation equation (2.11) reads

$$\frac{\partial}{\partial t}(\rho_f+\rho_s)+\frac{\partial}{\partial x}(V\rho_f)=0.$$

A second relation concerns the rate of deposition on the solid bed. The

exchange equation

$$\frac{\partial \rho_s}{\partial t}=k_1(A-\rho_s)\rho_f-k_2\rho_s(B-\rho_f)$$

is apparently the simplest equation with the required properties. The first term represents deposition from the fluid to the solid at a rate proportional to the amount in the fluid, but limited by the amount already on the solid up to a capacity A. The second term is the reverse transfer from the solid to the fluid. (In some processes, the second term is just proportional to ρ_s; this is the limit $B\to\infty$, k_2B finite.) In equilibrium, the right hand side of the equation vanishes and ρ_s is a definite function of ρ_f. In slowly varying conditions, with relatively large reaction rates k_1 and k_2, we may take a first approximation in which the right hand side still vanishes ("quasi-equilibrium") and we have

$$\rho_s=A\frac{k_1\rho_f}{k_2B+(k_1-k_2)\rho_f}.$$

Thus ρ_s is a function of ρ_f; hence q is a function of ρ. When changes become rapid, just before breaking, the term $\partial\rho_s/\partial t$ in the rate equation can no longer be neglected.

As a different type of example, the concept of group velocity can be fitted into this general scheme. In linear dispersive waves, as already noted following (1.26), there are oscillatory solutions with a local wave number $k(x,t)$ and a local frequency $\omega(x,t)$. Thus k is the density of the waves—the number of wave crests per unit length—and ω is the flux—number of wave crests crossing the position x per unit time. If we expect that wave crests will be conserved in the propagation, we have, in differential form, the conservation equation

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x}=0.$$

In addition, k and ω are related by the dispersion relation

$$\omega=\omega(k).$$

Hence

$$\frac{\partial k}{\partial t}+\omega'(k)\frac{\partial k}{\partial x}=0.$$

We have a wave propagation for the variations of the local wave number of the "carrier" wavetrain, and the propagation velocity is $d\omega/dk$. This is the group velocity. These ideas will be considered in full detail in the later discussion of dispersive waves.

The wave problems listed here depend primarily on the conservation equation (2.11), and for this reason they were given the name *kinematic waves* (Lighthill and Whitham, 1955) in contrast to the usual acoustic or elastic waves which depend strongly on how the acceleration is determined through the laws of dynamics.

After this review of some of the physical problems, we return to the study of breaking and shock waves in order to complete the theory. Further details of the physical problems are pursued in Chapter 3.

2.3 Shock Waves

When breaking occurs we question the assumption $q=Q(\rho)$ in (2.12) and also the differentiability of ρ and q in (2.11). But, provided the continuum assumption is adequate, we still insist on the conservation equation (2.10).

Consider first the mathematical question of whether discontinuities are possible. Certainly a simple jump discontinuity in ρ and in q is feasible as far as (2.10) is concerned; all the expressions in (2.10) have a meaning. Does (2.10) provide any restriction? To answer this, suppose there is a discontinuity at $x=s(t)$ and that x_1 and x_2 are chosen so that $x_1>s(t)>x_2$. Suppose ρ and q and their first derivatives are continuous in $x_1\geqslant x>s(t)$ and in $s(t)>x\geqslant x_2$, and have finite limits as $x\to s(t)$ from above and below. Then (2.10) may be written

$$q(x_2,t)-q(x_1,t)=\frac{d}{dt}\int_{x_2}^{s(t)}\rho(x,t)\,dx+\frac{d}{dt}\int_{s(t)}^{x_1}\rho(x,t)\,dx$$

$$=\rho(s^-,t)\dot{s}-\rho(s^+,t)\dot{s}+\int_{x_2}^{s(t)}\rho_t(x,t)\,dx+\int_{s(t)}^{x_1}\rho_t(x,t)\,dx,$$

where $\rho(s^-,t)$, $\rho(s^+,t)$ are the value of $\rho(x,t)$ as $x\to s(t)$ from below and above, respectively, and $\dot{s}=ds/dt$. Since ρ_t is bounded in each of the intervals separately, the integrals tend to zero in the limit as $x_1\to s^+$, $x_2\to s^-$. Therefore

$$q(s^-,t)-q(s^+,t)=\{\rho(s^-,t)-\rho(s^+,t)\}\dot{s}.$$

A conventional notation is to use a subscript 1 for the values ahead of the

shock and a subscript 2 for values behind. Then if U is the shock velocity, $\dot{s}$,

$$q_2-q_1=U(\rho_2-\rho_1). \tag{2.15}$$

The condition may also be written in the form

$$-U[\rho]+[q]=0, \tag{2.16}$$

where the brackets indicate the jump in the quantity. This form gives a nice correspondence between the shock condition and the differential equation (2.11), the correspondence being

$$\frac{\partial}{\partial t}\leftrightarrow-U[\ \ ],\qquad \frac{\partial}{\partial x}\leftrightarrow[\ \ ]. \tag{2.17}$$

We can now extend our solutions of (2.10) to allow such discontinuities. In any continuous part of the solution, (2.11) will still be satisfied and the assumption (2.12) may be retained. Since $q=Q(\rho)$ in the continuous parts, we have $q_2=Q(\rho_2)$ and $q_1=Q(\rho_1)$ on the two sides of any shock, and the shock condition (2.15) may be written

$$U=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}. \tag{2.18}$$

The problem then reduces to fitting shock discontinuities into the solution (2.5), (2.6) in such a way that (2.18) is satisfied and multivalued solutions are avoided.

The simplest case is the problem

$$\left.\begin{aligned}\rho&=\rho_1, & c&=c(\rho_1)=c_1, & x&>0,\\ \rho&=\rho_2, & c&=c(\rho_2)=c_2, & x&<0,\end{aligned}\right\}\qquad t=0,$$

with $c_2>c_1$. The breaking solution was indicated in Fig. 2.3. Now a single-valued solution is possible which is just a shock moving with velocity (2.18):

$$\begin{aligned}\rho&=\rho_1, & x&>Ut,\\ \rho&=\rho_2, & x&<Ut.\end{aligned}$$

This is represented schematically in Fig. 2.5.

A popular way to derive the shock condition is to view this particular solution from a frame of reference in which the shock is at rest, as shown

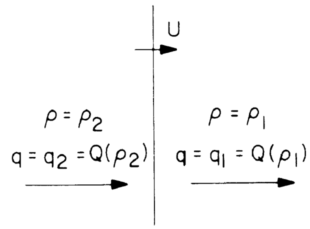

Fig. 2.5. Flow quantities for moving shock.

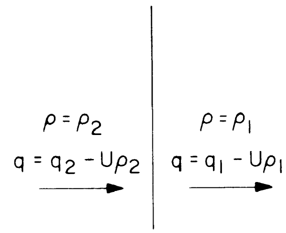

Fig. 2.6. Flow quantities relative to stationary shock.

in Fig. 2.6. The relative flows become $q_1 - U\rho_1$ and $q_2 - U\rho_2$. The conservation law may be stated immediately in the form

$$q_1 - U\rho_1 = q_2 - U\rho_2,$$

and (2.15) follows.

Before proceeding with the general problem of shock fitting, we consider the alternative view that the differential equation (2.11) is adequate but that the assumed relation (2.12) is insufficient.

2.4 Shock Structure

As a particular case, we need to find and examine a more accurate description of the simple discontinuous solution represented in Fig. 2.5. This is the problem of finding the "shock structure."

In many problems of kinematic waves, it would be a better approximation to suppose that q is a function of the density gradient ρ_x as well as ρ. A simple assumption is to take

$$q = Q(\rho) - \nu\rho_x, \tag{2.19}$$

where ν is a constant. In traffic flow, for example, we may argue that drivers will reduce their speed to account for an increasing density ahead,

and conversely. This argument would propose a positive value for ν, and we see below that the sign is important. If ν is small, in some suitable dimensionless measure, (2.12) is a good approximation provided ρ_x is not relatively large. At breaking, ρ_x becomes large and the correction term becomes crucial, however small ν may be. Now in *this* formulation, consider continuous solutions. From (2.11) and (2.19), they satisfy

$$\rho_t + c(\rho)\rho_x = \nu\rho_{xx}, \qquad c(\rho) = Q'(\rho). \tag{2.20}$$

The term $c(\rho)\rho_x$ in (2.20) leads to steepening and breaking. On the other hand, the term $\nu\rho_{xx}$ introduces diffusion typical of the heat equation

$$\rho_t = \nu\rho_{xx}.$$

For the heat equation, the solution of the initial step function problem

$$\left.\begin{array}{ll} \rho = \rho_1, & x > 0, \\ \rho = \rho_2, & x < 0, \end{array}\right\} \qquad t = 0$$

is

$$\rho = \rho_2 + \frac{\rho_1 - \rho_2}{\sqrt{\pi}} \int_{-\infty}^{x/\sqrt{4\nu t}} e^{-\zeta^2}\, d\zeta.$$

This represents a smoothed-out step approaching values ρ_1, ρ_2 as $x \to \pm\infty$, and with slope decreasing like $(\nu t)^{-1/2}$. The two opposite tendencies of nonlinear steepening and diffusion are combined in (2.20). The significance of $\nu > 0$ can be seen from the heat equation; solutions are unstable if $\nu < 0$.

We now look within the framework of this more accurate theory for the solution to replace the one shown in Fig. 2.5. One obvious idea is to look for a steady profile solution in which

$$\rho = \rho(X), \qquad X = x - Ut,$$

where U is a constant still to be determined. Then from (2.20),

$$\{c(\rho) - U\}\rho_X = \nu\rho_{XX}.$$

Integrating once, we have

$$Q(\rho) - U\rho + A = \nu\rho_X, \tag{2.21}$$

where A is a constant of integration. An implicit relation for $\rho(X)$ is obtained in the form

$$\frac{X}{\nu} = \int \frac{d\rho}{Q(\rho) - U\rho + A}, \tag{2.22}$$

but the qualitative behavior is more readily seen directly from (2.21). We are interested in the possibility of a solution which tends to constant states $\rho \to \rho_1$ as $X \to +\infty$, $\rho \to \rho_2$ as $X \to -\infty$. If such a solution exists with $\rho_X \to 0$ as $X \to \pm\infty$, the arbitrary parameters U, A must satisfy

$$Q(\rho_1) - U\rho_1 + A = Q(\rho_2) - U\rho_2 + A = 0.$$

In particular,

$$U = \frac{Q(\rho_2) - Q(\rho_1)}{\rho_2 - \rho_1}. \tag{2.23}$$

In such a solution, the relation between the velocity U and the two states at $\pm\infty$ is exactly the same as in the shock condition!

The values ρ_1, ρ_2 are zeros of $Q(\rho) - U\rho + A$, and in general they are simple zeros. As $\rho \to \rho_1$ or ρ_2 in (2.22), the integral diverges and $X \to \pm\infty$ as required. If $Q(\rho) - U\rho + A < 0$ between the two zeros, and if ν is positive, we have $\rho_X < 0$ and the solution is as shown in Fig. 2.7 with ρ increasing monotonically from ρ_1 at $+\infty$ to ρ_2 at $-\infty$. If $Q(\rho) - U\rho + A > 0$ and $\nu > 0$, the solution increases from ρ_2 at $-\infty$ to ρ_1 at $+\infty$. It is clear from (2.21) that if ρ_1, ρ_2 are kept fixed (so that U, A are fixed), a change in ν can be absorbed by a change in the X scale. As $\nu \to 0$, the profile in Fig. 2.7 is compressed in the X direction and tends in the limit to a step function increasing ρ from ρ_1 to ρ_2 and traveling with the velocity given by (2.23). This is exactly the discontinuous shock solution seen in Fig. 2.5. For small nonzero ν the shock is a rapid but continuous increase taking place over a narrow region. The breaking due to the nonlinearity is balanced by the diffusion in this narrow region to give a steady profile.

One very important point is the sign of the change in ρ. A continuous wave carrying an increase of ρ will break forward and require a shock with $\rho_2 > \rho_1$ if $c'(\rho) > 0$; it will break backward and require a shock with $\rho_2 < \rho_1$ if $c'(\rho) < 0$. The shock structure given by (2.21) must agree. As remarked above, ν is always positive for stability, so the direction of increase of ρ depends on the sign of $Q(\rho) - U\rho + A$ between the two zeros ρ_1 and ρ_2. But

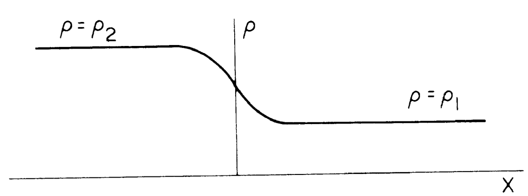

Fig. 2.7. Shock structure.

$c'(\rho) = Q''(\rho)$. Hence when $c'(\rho) > 0$, $Q(\rho) - U\rho + A < 0$ between zeros and the solution is as seen in Fig. 2.7 with $\rho_2 > \rho_1$ as required. If $c'(\rho) < 0$, the step is reversed and $\rho_2 < \rho_1$. The breaking argument and the shock structure agree.

In the special case of a quadratic expression for $Q(\rho)$, taken as

$$Q(\rho) = \alpha\rho^2 + \beta\rho + \gamma, \tag{2.24}$$

the integral in (2.22) is easily evaluated. The sign of α determines the sign of $c'(\rho) = Q''(\rho)$ and we consider $\alpha > 0$, for definiteness. We may write

$$Q - U\rho + A = -\alpha(\rho - \rho_1)(\rho_2 - \rho),$$

where

$$U = \beta + \alpha(\rho_1 + \rho_2), \qquad A = \alpha\rho_1\rho_2 - \gamma.$$

Then (2.22) becomes

$$\frac{X}{\nu} = -\int \frac{d\rho}{\alpha(\rho - \rho_1)(\rho_2 - \rho)} = \frac{1}{\alpha(\rho_2 - \rho_1)} \log \frac{\rho_2 - \rho}{\rho - \rho_1}. \tag{2.25}$$

As $X \to \infty$, $\rho \to \rho_1$ exponentially, and as $X \to -\infty$, $\rho \to \rho_2$ exponentially. There is no precise thickness to the transition region, but we can introduce various measures of the scale, such as the length over which 90% of the change occurs or $(\rho_2 - \rho_1)$ divided by the maximum slope $|\rho_X|$. Clearly all such measures of thickness are proportional to

$$\frac{\nu}{\alpha(\rho_2 - \rho_1)}. \tag{2.26}$$

If this is small compared with other typical lengths in the problem, the rapid shock transition is satisfactorily approximated by a discontinuity. We confirm that the thickness tends to zero as $\nu \to 0$ for fixed ρ_1, ρ_2, but it also should be noted that sufficiently weak shocks with $(\rho_2 - \rho_1)/\rho_1 \to 0$ ultimately become thick for fixed ν, however small. For weak shocks $Q(\rho)$ can always be approximated by a suitable quadratic over the range ρ_1 to ρ_2, so that (2.25) applies. Even for moderately strong shocks it is a good overall approximation to the shape.

The shock structure is only one special solution of (2.20), but from it we might expect in general that when $\nu \to 0$ in some suitable nondimensional form, solutions of (2.20) tend to solutions of

$$\rho_t + c(\rho)\rho_x = 0$$

together with discontinuous shocks satisfying

$$U=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}.$$

This is true when the solutions are compared at fixed (x,t) with $\nu\to 0$. However, the fact that the shock transition becomes very wide as $(\rho_2-\rho_1)/\rho_1\to 0$, for fixed ν, means that in any problem where the shocks ultimately tend to zero strength as $t\to\infty$, there may be some final stage with extremely weak shocks when the discontinuous theory will be invalid. This is often a very uninteresting stage, since the shocks must be very weak.

Otherwise, we can say that the two alternative ways of improving on the unacceptable multivalued solutions agree. The use of discontinuous shocks is the easier analytically and can be carried further in more complicated problems.

Confirmation in more detail would require some explicit solutions of (2.20) which involve shocks of varying strength. Although solutions are not known for a general $Q(\rho)$, it turns out that (2.20) can be solved explicitly when $Q(\rho)$ is once again a quadratic in ρ. If (2.20) is multiplied by $c'(\rho)$, it may be written

$$\begin{aligned} c_t+cc_x&=\nu c'(\rho)\rho_{xx}\\ &=\nu c_{xx}-\nu c''(\rho)\rho_x^2. \end{aligned} \tag{2.27}$$

If $Q(\rho)$ is quadratic, $c(\rho)$ is linear in ρ, then $c''(\rho)=0$ and we have

$$c_t+cc_x=\nu c_{xx}. \tag{2.28}$$

This is Burgers' equation and it can be solved explicitly. The main results are given in Chapter 4. For the present, we accept the evidence for pursuing discontinuous solutions of (2.2) bearing in mind that for extremely weak shocks it will not be appropriate. For the extremely weak shocks, $Q(\rho)$ can be approximated by a quadratic and Burgers' equation can be used.

The arguments in this section depend strongly on $\nu>0$. As noted previously, this is required for stability of the problem. Interesting cases of instability do occur, however, in traffic flow and flood waves. They are discussed in Chapter 3.

2.5 Weak Shock Waves

In a number of situations the shocks are weak in that $(\rho_2-\rho_1)/\rho_1$ is small, but they are not so extremely weak that they may no longer be treated as discontinuities. It is useful to note some approximations for such cases.

The shock velocity

$$U=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1}$$

tends to the characteristic velocity

$$c(\rho)=\frac{dQ}{d\rho}$$

in the limit as the shock strength $(\rho_2-\rho_1)/\rho_1\to 0$. For weak shocks the expression for the shock velocity U may be expanded in a Taylor series in $(\rho_2-\rho_1)/\rho_1$ as

$$U=Q'(\rho_1)+\frac{1}{2}(\rho_2-\rho_1)Q''(\rho_1)+O(\rho_2-\rho_1)^2.$$

The propagation velocity $c(\rho_2)=Q'(\rho_2)$ may also be expanded as

$$c(\rho_2)=c(\rho_1)+(\rho_2-\rho_1)Q''(\rho_1)+O(\rho_2-\rho_1)^2.$$

Therefore

$$U=\frac{1}{2}(c_1+c_2)+O(\rho_2-\rho_1)^2, \tag{2.29}$$

where $c_1=c(\rho_1)$ and $c_2=c(\rho_2)$. To this approximation, the shock velocity is the mean of the characteristic velocities on the two sides of it. In the (x,t) plane the shock curve bisects the angle between the characteristics which meet on the shock. This property is useful for sketching in the shocks, but it also simplifies the analytic determination of shock positions. Clearly the relation is exact when $Q(\rho)$ is a quadratic.

2.6 Breaking Condition

A continuous wave breaks and requires a shock if and only if the propagation velocity c decreases as x increases. Therefore when the shock is included we have

$$c_2>U>c_1, \tag{2.30}$$

where all velocities are measured positive in the direction of x increasing and the subscript 1 refers to the value of c just ahead of the shock (i.e., greater value of x) and the subscript 2 refers to the value of c just behind the shock. A shock produces an increase in c, and it is supersonic viewed from ahead and subsonic viewed from behind. As regards the jump condition alone, it would be feasible to fit in discontinuities with $c_2<c_1$. However, shocks with $c_2<c_1$ could never be formed from a continuous wave and they are never required; they are excluded from consideration for this reason.

One question about this argument is the point that the solution represented in Fig. 2.5 might be set up with $c_2<c_1$ (by some complicated but probably highly unrealistic device). Of course we have already noted in (2.9) and in Fig. 2.6 a satisfactory continuous solution for such initial conditions. Still, to be particularly awkward, one might insist that Fig. 2.5 gives an alternative solution. The answer is that this proposed solution is unstable. That is, small perturbations would change the flow into something quite different—the expansion fan solution of (2.9). This is a "disintegration argument" which is complementary to the "formation argument." The instability will not be considered in detail in this chapter since the formation argument is convincing and unambiguous. For higher order equations the shock formation becomes harder to study and the instability arguments sometimes give an easier method to decide whether a particular shock satisfying the shock conditions is really possible.

For gas dynamic shocks, the inequality corresponding to (2.30) is equivalent to the condition that the entropy of the gas *increases* as the gas passes through the shock. The entropy condition was the first argument for the irreversibility of shock waves, that is, that the shock transition goes only one way. However, conditions like (2.30) are more general. In some problems there is no obvious counterpart to entropy; in others, such as magnetogasdynamics, the entropy condition does not rule out some inadmissible shocks.

An alternative view of these criteria is that any acceptable discontinuous shock must have a satisfactory shock structure when described by more accurate equations. This is a more satisfactory point of view, since it appeals to a more realistic description of the phenomenon. However, the analysis may become prohibitive and one often resorts to the indirect arguments in the framework of the simpler theory.

This alternative approach was checked in the discussion of shock structure in Section 2.4. When $c'(\rho)>0$ we found only a shock structure for $\rho_2>\rho_1$; since $c'(\rho)>0$, this is equivalent to $c_2>c_1$. When $c'(\rho)<0$, we found $\rho_2<\rho_1$, but the change in sign of $c'(\rho)$ means that $c_2>c_1$. Since $c(\rho)=Q'(\rho)$, the shock velocity lies between the values c_1 and c_2 by Rolle's theorem.

2.7 Note on Conservation Laws and Weak Solutions

Mathematically, the composite solution composed of continuously differentiable parts satisfying

$$\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}=0 \tag{2.31}$$

together with jump discontinuities satisfying

$$-U[\rho]+[Q(\rho)]=0 \tag{2.32}$$

can be considered a weak solution of (2.31). Briefly, the idea is as follows. *Associate* with (2.31) the equation

$$-\iint_R \{\rho\phi_t+Q(\rho)\phi_x\}\,dx\,dt=0, \tag{2.33}$$

where R is an arbitrary rectangle in the (x,t) plane, and ϕ is an arbitrary "test" function with continuous first derivatives in R and $\phi=0$ on the boundary of R. If ρ and $Q(\rho)$ are continuously differentiable, (2.31) and (2.33) are equivalent. On the one hand, if (2.31) is multiplied by ϕ and integrated over R, we may deduce (2.33) after integration by parts. On the other hand, integration by parts on (2.33) leads to

$$\iint_R \left\{\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right\}\phi\,dx\,dt=0,$$

and, since this must hold for all arbitrary continuous ϕ, (2.31) follows. However, (2.33) allows more general possibilities, since the admissible functions $\rho(x,t)$ need not have derivatives. Functions $\rho(x,t)$ which satisfy (2.33) for all test functions ϕ are called "weak solutions" of (2.31).

We now investigate what this extended meaning of solution has achieved. Consider the possibility of a weak solution $\rho(x,t)$, that is, one satisfying (2.33), which is continuously differentiable in two parts R_1 and R_2 of R, but with a simple jump discontinuity across the dividing boundary, S, between R_1 and R_2. We may integrate by parts in each of the separate regions R_1, R_2, and deduce from (2.33) that

$$\iint_{R_1} \left\{\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right\}\phi\,dx\,dt+\iint_{R_2} \left\{\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}\right\}\phi\,dx\,dt$$
$$+\int_S\{[\rho]l+[Q(\rho)]m\}\phi\,ds=0,$$

where (l,m) is the normal to S and $[\rho]$, $[Q(\rho)]$ denote the jumps across S. The line integral on S consists of the two contributions from the boundary terms of R_1 and R_2 obtained in the integration by parts. Since this equation must hold for all test functions ϕ, we deduce that (2.31) holds inside each of the regions R_1 and R_2, but in addition we deduce

$$[\rho]l+[Q(\rho)]m=0 \qquad \text{on} \quad S.$$

This is the shock condition (2.32), since $U=-l/m$. Thus weak solutions of this type would satisfy (2.31) at points of continuity and allow jump discontinuities satisfying the shock condition. Just what we want!

At first sight, the weak solution concept appears to bypass the more involved and less precise discussion of the real physical processes. But it is not really so. Corresponding to the differential equation

$$\frac{\partial\rho}{\partial t}+c(\rho)\frac{\partial\rho}{\partial x}=0$$

there are an infinite number of conservation equations

$$\frac{\partial f(\rho)}{\partial t}+\frac{\partial g(\rho)}{\partial x}=0. \tag{2.34}$$

Any choice which satisfies

$$g'(\rho)=f'(\rho)c(\rho) \tag{2.35}$$

will do. For differentiable functions $\rho(x,t)$, these are all equivalent. However, their integrated forms are *not* equivalent and lead to different jump conditions. The weak solution of (2.34) will require the shock condition

$$-U[f(\rho)]+[g(\rho)]=0; \tag{2.36}$$

different choices of f and g lead to different relations between ρ_1, ρ_2, and U. Therefore a discussion of the physical processes is still necessary in order to pick out which weak solution is relevant to the particular physical problem at hand.

From the differential equation 2.34 we can propose a candidate for a conservation equation in integrated form:

$$\frac{d}{dt}\int_{x_2}^{x_1}f(\rho)\,dx+[g(\rho)]_{x_2}^{x_1}=0. \tag{2.37}$$

But whether this holds for nondifferentiable ρ can be decided only by returning to the original formulation of the problem. In Section 2.2 we argued in the correct order: first (2.10) then (2.11). The reverse order, going from an equivalent partial differential equation to an integrated form, introduces the lack of uniqueness.

If (2.37) is the true conservation equation, then (2.36) may be deduced as the shock condition by the same argument that was used in Section 2.3. Thus the correct choice of weak solution is made on the basis of which quantities are really conserved across the shock. In view of the lack of uniqueness and possible confusion, it is felt that the weak solution concept is not particularly valuable in this context and it is better to stress that physical problems are first formulated in the basic integrated forms from which both the partial differential equations and the appropriate jump conditions follow.

A looser form of the weak solution idea is sometimes useful in a preliminary look at a problem. If, for example, we ask whether (2.34) might admit moving discontinuities as part of the solution, we might try

$$f(\rho)=f_0(x)H(x-Ut)+f_1,$$

$$g(\rho)=g_0(x)H(x-Ut)+g_1,$$

where $H(x)$ is the Heaviside step function and f_1, g_1 are continuous functions. On substitution in the equation we obtain δ function terms

$$(-Uf_0+g_0)\delta(x-Ut)$$

plus less singular terms. We deduce that

$$-Uf_0+g_0=0,$$

and this is the shock condition (2.36), since $f_0=[f]$, $g_0=[g]$. This does not avoid the lack of uniqueness, of course, and it also uses δ functions in a slightly dubious way. The use of δ functions in nonlinear problems usually is excluded because there is no satisfactory meaning to powers and products of such generalized functions; we have retained an artificial linearity by expressing $f(\rho)$ and $g(\rho)$ separately, rather than using a single expression for ρ. Of course, the justification of the δ function argument is via the weak solution.

In contrast, consider the same question of the possibility of moving discontinuities for (2.20) written in the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial Q(\rho)}{\partial x}=\nu\frac{\partial^2\rho}{\partial x^2}.$$

If ρ and $Q(\rho)$ are expressed in terms of $H(x-Ut)$, the term $\partial^2\rho/\partial x^2$ will have a term $[\rho]\delta'(x-Ut)$ and there is no other term as singular as $\delta'(x-Ut)$ to balance it. We conclude that $[\rho]=0$ and that discontinuities are not possible. This is clearly a useful tool for a preliminary assessment.

2.8 Shock Fitting; Quadratic $Q(\rho)$

After discussion of these various points of view we now turn to the analytic problem of fitting discontinuous shocks satisfying

$$U=\frac{Q(\rho_2)-Q(\rho_1)}{\rho_2-\rho_1} \tag{2.38}$$

into the continuous solution

$$\begin{aligned} \rho &= f(\xi), \\ x &= \xi+F(\xi)t. \end{aligned} \tag{2.39}$$

Any multivalued part of the wave profile must be replaced by an appropriate discontinuity, as shown in Fig. 2.8.* The correct position for the discontinuity may be determined by the followng ingenious argument. Both the multivalued curve and the discontinuous curve satisfy conservation. Therefore $\int \rho\, dx$ under each curve must be the same; hence the discontinuity must cut off lobes of equal area, shown shaded in Fig. 2.8.

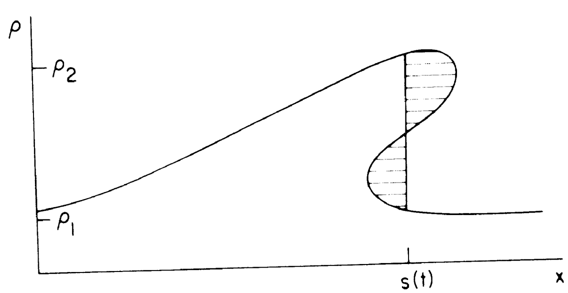

Fig. 2.8. Equal area construction for the position of the shock in a breaking wave.

*The figure is drawn for the case $c'(\rho)>0$ but all the formulas in this section are correct for either case.

This determination, although quite general, is not in a convenient form for analytic work. The general case gets complicated and it is worthwhile to do a special case first. The special case is again a quadratic expression for $Q(\rho)$. This includes the case of weak disturbances about a value $\rho=\rho_0$, since $Q(\rho)$ can then be approximated by

$$Q=Q(\rho_0)+Q'(\rho_0)(\rho-\rho_0)+\frac{1}{2}Q''(\rho_0)(\rho-\rho_0)^2,$$

and for this reason it has considerable generality.

We consider

$$Q(\rho)=\alpha\rho^2+\beta\rho+\gamma.$$

Then

$$c(\rho)=Q'(\rho)=2\alpha\rho+\beta$$

and the shock velocity (2.38) becomes

$$U=\frac{1}{2}(c_1+c_2),$$

where $c_1=c(\rho_1)$, $c_2=c(\rho_2)$.

The simplicity of this case is that the whole problem can be written in terms of c. The continuous solution is

$$\begin{aligned} c &= F(\xi), \\ x &= \xi+F(\xi)t, \end{aligned} \tag{2.40}$$

and shocks must be fitted in such that

$$U=\frac{1}{2}(c_1+c_2)=\frac{1}{2}\{F(\xi_1)+F(\xi_2)\}, \tag{2.41}$$

where ξ_1 and ξ_2 are the values of ξ on the two sides of the shock. Since ρ and c are linearly related, the conservation of ρ implies conservation of c; that is, $\int c\,dx$ is conserved in the solution. Therefore for this special case the shock construction for the (ρ,x) curve in Fig. 2.8 applies equally well to the (c,x) curve.

It is convenient to note for future reference that this solution in terms of c solves the equation

$$c_t+cc_x=0, \tag{2.42}$$

with weak solutions chosen to satisfy the conservation law

$$c_t+\left(\tfrac{1}{2}c^2\right)_x=0 \tag{2.43}$$

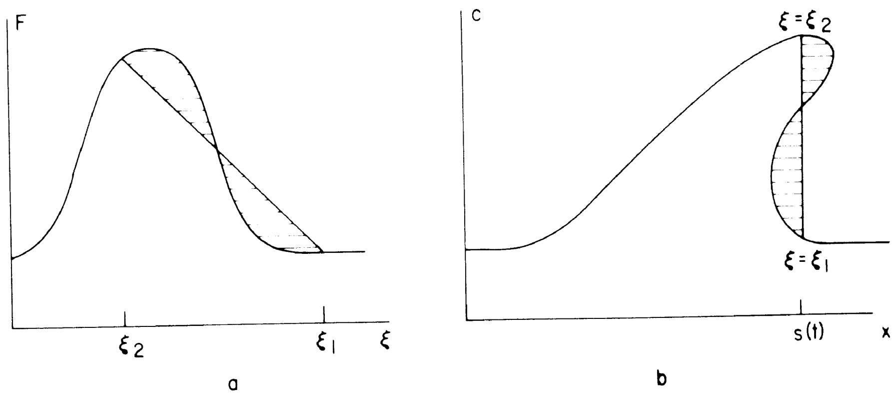

Fig. 2.9. Equal area construction: (*a*) on the initial profile; (*b*) on the transformed breaking profile.

so that the shock condition is

$$U=\frac{\frac{1}{2}c_2^2-\frac{1}{2}c_1^2}{c_2-c_1}=\frac{1}{2}(c_1+c_2). \tag{2.44}$$

Equation 2.42 is true for general $Q(\rho)$, since it is $c'(\rho)$ times $\rho_t+c(\rho)\rho_x=0$; (2.44) is always a possible weak solution, but it is the *correct* choice only when $Q(\rho)$ is quadratic or approximated by a quadratic since it is only in that case that the integrated form of (2.43) holds across discontinuities.

The shock construction can now be combined with the continuous solution (2.40). Since we now work with c the awkward distinction between the two cases $c'(\rho)\gtrless 0$ does not arise. According to (2.40) the solution at time t is obtained from the initial profile $c=F(\xi)$ by translating each point a distance $F(\xi)t$ to the right, as shown in Fig. 2.9. The shock cuts out the

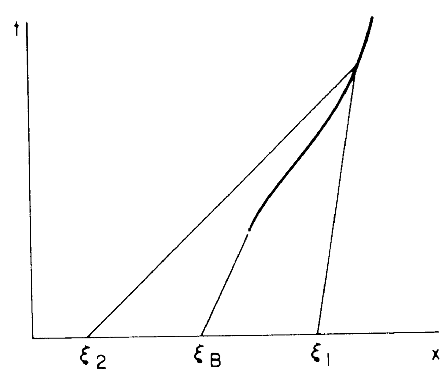

Fig. 2.10. The (x,t) diagram associated with the shock construction in Fig. 2.9.

part corresponding to $\xi_2 \geqslant \xi \geqslant \xi_1$. If the discontinuity line in Fig. 2.9*b* is also mapped back as in Fig. 2.9*a*, it is a straight line chord between the points $\xi=\xi_1$ and $\xi=\xi_2$ on the curve $F(\xi)$. Moreover, since areas are preserved under the mapping, the equal area property still holds in Fig. 2.9*a*; the chord on the F curve cuts off lobes of equal area. The shock determination can then be described entirely on the fixed $F(\xi)$ curve by drawing all the chords with the equal area property. The pairs $\xi=\xi_1$, $\xi=\xi_2$ at the ends of each chord relate characteristics which meet on the shock. The (x,t) plane is shown in Fig. 2.10. The equal area property can be written analytically as

$$\frac{1}{2}\{F(\xi_1)+F(\xi_2)\}(\xi_1-\xi_2)=\int_{\xi_2}^{\xi_1}F(\xi)\,d\xi, \tag{2.45}$$

since the left hand side is the area under the chord and the right hand side is the area under the F curve. If the shock is at $x=s(t)$ at time t, we also have

$$s(t)=\xi_1+F(\xi_1)t \tag{2.46}$$

$$s(t)=\xi_2+F(\xi_2)t \tag{2.47}$$

from the second of (2.40). The three equations (2.45)–(2.47) determine the three functions $s(t)$, $\xi_1(t)$, and $\xi_2(t)$. The determination of $s(t)$ is implicit involving the two additional functions $\xi_1(t)$ and $\xi_2(t)$ which determine the characteristics meeting tne shock at time t. The values of c on the two sides of the shock are $c_1=F(\xi_1)$ and $c_2=F(\xi_2)$; the values of ρ are obtained from c.

Since the shock determination (2.45)–(2.47) was obtained geometrically, it is interesting to check directly that it does indeed satisfy the shock condition (2.41). We may write this verification as an independent derivation of the result in (2.45). We have to find three functions $s(t)$, $\xi_1(t)$, $\xi_2(t)$ which satisfy (2.46), (2.47), and

$$\dot{s}(t)=\frac{1}{2}\{F(\xi_1)+F(\xi_2)\}. \tag{2.48}$$

(Dots denote t derivatives.) From (2.46) and (2.47), we have

$$t=-\frac{\xi_1-\xi_2}{F(\xi_1)-F(\xi_2)}, \tag{2.49}$$

and

$$\dot{s}(t) = \{1 + tF'(\xi_1)\}\dot{\xi}_1 + F(\xi_1),$$

$$\dot{s}(t) = \{1 + tF'(\xi_2)\}\dot{\xi}_2 + F(\xi_2).$$

If we take the mean of these last two expressions for $\dot{s}$ in order to preserve symmetry, substitute for t from (2.49), and then substitute in (2.48), we obtain

$$\frac{1}{2}\{F'(\xi_1)\dot{\xi}_1 + F'(\xi_2)\dot{\xi}_2\}(\xi_1-\xi_2) + \frac{1}{2}\{F(\xi_1)+F(\xi_2)\}(\dot{\xi}_1-\dot{\xi}_2)$$
$$= F(\xi_1)\dot{\xi}_1 - F(\xi_2)\dot{\xi}_2.$$

This may be integrated to (2.45); the constant of integration may be dropped since the starting point of the shock, $\xi_1=\xi_2$, must be a solution.

The expression (2.49) for the time can be used to follow the development of the shock. Since $t>0$, all the relevant chords in Fig. 2.9a must have negative slope. Since $\xi_1>\xi_2$ by the choice of notation, $F(\xi_2)>F(\xi_1)$, that is, $c_2>c_1$ as we decided from the breaking condition. The earliest time for the shock corresponds to the steepest chord. This is the limit when the chord is tangent at the point of inflexion $\xi=\xi_B$, say. Then $F(\xi_1)=F(\xi_2)$ so the shock starts with zero strength and the time is

$$t_B = -\frac{1}{F'(\xi_B)}.$$

This all fits with the conditions for the first point of breaking discussed in (2.8). For an F curve like Fig. 2.9a, the chords tend to the horizontal as $t\to\infty$, with $F(\xi_2)-F(\xi_1)\to 0$; hence $c_2-c_1\to 0$ and the shock strength tends to zero as $t\to\infty$.

Single Hump.

To study the shock in detail, we suppose first that $F(\xi)$ is equal to a constant c_0 outside the range $0<\xi<L$, and $F(\xi)>c_0$ in the range. Equation 2.45 may be written as

$$\frac{1}{2}\{F(\xi_1)+F(\xi_2)-2c_0\}(\xi_1-\xi_2) = \int_{\xi_2}^{\xi_1}\{F(\xi)-c_0\}\,d\xi.$$

As time goes on, ξ_1 increases and eventually exceeds L. At this stage $F(\xi_1)=c_0$ and the shock is moving into the constant region $c=c_0$. The function $\xi_1(t)$ can then be eliminated, for we have

$$\frac{1}{2}\{F(\xi_2)-c_0\}(\xi_1-\xi_2) = \int_{\xi_2}^{L}\{F(\xi)-c_0\}\,d\xi, \qquad t = \frac{\xi_1-\xi_2}{F(\xi_2)-c_0}.$$

Therefore

$$\frac{1}{2}\{F(\xi_2)-c_0\}^2 t = \int_{\xi_2}^{L}\{F(\xi)-c_0\}\,d\xi.$$

At this stage, the shock position and the value of c just behind the shock are given by

$$s(t) = \xi_2 + F(\xi_2)t, \qquad c = F(\xi_2), \tag{2.50}$$

where $\xi_2(t)$ satisfies

$$\frac{1}{2}\{F(\xi_2)-c_0\}^2 t = \int_{\xi_2}^{L}\{F(\xi)-c_0\}\,d\xi.$$

As $t\to\infty$, we have $\xi_2\to 0$ and $F(\xi_2)\to c_0$; hence the equation for $\xi_2(t)$ takes the limiting form

$$\frac{1}{2}\{F(\xi_2)-c_0\}^2 t \sim A,$$

where

$$A = \int_0^L\{F(\xi)-c_0\}\,d\xi$$

is the area of the hump above the undisturbed value c_0. We have $\xi_2\to 0$, $F(\xi_2)\sim c_0+\sqrt{2A/t}$. Therefore the asymptotic formulas for $s(t)$ and c in (2.50) are

$$s \sim c_0 t + \sqrt{2At}, \qquad c - c_0 \sim \sqrt{\frac{2A}{t}}, \tag{2.51}$$

at the shock. The shock curve is asymptotically parabolic and the shock strength $(c-c_0)/c_0$ tends to zero like $t^{-1/2}$.

The solution behind the shock is given by (2.40) with $0<\xi<\xi_2$. Since

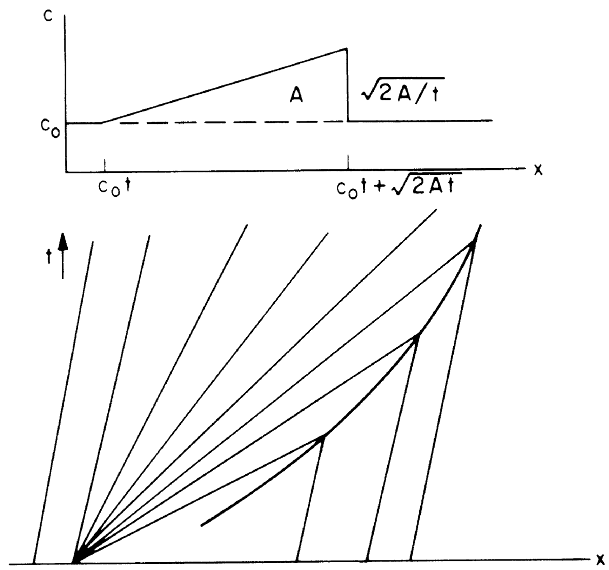

Fig. 2.11. The asymptotic triangular wave.

$\xi_2 \to 0$ as $t \to 0$, all the relevant values of ξ also tend to zero and the asymptotic form is

$$c \sim \frac{x}{t}, \qquad c_0 t < x < c_0 t + \sqrt{2At}\,. \tag{2.52}$$

The asymptotic solution and the corresponding (x,t) diagram are shown in Fig. 2.11. Notice that the details of the initial distribution are lost; only $A = \int_0^L \{F(\xi) - c_0\}\, d\xi$ appears in the ultimate asymptotic behavior.

N Wave.

Other problems can be worked out in a similar way. One important case is when $F(\xi)$ has a positive and a negative phase about an undisturbed value c_0, as in Fig. 2.12. There are now two shocks, corresponding to the two compression phases at the front and at the back where $F'(\xi) < 0$. The families of chords for each are shown in the figure. As $t \to \infty$, the pair (ξ_2, ξ_1) for the front shock approach $(0, \infty)$, whereas for the rear shock (ξ_2, ξ_1) approach $(-\infty, 0)$. Asymptotically the front shock is

$$s \sim c_0 t + \sqrt{2At}$$

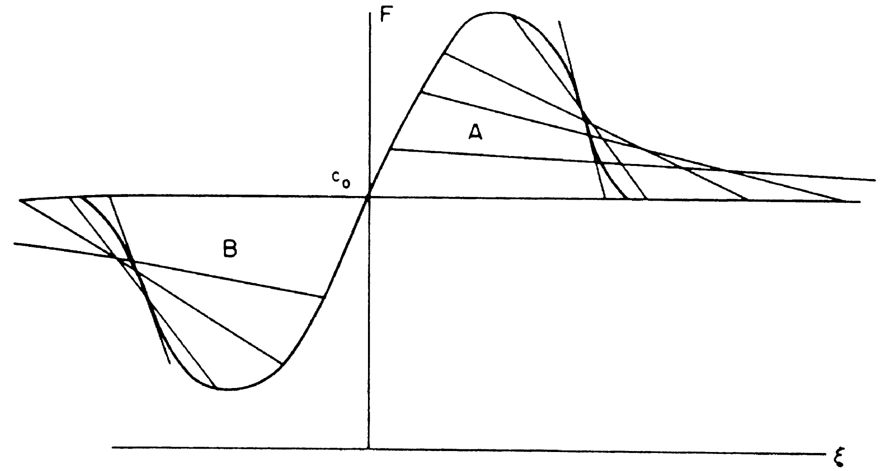

Fig. 2.12. Shock construction for an N wave.

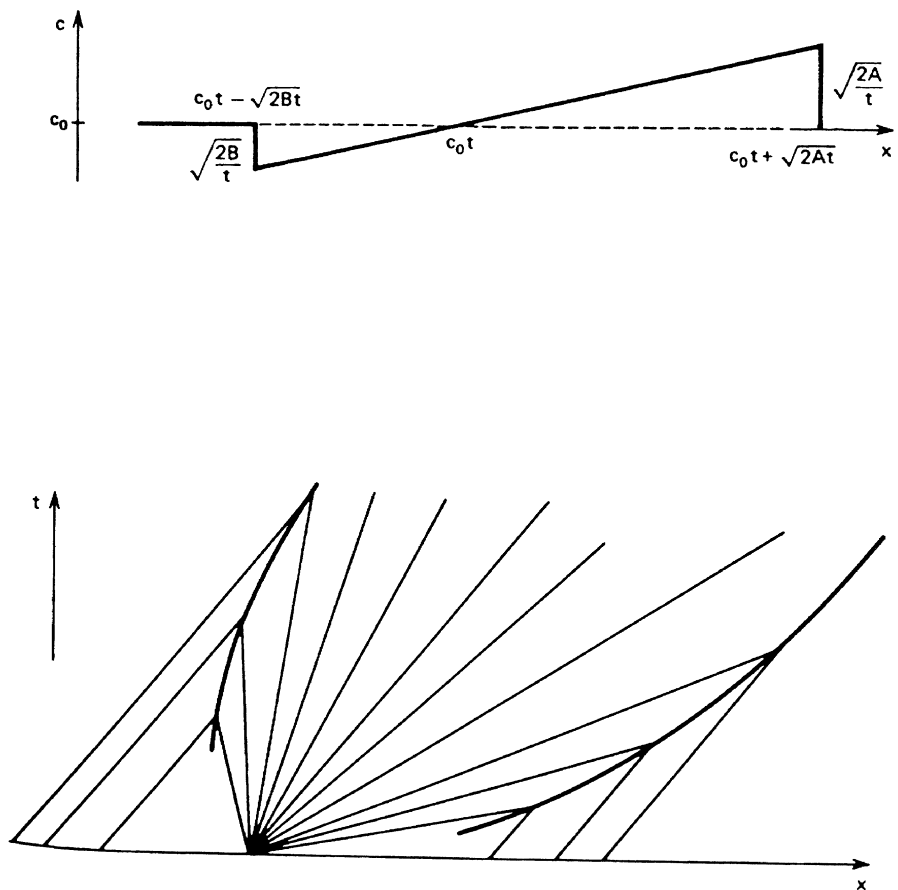

Fig. 2.13. The asymptotic N wave.

and the jump of c is

$$c - c_0 \sim \sqrt{\frac{2A}{t}},$$

where A is the area of the F curve above $c = c_0$. The rear shock has

$$x \sim c_0 t - \sqrt{2Bt},$$

$$c - c_0 \sim -\sqrt{\frac{2B}{t}},$$

where B is the area below $c = c_0$. The solution between the shocks is again asymptotically

$$c \sim \frac{x}{t}, \qquad c_0 t - \sqrt{2Bt} < x < c_0 t + \sqrt{2At}. \tag{2.53}$$

The asymptotic form and the (x,t) diagram are shown in Fig. 2.13. Because of the shape of the wave profile, it is known as the N wave.

Periodic Wave.

Another interesting problem is that of an initial distribution

$$c = F(\xi) = c_0 + a \sin \frac{2\pi\xi}{\lambda}. \tag{2.54}$$

In this case, the shock equations (2.45)–(2.47) simplify considerably for all times t. Consider one period $0 < \xi < \lambda$ as in Fig. 2.14. Relation 2.47 becomes

$$(\xi_1 - \xi_2) \sin \frac{\pi}{\lambda}(\xi_1 + \xi_2) \cos \frac{\pi}{\lambda}(\xi_1 - \xi_2) = \frac{\lambda}{\pi} \sin \frac{\pi}{\lambda}(\xi_1 - \xi_2) \sin \frac{\pi}{\lambda}(\xi_1 + \xi_2),$$

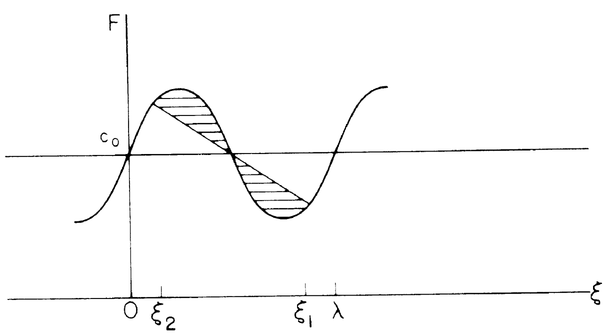

Fig. 2.14. Shock construction for a periodic wave.

and the relevant choice is the trivial one

$$\sin \frac{\pi(\xi_1 + \xi_2)}{\lambda} = 0, \qquad \text{that is,} \quad \xi_1 + \xi_2 = \lambda.$$

From the difference and sum of (2.46) and (2.47), we have

$$t = \frac{\xi_1 - \xi_2}{2a \sin \frac{\pi}{\lambda}(\xi_1 - \xi_2)},$$

$$s = c_0 t + \frac{\lambda}{2},$$

respectively. The discontinuity in c at the shock is

$$c_2 - c_1 = a \sin \frac{2\pi\xi_1}{\lambda} - a \sin \frac{2\pi\xi_2}{\lambda}$$

$$= 2a \sin \frac{\pi}{\lambda}(\xi_1 - \xi_2).$$

If we introduce

$$\xi_1 - \xi_2 = \frac{\lambda\theta}{\pi}, \qquad \xi_1 + \xi_2 = \lambda,$$

we have

$$t = \frac{\lambda}{2\pi a} \frac{\theta}{\sin\theta},$$

$$s = c_0 t + \frac{\lambda}{2}, \tag{2.55}$$

$$\frac{c_2 - c_1}{c_0} = \frac{2a}{c_0} \sin\theta.$$

The shock has constant velocity c_0 and this result could have been deduced in advance from the symmetry of the problem. The shock starts with zero strength corresponding to $\theta = 0$ at time $t = \lambda/2\pi a$. It reaches a maximum strength of $2a/c_0$ for $\theta = \pi/2$, $t = \lambda/4a$, and decays ultimately with $\theta \to \pi$, $t \to \infty$,

$$\frac{c_2 - c_1}{c_0} \sim \frac{\lambda}{c_0 t}. \tag{2.56}$$

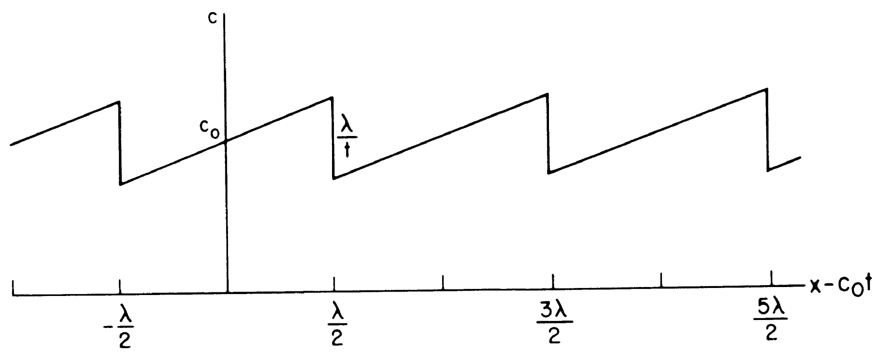

Fig. 2.15. Asymptotic form of a periodic wave.

It is interesting that the final decay formula does not even depend explicitly on the amplitude a. However, the condition for its applicability is $t \gg \lambda/a$. For any periodic $F(\xi)$, sinusoidal or not, $\xi_1 - \xi_2 \to \lambda$ as $t \to \infty$; hence from (2.49)

$$\frac{c_2 - c_1}{c_0} = \frac{F(\xi_2) - F(\xi_1)}{c_0} \sim \frac{\lambda}{c_0 t}.$$

Between successive shocks, the solution for c is linear in x with slope $1/t$ as before, and the asymptotic form of the entire profile is the sawtooth shown in Fig. 2.15.

Confluence of Shocks.

When a number of shocks are produced it is possible in general for one of them to overtake the shock ahead; they then combine and continue as a single shock. This is also described by our shock solution. Consider the F curve in Fig. 2.16. Two shocks are formed corresponding to the points of inflexion P and Q with families of equal area chords typified by P_1P_2 and Q_1Q_2. As time goes on the points Q_1 and P_2 approach each other until the stage in Fig. 2.16b is reached where a common chord cuts off lobes of equal area for both humps. At this stage the characteristics corresponding to P_2' and Q_1' are the same, and therefore the shocks have just combined into one as shown in the (x,t) diagram Fig. 2.17. All the characteristics between Q_2' and P_1' have now been absorbed by one or other of the shocks; a single shock proceeds using chords $P_1''Q_2''$ as in Fig. 2.16c, counting only *total* areas above and below the chord in the equal area construction.

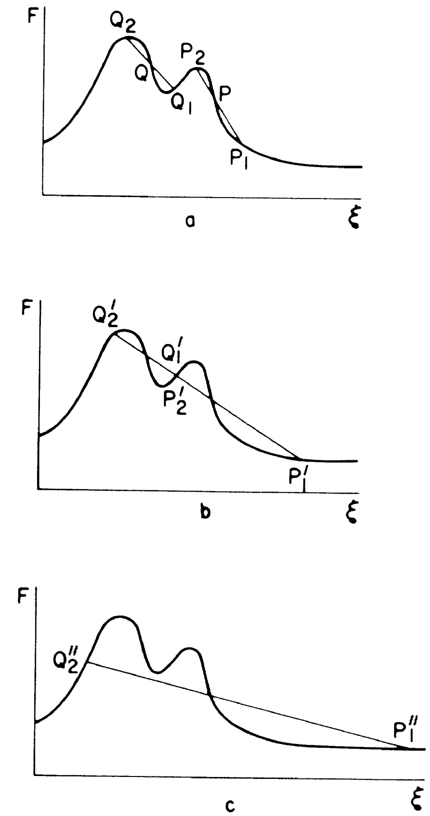

Fig. 2.16. Construction for merging shocks.

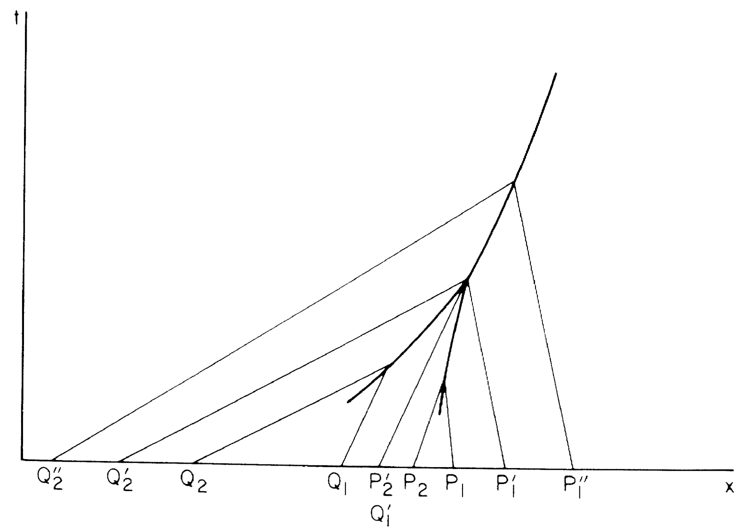

Fig. 2.17. The (x,t) diagram for merging shocks corresponding to Figs. 2.16.

2.9 Shock Fitting; General $Q(\rho)$

For the general dependence of q on ρ, the shock determination can be put into an analytic form similar to (2.45)–(2.47). The complication is the nonlinear relation between c and ρ, so that the construction in Fig. 2.8 is correct for ρ but not for c. Accordingly, we must work with ρ. But if we then plot (ρ, x) curves similar to Fig. 2.9, the discontinuity line does not map into a straight chord because the translation is proportional to c and not to ρ. Thus the mapping back onto the initial curve is no particular advantage.

However, we can proceed as follows. Introduce the function $\xi(\rho)$, which is the inverse of

$$\rho = f(\xi),$$

and introduce also the function $X(\rho, t)$, which is the inverse of the function $\rho = \rho(x,t)$ in the multivalued solution. That is, we fix attention on a particular value of ρ and note where it is now, $X(\rho, t)$, and where it was initially, $\xi(\rho)$. From the equation for the characteristics we have

$$X(\rho, t) = c(\rho)t + \xi(\rho). \tag{2.57}$$

Consider the shock at $s(t)$ and let ρ_1 and ρ_2 be the values ahead and behind the shock, respectively. The equal area construction in Fig. 2.8 may be written

$$\int_{\rho_2}^{\rho_1} X(\rho, t)\, d\rho = (\rho_1 - \rho_2)s(t).$$

[This is true for either case $c'(\rho) \gtrless 0$. We always take ρ_1 to be the value ahead of the shock and ρ_2 to be the value behind. If $c'(\rho) > 0$, then $\rho_2 > \rho_1$; if $c'(\rho) < 0$, then $\rho_2 < \rho_1$.] Hence from (2.57),

$$\int_{\rho_2}^{\rho_1} \{ c(\rho)t + \xi(\rho) \}\, d\rho = (\rho_1 - \rho_2)s(t).$$

Since $c(\rho) = Q'(\rho)$, this may be written

$$(q_1 - q_2)t - (\rho_1 - \rho_2)s(t) = -\int_{\rho_2}^{\rho_1} \xi(\rho)\, d\rho. \tag{2.58}$$

The right hand side can be integrated by parts and rewritten as

$$-\rho_1\xi_1 + \rho_2\xi_2 + \int_{\xi_2}^{\xi_1} \rho(\xi)\, d\xi.$$

The shock position $s(t)$ is given by

$$\begin{aligned} s(t) &= \xi_1 + c_1 t, \\ s(t) &= \xi_2 + c_2 t; \end{aligned} \tag{2.59}$$

these may be solved for $s(t)$ and t and substituted in (2.58). Finally, (2.58) becomes

$$\{ (q_2 - q_1) - (\rho_2 c_2 - \rho_1 c_1) \} \frac{\xi_1 - \xi_2}{c_1 - c_2} = \int_{\xi_2}^{\xi_1} \rho\, d\xi, \tag{2.60}$$

where ρ, q, and c are all to be evaluated as functions of ξ through the relations

$$\rho = f(\xi), \qquad q = Q(f(\xi)), \qquad c = Q'(f(\xi)) \tag{2.61}$$

and subscripts indicate values for $\xi = \xi_1$ and $\xi = \xi_2$. [This makes it a little clearer than using $f(\xi)$ for ρ, $F(\xi)$ for c and introducing a new symbol for q as a function of ξ.] Equations 2.59 and 2.60 give three relations for $s(t)$, $\xi_1(t)$, $\xi_2(t)$. Again it may be verified directly by differentiation that the shock condition

$$\dot{s} = \frac{q_2 - q_1}{\rho_2 - \rho_1}$$

is satisfied. When q is quadratic in ρ, it is easily verified that (2.60) reduces to (2.45). The problems like a single hump or an N wave can be analyzed as before and are qualitatively similar. The asymptotic formulas (2.51), (2.52), and (2.53) still apply with the modification that

$$A = c'(\rho_0) \int_0^L (\rho - \rho_0)\, d\xi,$$

and B is changed similarly. The expressions for ρ may be deduced from those for c, since the disturbance is weak in the asymptotic limit and $\rho - \rho_0 = (c - c_0)/c'(\rho_0)$ to first order.

2.10 Note on Linearized Theory

When disturbances are weak, nonlinear equations are often "linearized" by neglecting all but the first order powers of the perturbations. For weak disturbances with $(c - c_0)/c_0 \ll 1$, the equation

$$c_t + c c_x = 0$$

would be linearized to

$$c_t + c_0 c_x = 0.$$

As noted earlier the solution of this equation is $c - c_0 = f(x - c_0 t)$. The breaking effect and the formation of shocks are completely absent, yet we see from Figs. 2.11, 2.13, and 2.15 that these become crucial after a sufficient time, *however weak the initial disturbance may be*. Thus it is clear from comparison of the answers that the linearized approximation cannot be uniformly valid as $t \to \infty$.

This may also be seen directly, by looking at the linear theory as the first term in a naive expansion in powers of a small parameter. Suppose ϵ measures the maximum initial value of $(c - c_0)/c_0$, and a solution is sought in the form

$$c = c_0 + \epsilon c_1(x,t) + \epsilon^2 c_2(x,t) + \cdots.$$

When this expansion is substituted in $c_t + c c_x = 0$ and coefficients of ϵ^n are equated to zero, we have a hierarchy of equations starting with

$$c_{1t} + c_0 c_{1x} = 0,$$

$$c_{2t} + c_0 c_{2x} = -c_1 c_{1x},$$

$$c_{3t} + c_0 c_{3x} = -c_2 c_{1x} - c_1 c_{2x}.$$

These are easily solved successively since at each step we have

$$\phi_t + c_0 \phi_x = \Phi(x,t),$$

where Φ is known from the previous step. If we introduce the characteristic coordinate $y = x - c_0 t$, this may be written

$$\left(\frac{\partial \phi}{\partial t}\right)_{y=\text{const.}} = \Phi(y + c_0 t, t).$$

Therefore

$$\phi = \int_0^t \Phi(y + c_0\tau, \tau)\, d\tau + \Psi(y).$$

The initial condition on c may be written

$$c = c_0 + \epsilon P(x) \qquad \text{at } t = 0,$$

and it is satisfied by

$$c_1 = P(x), \quad c_n = 0 \quad (n > 1) \qquad \text{at } t = 0.$$

Hence the complementary functions $\Psi(y)$ are zero in the solutions for the c_n, $n > 1$. The first three c_n are found to be

$$c_1 = P(y),$$

$$c_2 = -tP(y)P'(y),$$

$$c_3 = \frac{t^2}{2}(P^2 P')'.$$

It is clear that in general c_n will contain a term of the form $t^{n-1}R_n(y)$. Therefore the successive terms in the assumed series for c are of order $\epsilon^n t^{n-1}$, and the series is not uniformly valid as $t \to \infty$.

The failure of the linearized theory, brought out strongly in the solution for the higher order terms, is that it approximates the characteristics as lines $x - c_0 t = \text{constant}$. The slight inclination of the true characteristic lines, relative to each other, accumulates to a large displacement as $t \to \infty$. The correct solution may be written as a telescoping function:

$$c = c_0 + \epsilon P(x - ct),$$

$$= c_0 + \epsilon P(x - [c_0 + \epsilon P]t),$$

and so on. The naive perturbation expansion can then be obtained by an inadvisable use of Taylor series!

2.11 Other Boundary Conditions; The Signaling Problem

The solution for the initial value problem has been given in great detail. Other boundary value problems can be solved in similar fashion. It is clear from the characteristic form (2.4) that the solution is determined once the value of ρ is given on any curve that intersects each characteristic once. Such a boundary value provides the initial conditions for integrating the two ordinary equations in (2.4) along the characteristic through that point. In principle, this is repeated at each point of the boundary curve to build up the solution in the whole region covered by the characteristics through the boundary curve. If the curve intersects characteristics twice, as does curve ABC in Fig. 2.18, the data can only be posed on AB or BC, otherwise the integration starting from AB, say, will conflict with the data on arrival at BC. Since the characteristics may depend on the solution, the region covered and the admissibility of the boundary curve cannot always be decided in advance.

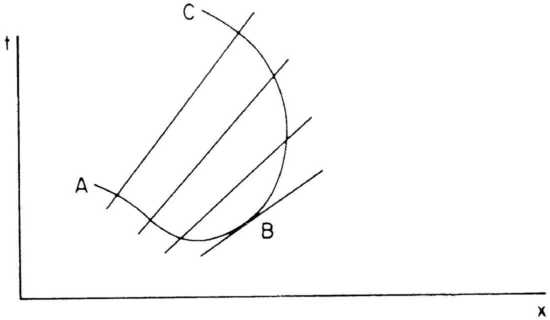

Fig. 2.18. Characteristics and initial data.

A standard boundary value problem is the so-called signaling problem for which

$$\rho=\rho_0 \qquad \text{for } x>0,\ t=0,$$

$$\rho=g(t) \qquad \text{for } t>0,\ x=0,$$

and the solution is required in $x>0$, $t>0$. Of course, this problem only arises in the case $c=Q'(\rho)>0$. The (x,t) diagram is shown in Fig. 2.19. Characteristics start from the positive x axis and the positive t axis. Those from the t axis have $\rho=\rho_0$, $c=c(\rho_0)=c_0$ and are straight lines $x-c_0t=\text{constant}$. Thus they predict

$$\rho=\rho_0, \quad c=c_0 \qquad \text{in } x>c_0t. \tag{2.62}$$

For the characteristics starting from the t axis, let a typical one start at $t=\tau$. Then

$$\begin{aligned}\rho&=g(\tau),\\ x&=G(\tau)(t-\tau),\end{aligned} \tag{2.63}$$

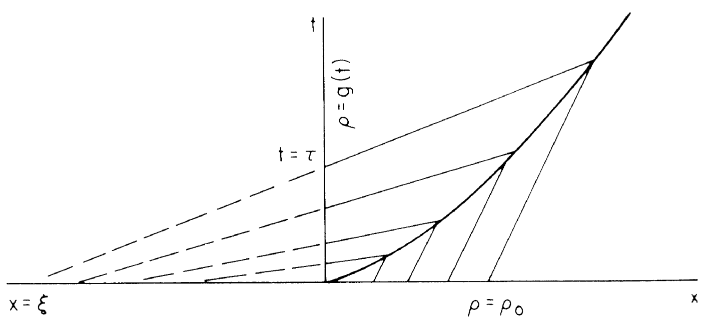

Fig. 2.19. The (x,t) diagram for the signaling problem.

where $G(\tau)=c\{g(\tau)\}$. This gives the solution implicitly in terms of $\tau(x,t)$.

The solution can be related to the solution of the initial value problem in two ways. The first method is to note that the two solutions agree if

$$\xi=-\tau G(\tau), \qquad f(\xi)=g(\tau), \qquad F(\xi)=G(\tau). \tag{2.64}$$

This corresponds to continuing the characteristic through $t=\tau$, $x=0$, back to the x axis and denoting the point of intersection as $x=\xi$; in this way the signaling problem is formulated as an initial value problem. The alternative method is to interchange the roles of x and t, and of q and ρ; in the formulas, $d\rho/dq=1/c$ will appear in place of $dq/d\rho=c$.

Any multivalued overlap in the solution (2.63) has to be resolved by shocks. If $G(0+)>c_0$, there will be an overlap immediately since the first characteristic $x=tG(0+)$ of the disturbed region is ahead of the last characteristic $x=c_0t$ of the undisturbed region. In that case a shock of finite strength will start from the origin. The shock determination can be taken from the results for the initial value problem using either of the above methods, or it can be developed independently. If characteristics $\tau_1(t)$ and $\tau_2(t)$ meet the shock at the time t, then from (2.63),

$$\begin{aligned} s(t)&=(t-\tau_1)c_1, \qquad c_1=G(\tau_1),\\ s(t)&=(t-\tau_2)c_2, \qquad c_2=G(\tau_2),\end{aligned} \tag{2.65}$$

and the formula corresponding to (2.60) may be written

$$\{(q_2-\rho_2c_2)c_1-(q_1-\rho_1c_1)c_2\}\frac{\tau_2-\tau_1}{c_2-c_1}=-\int_{\tau_1}^{\tau_2}q(\tau)\,d\tau. \tag{2.66}$$

Equations 2.65 and 2.66 provide the three implicit equations for the functions $\tau_1(t)$, $\tau_2(t)$, $s(t)$. The most important case is that of a front shock formed at the origin [i.e., $G(0+)>c_0$] and propagating into the undisturbed region. Then we have $\rho_1=\rho_0$, $c_1=c_0$, $q_1=q_0$, and τ_1 can be eliminated from (2.65) and (2.66). At the same time we drop the subscript 2 and reduce the shock relation (2.66) to

$$\{(q-q_0)-(\rho-\rho_0)c\}(t-\tau)=-\int_0^{\tau}\{q(\tau')-q_0\}\,d\tau'. \tag{2.67}$$

Here ρ, q, and c are functions of τ determined from

$$\rho=g(\tau), \qquad q=Q(g(\tau)), \qquad c=Q'(g(\tau));$$

they are all known functions of each other and when one is prescribed as a function of τ the others follow. Equations 2.63 determine the solution in the disturbed region behind the shock; (2.67) determines the appropriate value $\tau(t)$ at the shock and on substitution in (2.63) we have both the position of the shock and the value of ρ just behind it.

In the initial motion of the shock, the value of $\tau(t)$ in (2.67) is small and we have

$$\{(q_i-q_0)-(\rho_i-\rho_0)c_i\}(t-\tau)=-(q_i-q_0)\tau+O(\tau^2),$$

where ρ_i, q_i, and c_i are the initial values on $x=0$; that is, $\rho_i=g(0+)$, and so on. Therefore

$$\tau=\left\{1-\frac{q_i-q_0}{(\rho_i-\rho_0)c_i}\right\}t+O(t^2).$$

From (2.63), the shock position is

$$x=(t-\tau)c_i+O(t^2)$$

$$=\frac{q_i-q_0}{\rho_i-\rho_0}t+O(t^2).$$

The shock starts with velocity $(q_i-q_0)/(\rho_i-\rho_0)$, and this result can be seen directly from the shock condition. If $g(\tau)$ remains constant and equal to ρ_i, this is exact for all t and the solution is a shock of constant velocity separating the two uniform regions $\rho=\rho_0$ and $\rho=\rho_i$.

If $g(\tau)$ returns to ρ_0, the shock ultimately decays. For a single positive phase with $g(\tau)$ returning to ρ_0 at $\tau=T$, the asymptotic behavior corresponds to $\tau\to T$, $t\to\infty$, $\rho\to\rho_0$ in (2.67). In this limit, (2.67) becomes

$$\frac{1}{2}c'(\rho_0)(\rho-\rho_0)^2t\sim\int_0^T\{q(\tau')-q_0\}\,d\tau',$$

and the expression for the shock position in (2.63) becomes

$$x\sim c_0t+c'(\rho_0)(\rho-\rho_0)t.$$

Therefore at the shock we have

$$\rho-\rho_0\sim\frac{1}{c'(\rho_0)}\sqrt{\frac{2A}{t}}\,,\qquad c-c_0\sim\sqrt{\frac{2A}{t}}\,, \tag{2.68}$$

$$x\sim c_0t+\sqrt{2At}\,,$$

where

$$A=c'(\rho_0)\int_0^T(q-q_0)\,d\tau.$$

In the region behind the shock,

$$c\sim\frac{x}{t},\qquad c_0t<x<c_0t+\sqrt{2At}\,,$$
$$\rho-\rho_0\sim\frac{(c-c_0)}{c'(\rho_0)}\sim\frac{1}{c'(\rho_0)}\,\frac{x-c_0t}{t}. \tag{2.69}$$

These results are very similar to those for the initial value problem. Other cases may be studied in the same way. If the positive phase is followed by a negative phase, there is a second shock whose asymptotic behavior is given by (2.68) with the modifications that A is replaced by the corresponding integral over the negative phase and the signs are changed appropriately. The ultimate form is an N wave with the formulas (2.69) extended back to the rear shock.

2.12 More General Quasi-Linear Equations

The general quasi-linear equation of first order is linear in ρ_t and ρ_x but may also have an undifferentiated term. The coefficients of ρ_t, ρ_x and the undifferentiated term may be any functions of ρ, x, t. If the coefficient of ρ_t is nonzero, the equation may be divided by this quantity and written in the form

$$\rho_t+c\rho_x=b, \tag{2.70}$$

where b and c are functions of ρ, x, and t. Such equations can again be reduced to the integration of ordinary differential equations along characteristic curves by writing (2.70) in the characteristic form

$$\frac{d\rho}{dt}=b(\rho,x,t),\qquad\frac{dx}{dt}=c(\rho,x,t). \tag{2.71}$$

In particular the initial value problem with initial data

$$\rho=f(x),\qquad t=0$$

is solved by integrating the coupled ordinary differential equations in

(2.71) subject to the initial conditions

$$\rho = f(\xi), \quad x = \xi, \qquad \text{at } t = 0.$$

Each choice of ξ leads to the determination of the characteristic through $x = \xi$ and the value of ρ along it. The solution in a whole region is obtained by varying the parameter ξ.

When $b \neq 0$, ρ is not constant along the characteristics and generally the characteristics are not straight lines. But the method of determination is qualitatively the same. Again waves may break with the characteristics overlapping in the (x,t) plane. Again the multivalued solutions may be avoided by including suitable discontinuities.

Some interesting cases concerning breaking arise and we will consider two examples here.

Damped Waves.

Consider as a first example the case

$$c_t + cc_x + ac = 0, \tag{2.72}$$

where a is a positive constant. In characteristic form it is written

$$\frac{dc}{dt} = -ac, \qquad \frac{dx}{dt} = c. \tag{2.73}$$

If we take the initial value problem, the first equation may be integrated to

$$c = e^{-at} f(\xi). \tag{2.74}$$

Then the second equation is

$$\frac{dx}{dt} = e^{-at} f(\xi),$$

and we require $x = \xi$ at $t = 0$. The solution is

$$x = \xi + \frac{1 - e^{-at}}{a} f(\xi). \tag{2.75}$$

The nonlinearity gives the typical distortion of the wave profile, but simultaneously the wave is damped due to the presence of the undifferentiated term in the equation.

Consider now the question of breaking. This is most easily investigated by seeing whether the characteristic curves (2.75) have an envelope. An envelope of these curves satisfies the derivative of (2.75) with respect to the parameter ξ:

$$0 = 1 + \frac{1 - e^{-at}}{a} f'(\xi). \tag{2.76}$$

Since $a > 0$, $t > 0$, this is possible if and only if

$$f'(\xi) < -a. \tag{2.77}$$

Thus breaking occurs if and only if the initial curve has a large enough negative slope; the damping may prevent breaking if the compressive phase is not steep enough.

Although the appropriate equations are more complicated than those just considered (see Chapter 3), this type of inequality determines whether the tidal variation propagating up a river will be strong enough to produce breaking into a bore, or whether the friction will dominate. For most rivers the frictional effects dominate. However, those famous rivers that have a bore have high enough tides at the mouth and additional reinforcement from rapid narrowing of the river to overcome the various frictional effects. This theory has been discussed and applied by Abbott (1956). It will be referred to again in Section 5.7.

Waves Produced by a Moving Source.

If b is independent of ρ in (2.70), it may be interpreted as an external source of the fluid. A particularly interesting case is when the source distribution moves with constant velocity V. There is a recent example, in the more complicated context of magnetogasdynamics, where a wave motion is produced by applying a moving force to the fluid (Hoffman, 1967). We can examine some of the qualitative effects in our simple model.

We take

$$b = B(x - Vt),$$

where V is constant and $B(x)$ is a positive function tending rapidly to zero as $|x| \to \infty$. We assume that ρ has a constant value $\rho = \rho_0$ at $t = 0$. If $c_0 = c(\rho_0)$, there are important differences depending on whether the source moves supersonically with $V > c_0$ or subsonically with $V < c_0$.

The surprising result is that a supersonic source need not produce a shock, whereas a subsonic source always does. This can be seen quite simply by looking for a steady profile solution with

$$\rho=\rho(X), \qquad X=x-Vt. \tag{2.78}$$

Since we are only looking at models anyway, with the aim of showing qualitative effects, let us take the special case

$$c_t+cc_x=B(x-Vt). \tag{2.79}$$

Then in the steady profile solution

$$(c-V)c_X=B(X),$$

$$\tfrac{1}{2}(V-c)^2-\tfrac{1}{2}(V-c_0)^2=-\int_X^\infty B(y)\,dy.$$

In the supersonic case, $V>c_0$, the solution for c is

$$c=V-\left\{(V-c_0)^2-2\int_X^\infty B(y)\,dy\right\}^{1/2}. \tag{2.80}$$

If

$$V-c_0>\left\{2\int_{-\infty}^\infty B(y)\,dy\right\}^{1/2}, \tag{2.81}$$

(2.80) is a satisfactory single-valued solution for all X and no shock is required. The criterion (2.81) is an inequality between the speed $V-c_0$ and the total source strength

$$\int_{-\infty}^\infty B(y)\,dy.$$

We can get a feeling for the result by the following argument. If the source moves with a large supersonic speed, the only shock that could keep up with it would be strong. But if the source is relatively small, a strong shock cannot be produced and is not required.

When the inequality (2.81) does not hold, (2.80) breaks down for $X \leqslant X_0$, where

$$V-c_0=\left\{2\int_{X_0}^\infty B(y)\,dy\right\}^{1/2}.$$

At $X=X_0$, $c=V$ and transients from the starting conditions can and do overtake the wave. The solution cannot be completed without a detailed discussion of the transients. Similarly, in the subsonic case the solution cannot be established without full discussion of the transients. In both these cases shocks are found to occur. A detailed discussion is given by Hoffman (1967).

2.13 Nonlinear First Order Equations

The discussion of quasi-linear equations has raised many questions that require further consideration. Before proceeding, however, we note briefly that similar constructions using characteristics go through in the general case of fully nonlinear first order equations. These results will also be needed later.

It will be useful to have the characteristic form for an equation in n independent variables $(x_1,\dots,x_n)$. We consider, then, a function $\phi(x_1,\dots,x_n)$ which satisfies a differential equation

$$H(\mathbf{p},\phi,\mathbf{x})=0, \tag{2.82}$$

where $\mathbf{p}$ and $\mathbf{x}$ denote the vectors with components p_i and x_i, $i=1,\dots,n$, and

$$p_i=\frac{\partial\phi}{\partial x_i}. \tag{2.83}$$

We may motivate the characteristic form by asking whether there are curves in x space with special properties akin to the characteristics of the quasi-linear equations. Any curve $\mathcal{C}$ in x space may be written in parametric form

$$\mathbf{x}=\mathbf{x}(\lambda).$$

The total derivative of ϕ along the curve $\mathcal{C}$ is*

$$\frac{d\phi}{d\lambda}=\frac{\partial\phi}{\partial x_j}\frac{dx_j}{d\lambda}=p_j\frac{dx_j}{d\lambda}.$$

Is there a choice of the direction vector $dx_j/d\lambda$ which has special significance for the solution of (2.82)? In the quasi-linear case where we have

*We use the summation convention that a repeated subscript is automatically summed over $1,\dots,n$.

$H \equiv c_j(\phi, \mathbf{x})p_j - b(\phi, \mathbf{x})$, we choose

$$\frac{dx_j}{d\lambda} = c_j(\phi, \mathbf{x})$$

so that $d\phi/d\lambda = c_j p_j$; we then use the equation to obtain

$$\frac{d\phi}{d\lambda} = b(\phi, \mathbf{x}).$$

But generally the p_i cannot be eliminated in the expression for $d\phi/d\lambda$ whatever the choice of the $dx_j/d\lambda$. We do not have an ordinary differential equation for ϕ alone; the p_i are involved. However, consider in addition the total derivatives of the p_i on $\mathcal{C}$. We have

$$\frac{dp_i}{d\lambda} = \frac{d}{d\lambda}\left(\frac{\partial\phi}{\partial x_i}\right) = \frac{\partial^2\phi}{\partial x_i \partial x_j}\frac{dx_j}{d\lambda}, \tag{2.84}$$

and the x_i derivative of (2.82) yields

$$\frac{\partial^2\phi}{\partial x_i \partial x_j}\frac{\partial H}{\partial p_j} + \frac{\partial H}{\partial \phi}\frac{\partial \phi}{\partial x_i} + \frac{\partial H}{\partial x_i} = 0. \tag{2.85}$$

Comparing the two, we see the special advantage in choosing curves $\mathcal{C}$ defined by

$$\frac{dx_i}{d\lambda} = \frac{\partial H}{\partial p_i}. \tag{2.86}$$

For in that case (2.84) may be calculated from (2.85) as

$$\frac{dp_i}{d\lambda} = -p_i\frac{\partial H}{\partial \phi} - \frac{\partial H}{\partial x_i}. \tag{2.87}$$

Then if we add

$$\frac{d\phi}{d\lambda} = p_j\frac{\partial H}{\partial p_j}. \tag{2.88}$$

(2.86)–(2.88) are a complete set of $(2n+1)$ ordinary differential equations for determining a "characteristic curve" $x_i(\lambda)$ and the values of ϕ and p_i along it. In principle, the solution in a whole region can be obtained by integrating these characteristic equations along the characteristics covering the region.

In the special case of the quasi-linear equation, $H \equiv c_i(\phi, \mathbf{x})p_i - b(\phi, \mathbf{x})$, (2.86) and (2.88) reduce to

$$\frac{dx_i}{d\lambda} = c_i(\phi, \mathbf{x}),$$

$$\frac{d\phi}{d\lambda} = p_j c_j = b(\phi, \mathbf{x}),$$

which may be solved independently of (2.87). In the earlier discussion one of the x_i was the time t, the corresponding c_i was unity and the parameter λ was t itself.

CHAPTER 3

Specific Problems

In this chapter the basic ideas developed so far are applied in more detail to the particular cases raised in Section 2.2. At the same time, the general ideas can be taken further on the basis of specific sets of equations.

3.1 Traffic Flow

The application of these ideas to traffic flow was formulated and discussed independently by Lighthill and Whitham (1955) and Richards (1956). It is clear in this case that the flow velocity

$$V(\rho)=\frac{Q(\rho)}{\rho}$$

must be a decreasing function of ρ which starts from a finite maximum value at $\rho=0$ and decreases to zero as $\rho\to\rho_j$, the value for which the cars are bumper to bumper. Thus $Q(\rho)$ is zero at both $\rho=0$ and $\rho=\rho_j$, and has a maximum value q_m at some intermediate density ρ_m. It has the general convex form shown in Fig. 3.1. Actual observations of traffic flow indicate that typical values for a single lane are $\rho_j\sim 225$ vehicles per mile, $\rho_m\sim 80$ vehicles per mile, $q_m\sim 1500$ vehicles per hour. It appears to be roughly correct to multiply these values by the number of lanes for multilane highways. It is interesting that, according to these figures, the maximum flow rate q_m is attained at a low velocity in the neighborhood of 20 miles per hour.

The propagation velocity for the waves is

$$c(\rho)=Q'(\rho)=V(\rho)+\rho V'(\rho).$$

Since $V'(\rho)<0$, the propagation velocity is less than the car velocity; waves propagate backward through the stream of traffic and drivers are warned of disturbances ahead. The velocity c is the slope of the (q,ρ) curve so the

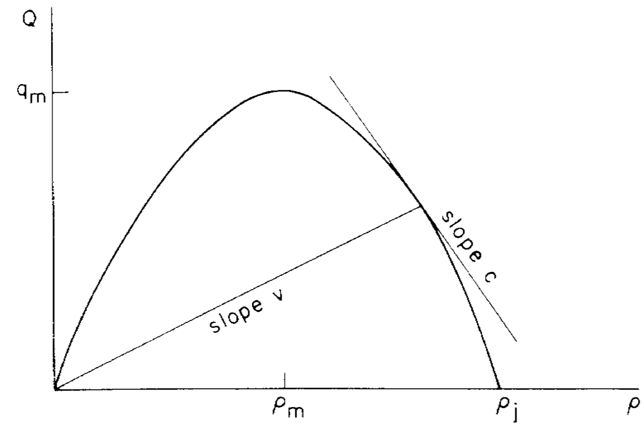

Fig. 3.1. Flow-density curve in traffic flow.

waves move forward or backward relative to the road depending on whether $\rho<\rho_m$ or $\rho>\rho_m$. At the maximum flow rate, $\rho=\rho_m$, the waves are stationary relative to the road, so the propagation velocity relative to the cars is then the same as $q_m/\rho_m\sim 20$ mph.

Near $\rho=\rho_j$, we can make a rough estimate on the basis of a simple reaction time argument. If we assume that a driver and his car take a time δ to react to any change ahead, then the gap between cars should be kept at $V\delta$ for safety. If h is the headway, defined as the distance between the front ends of successive cars, and L is the typical car length, this leads to

$$V=\frac{h-L}{\delta}.$$

Since $h=1/\rho$, $L=1/\rho_j$, we have

$$V(\rho)=\frac{L}{\delta}\left(\frac{\rho_j}{\rho}-1\right),\qquad Q(\rho)=\frac{L}{\delta}(\rho_j-\rho).$$

One should probably interpret this as an estimate of the slope of the $Q(\rho)$ curve at ρ_j, rather than as a realistic prediction of a linear dependence on ρ. In any event, it gives $c_j=-L/\delta$ for the propagation velocity there. In the traffic flow context δ is usually estimated in the range 0.5–1.5 sec, although in other circumstances the human reaction time can be much faster. With $L=20$ ft, $\delta=1$ sec, we have $c_j\sim -14$ mph.

Greenberg (1959) found a good fit with data for the Lincoln Tunnel in

New York by taking

$$Q(\rho) = a\rho \log \frac{\rho_j}{\rho},$$

with $a = 17.2$ mph, $\rho_j = 228$ vpm (vehicles per mile). For this formula, the relative propagation velocity $V - c$ is equal to the constant value a at all densities. The values of ρ_m and q_m are $\rho_m = 83$ vpm, $q_m = 1430$ vph (vehicles per hour). The logarithmic formula does not give a finite value for V as $\rho \to 0$, but the theory would be on dubious ground for very light traffic so this point alone is not important. With a finite maximum V and a finite $V'(\rho)$, we have $c \to V$ as $\rho \to 0$, so one should expect $V - c$ to decrease at the lighter densities.

Since $Q(\rho)$ is convex with $Q''(\rho) < 0$, c itself is always a decreasing function of ρ. This means that a local increase of density propagates as shown in Fig. 3.2 with a shock forming at the back. Individual cars move faster than the waves, so that a driver enters such a local density increase from behind; he must decelerate rapidly through the shock but speeds up only slowly as he leaves the congestion. This seems to accord with experience. The details can be analyzed by the theory of Chapter 2. In particular the final asymptotic behavior is the triangular wave which is the last profile in Fig. 3.2. The length of the wave increases like $t^{1/2}$ and the shock decays like $t^{-1/2}$. The actual analytic expressions are

$$c \sim \frac{x}{t}, \qquad \rho - \rho_0 \sim \frac{x - c_0 t}{c'(\rho_0)\, t} \qquad \text{for} \quad c_0 t - \sqrt{2Bt} < x < c_0 t,$$

where

$$B = |c'(\rho_0)| \int_{-\infty}^{\infty} (\rho - \rho_0)\, dx.$$

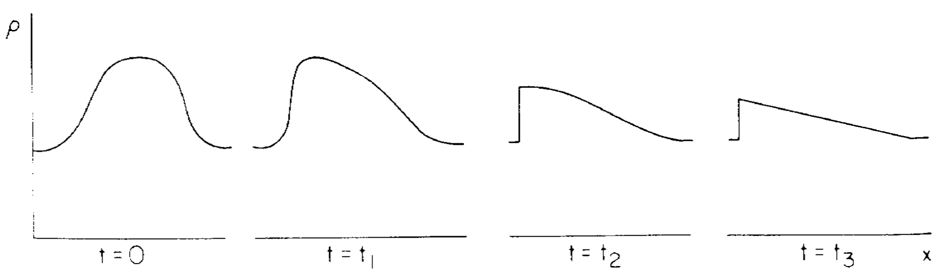

Fig. 3.2. Breaking wave in traffic flow.

The shock is at

$$x = c_0 t - \sqrt{2Bt},$$

and the jumps of c and ρ at the shock are

$$c - c_0 \sim -\sqrt{\frac{2B}{t}}, \qquad \rho - \rho_0 \sim \frac{1}{|c'(\rho_0)|} \sqrt{\frac{2B}{t}}.$$

Traffic Light Problem.

A more complicated problem is the analysis of the flow at a traffic light. We construct the characteristics in the (x, t) diagram. These are lines of constant density and their slopes $c(\rho)$ determine the corresponding values of ρ on them. So the problem is solved once the (x, t) diagram has been obtained.

Suppose first that the red period of the light is long enough to allow the incoming traffic to flow freely at some value $\rho_i < \rho_m$. Then we may start with characteristics of slope $c(\rho_i)$ intersecting the t axis in the interval AB in Fig. 3.3; AB is part of a green period. [The (x, t) diagram is plotted with x vertical and t horizontal since this is the usual practice in the references on traffic flow.] Just below the red period BC, the cars are stationary with $\rho = \rho_j$; hence the characteristics have the negative slope $c(\rho_j)$. The line of separation between the stopped queue at the traffic light and the free flow must be a shock BP, and from the shock condition its velocity is

$$-\frac{q(\rho_i)}{\rho_j - \rho_i}.$$

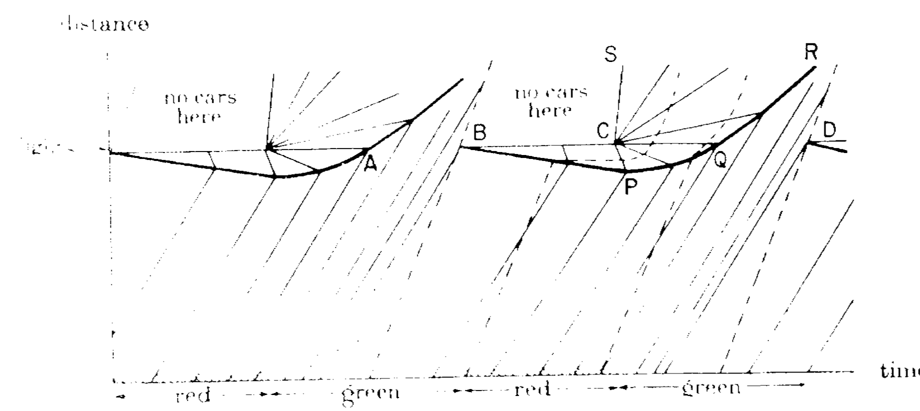

Fig. 3.3. Wave diagram for an efficient traffic light.

When the light turns green at C, the leading cars can go at the maximum speed since $\rho=0$ ahead of them. (The finite acceleration could be allowed for roughly by extending the effective red period.) This is represented by the characteristic CS with maximum slope $c(0)$. Between CS and CP we have an expansion fan with all values of c being taken. Exactly at the intersection CQ, the slope c must be zero. But this corresponds to the maximum $q=q_m$. Therefore we have the interesting result that q attains its maximum value right at the traffic light. The shock $BPQR$ is weakened by the expansion fan and ultimately accelerates through the intersection, provided the green period is long enough. The criterion for whether the shock gets through is easily established. The total incoming flow for the time BQ is $(t_r+t_s)q_i$ where t_r is the red period BC and t_s is the part of the green period before the shock gets through. The flow across the intersection in this time is $t_s q_m$. These two must be equal; therefore

$$t_s=\frac{t_r q_i}{q_m-q_i}.$$

For the shock to get through and the light to operate freely, the green period must exceed this critical value.

If the shock does not get through, the flow never becomes free and the notorious traffic crawl develops. It is perhaps sufficient to show the corresponding (x,t) diagram Fig. 3.4 without comment!

Higher Order Effects; Diffusion and Response Time.

There are two obvious additional effects one may wish to include in the theory. One was mentioned in Section 2.4: the dependence of q on ρ_x as well as ρ. This introduces in a rough way the drivers' awareness of conditions ahead, and it produces a *diffusion* of the waves. The simplest assumption with the correct qualitative behavior is

$$q=Q(\rho)-\nu\rho_x, \qquad v=V(\rho)-\frac{\nu}{\rho}\rho_x, \tag{3.1}$$

and one does not have much basis for any more complicated choice.

The second effect is the time lag in the response of the driver and of his car to any changes in the flow conditions. One way to introduce this effect is to consider the expression for v in (3.1) as a desired velocity which the driver accelerates toward: therefore the equation

$$v_t+vv_x=-\frac{1}{\tau}\left\{v-V(\rho)+\frac{\nu}{\rho}\rho_x\right\} \tag{3.2}$$

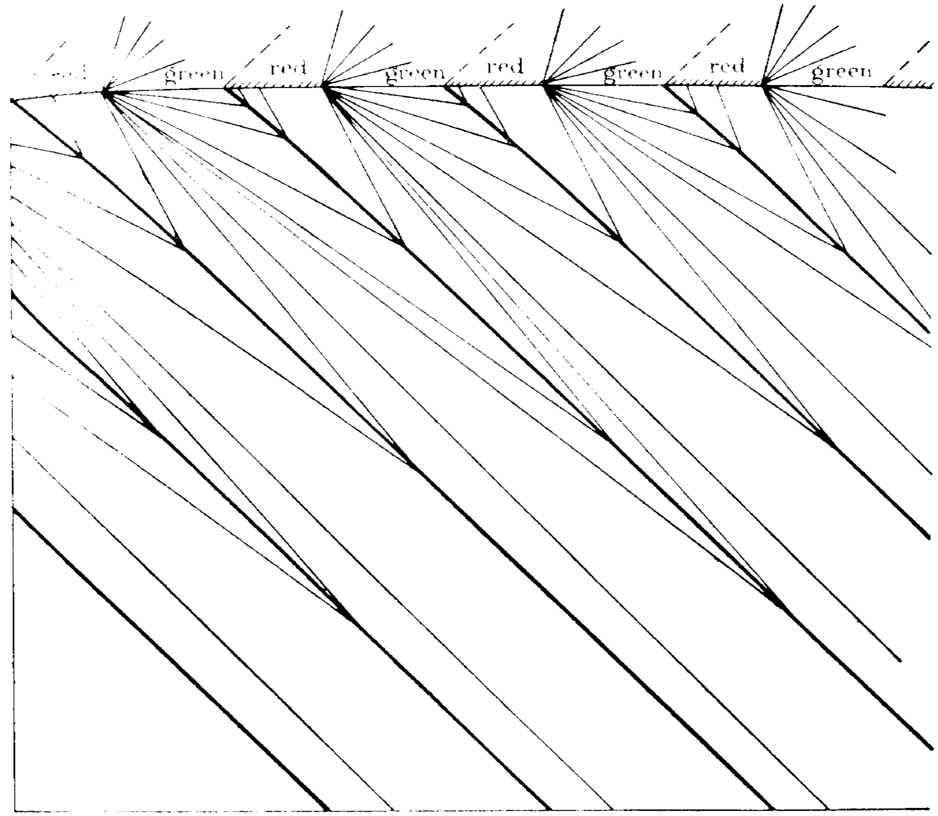

Fig. 3.4. Wave diagram for the slow crawl at an overcrowded traffic light.

may be introduced for the acceleration. The coefficient τ is a measure of the response time and is akin to the quantity δ mentioned earlier. Equation 3.2 is to be solved together with the conservation equation

$$\rho_t+(\rho v)_x=0. \tag{3.3}$$

When ν, τ are both small in a suitable nondimensional measure, (3.2) is approximated by $v=V(\rho)$, and we have the simpler theory. With the higher order terms included in (3.2), we expect shocks to appear as smooth steps and so on. This is true on the whole, but the situation turns out to be more complicated.

It is always helpful to get a first feel for a nonlinear equation by looking at the linearized theory, even though the linearization may have its own shortcomings, as we discussed in Section 2.10. If (3.2) and (3.3) are linearized for small perturbations about $\rho=\rho_0$, $v=v_0=V(\rho_0)$, by substituting

$$\rho=\rho_0+r, \qquad v=v_0+w,$$

and retaining only first powers of r and w, we have

$$\tau(w_t+v_0w_x)=-\left\{w-V'(\rho_0)r+\frac{\nu}{\rho_0}r_x\right\},$$

$$r_t+v_0r_x+\rho_0w_x=0.$$

The kinematic wave speed is

$$c_0=\rho_0V'(\rho_0)+V(\rho_0);$$

hence $V'(\rho_0)=-(v_0-c_0)/\rho_0$. Introducing this expression and then eliminating w, we have

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=\nu\frac{\partial^2 r}{\partial x^2}-\tau\left(\frac{\partial}{\partial t}+v_0\frac{\partial}{\partial x}\right)^2 r. \tag{3.4}$$

When $\nu=\tau=0$, we have the linearized approximation to the kinematic waves: $r=f(x-c_0t)$. The term proportional to ν introduces typical diffusion of the heat equation type. The effect of the finite response time τ is more complicated, but a quick insight can be gained as follows. In the basic wave motion governed by the left hand side, $r=f(x-c_0t)$, so that t derivatives are approximately equal to $-c_0$ multiplied by x derivatives:

$$\frac{\partial}{\partial t}\simeq -c_0\frac{\partial}{\partial x}. \tag{3.5}$$

If this approximation is used in the right hand side of (3.4), the equation reduces to

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=\left\{\nu-(v_0-c_0)^2\tau\right\}\frac{\partial^2 r}{\partial x^2}. \tag{3.6}$$

There is a combined diffusion when

$$\nu>(v_0-c_0)^2\tau \tag{3.7}$$

but instability if

$$\nu<(v_0-c_0)^2\tau. \tag{3.8}$$

This is reasonable: for stability a driver should look far enough ahead to make up for his response time.

The stability criterion can be verified directly from the complete equation (3.4) in the traditional way. There are exponential solutions of (3.4) with

$$r\propto e^{ikx-i\omega t}$$

provided that

$$\tau(\omega-v_0k)^2+i(\omega-c_0k)-\nu k^2=0.$$

The exponential solutions will be stable provided $\mathscr{I}\omega<0$ for both of the roots ω. It is easily verified that the requirement for this is (3.7), so the result of the approximate procedure is confirmed and extended to all wavelengths.

Higher Order Waves.

It is important to note that the right hand side of (3.4) is itself a wave operator and we may write the equation as

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=-\tau\left(\frac{\partial}{\partial t}+c_+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}+c_-\frac{\partial}{\partial x}\right)r, \tag{3.9}$$

where

$$c_+=v_0+\sqrt{\nu/\tau}\,, \qquad c_-=v_0-\sqrt{\nu/\tau}\,.$$

It would be expected therefore that waves traveling with speeds c_+ and c_- also play some role. It would be premature to go deeply into this question at this stage, but one remark has great significance in interpreting the stability condition. We shall see later in our discussion of higher order equations that the propagation speeds in the highest order derivatives always determine the fastest and slowest signals. Thus in the present case however small τ may be provided it is nonzero, the fastest signal travels with speed c_+ and the slowest with speed c_-. It is clear therefore that the approximation

$$\frac{\partial r}{\partial t}+c_0\frac{\partial r}{\partial x}=0 \tag{3.11}$$

could only make sense if

$$c_-<c_0<c_+. \tag{3.12}$$

But this is exactly the stability criterion (3.7). So the flow is stable only if (3.12) holds, and then it is appropriate to approximate (3.9) by (3.11) for small τ. There is a nice correspondence between stability and wave interaction.

Equation 3.9 arises in several applications and a full discussion is given in Chapter 10.

Shock Structure.

The more complicated form of the higher order corrections introduces a new possibility in the shock structure. For the simple diffusion term used in Section 2.4 with $\nu > 0$, a continuous shock structure was obtained. We shall see now that this is not always the case when there are additional higher terms. We look for a steady profile solution of (3.2)–(3.3) with

$$\rho = \rho(X), \qquad v = v(X), \qquad X = x - Ut,$$

where U is the constant translational velocity. Equation 3.3 becomes

$$-U\rho_X + (v\rho)_X = 0 \tag{3.13}$$

and may be integrated to

$$\rho(U - v) = A, \tag{3.14}$$

where A is a constant. Equation 3.2 becomes

$$\tau\rho(v - U)v_X + \nu\rho_X + \rho v - Q(\rho) = 0. \tag{3.15}$$

Since $v = U - A/\rho$, this may be reduced to

$$\left(\nu - \frac{A^2}{\rho^2}\tau\right)\rho_X = Q(\rho) - \rho U + A. \tag{3.16}$$

For $\tau = 0$, it is the same as (2.21), as it should be. For $\tau \neq 0$, the possibility that $\nu - A^2\tau/\rho^2$ may vanish introduces the new effects.

As before we are interested in solution curves between ρ_1 at $X = +\infty$ and ρ_2 at $X = -\infty$. These values will be zeros of the right hand side of (3.16). For traffic flow $c'(\rho) = Q''(\rho) < 0$, so $\rho_2 < \rho_1$ and the right hand side of (3.16) is positive for $\rho_2 < \rho < \rho_1$. If $\nu - A^2\tau/\rho^2$ remains positive in this range, then $\rho_X > 0$ and we have a smooth profile as in Fig. 3.5. In view of (3.14), the condition for $\nu - A^2\tau/\rho^2$ to remain positive may be written

$$\nu > (v - U)^2\tau, \qquad \text{that is,} \quad v - \sqrt{\nu/\tau} < U < v + \sqrt{\nu/\tau}. \tag{3.17}$$

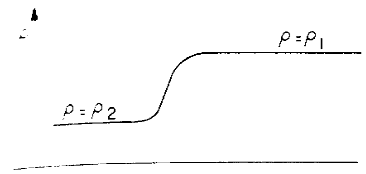

Fig. 3.5 Continuous shock structure.

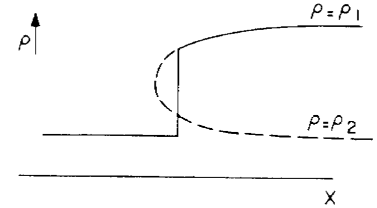

Fig. 3.6. Shock structure with an inner discontinuity.

This is similar in form to the linearized stability criterion (3.7) with v_0 replaced by the local velocity v and c_0 replaced by the shock velocity U. We might also interpret it in a way similar to (3.12) as a warning of possible complications if a shock tries to violate the higher order signal speeds. However, it is not necessarily an unstable situation. The conditions for the uniform states at $\pm\infty$ to be stable are

$$v_1 - \sqrt{\nu/\tau} < c_1 < v_1 + \sqrt{\nu/\tau}, \qquad v_2 - \sqrt{\nu/\tau} < c_2 < v_2 + \sqrt{\nu/\tau}. \tag{3.18}$$

It is possible, in general, for these to be satisfied and yet (3.17) to be violated. When this is the case, $\nu - A^2\tau/\rho^2$ changes sign in the profile, as in Fig. 3.6, and a single-valued continuous profile is no longer possible.

In most problems of shock structure, when the profile turns back on itself in this way, it is rectified by fitting in an appropriate discontinuity. The situation again corresponds, strictly speaking, to a breakdown of the assumptions for the particular level of description, but the introduction of a discontinuity, *provided it corresponds to a valid integrated form of the basic equations*, avoids an explicit discussion of yet higher order effects. In the case of (3.2) and (3.3), it is not clear which conservation forms are appropriate for the discontinuity conditions nor what additional effects should be introduced. One expects a discontinuous profile shown by the full curve in Fig. 3.6, but the precise determination of the discontinuity is not clear for this case. In other cases discussed later the details can be completed. The point to stress here is that the discontinuities in the simple theory using

$$\rho_t + c(\rho)\rho_x = 0$$

may be only partially resolved into continuous transitions in a more accurate formulation.

A Note on Car-following Theories.

Considerable work has been done on discrete models where the motion of the nth car in a line of cars is prescribed in terms of the motion of the other cars. [See, for example, Newell (1961) and the earlier references given there.] If the position of the nth car is $s_n(t)$ at time t, the assumed laws of motion usually take the form

$$\dot{s}_n(t+\Delta) = G\{s_{n-1}(t) - s_n(t)\}, \tag{3.19}$$

between velocity $\dot{s}_n$ and headway $h_n = s_{n-1} - s_n$ with a time lag Δ to account for the driver response time. If $G(h_n)$ is chosen to be linear in h_n, or if the equation is linearized to study fluctuations about a uniform state, solutions can be obtained by Laplace transforms. In general, however, one must appeal to computer studies.

This type of model takes a more rigid view of how each individual car moves, so it is narrower in scope than the continuum theory, where the whole complicated behavior of the individuals is lumped together in the function $Q(\rho)$ and the parameters ν and τ. But each model leads to a particular form for these quantities, which may be helpful in interpreting observational data. Moreover, such models may lead to additional effects that cannot be seen in the continuum theory.

To see the correspondence of the particular car-following model in (3.19) with the continuum theory, we first note the relation of $G(h)$ to $Q(\rho)$. In a uniform stream with equal spacing h, the velocities in (3.19) are all equal and are given by the relation $v = G(h)$. Since $h = 1/\rho$, $v = q/\rho$, the function $Q(\rho)$ in the corresponding continuum equations is

$$Q(\rho) = \rho G\left(\frac{1}{\rho}\right).$$

If empirical or other information is known about $G(h)$, it may be transferred to information about $Q(\rho)$ near $\rho = \rho_j$. Of course at lower densities $Q(\rho)$ will be affected more by cars overtaking and changing lanes.

The wave propagation described by (3.19), in which the motion of a lead car is transmitted successively back through the stream, should be a typical finite difference version of the earlier continuum results with this choice of $Q(\rho)$. The finite difference form of (3.19) also introduces higher order effects equivalent to those in (3.2) and we can make a detailed comparison. If we let

$$v_n(t) = \dot{s}_n(t), \qquad s_{n-1}(t) - s_n(t) = h_n(t), \tag{3.20}$$

(3.19) is equivalent to the pair of equations

$$v_n(t+\Delta) = G(h_n), \tag{3.21}$$

$$\frac{dh_n}{dt} = v_{n-1}(t) - v_n(t). \tag{3.22}$$

In this form we introduce continuous functions $v(x,t)$ and $h(x,t)$ such that

$$v(s_n, t) = v_n(t), \tag{3.23}$$

$$h\left(\frac{s_{n-1}+s_n}{2}, t\right) = h_n(t), \tag{3.24}$$

and obtain corresponding partial differential equations in the approximations of small Δ and small h_n. Equation 3.21 may be written

$$v\{s_n(t+\Delta), t+\Delta\} = G\left\{h\left(s_n + \frac{1}{2}h_n, t\right)\right\},$$

and it may be approximated by

$$v + (v_t + vv_x)\Delta = G(h) + \frac{1}{2}hG'(h)h_x, \tag{3.25}$$

where the functions are evaluated at $x = s_n(t)$ and the errors are of order Δ^2, h^2. Equation 3.22 may be written

$$\frac{d}{dt}h\left(\frac{s_{n-1}+s_n}{2}, t\right) = v(s_{n-1}, t) - v(s_n, t)$$

and approximated by

$$h_t + vh_x = hv_x \qquad \text{at } x = \frac{s_{n-1}+s_n}{2}. \tag{3.26}$$

The error in (3.26) is *third* order in h [due to the centering of h at the midpoint $(s_{n-1}+s_n)/2$], so the equation is correct to both first and second orders. In terms of $\rho = 1/h$, $V(\rho) = G(h)$, (3.25)–(3.26) become

$$v + (v_t + vv_x)\Delta = V(\rho) + \frac{1}{2}\frac{V'(\rho)}{\rho}\rho_x, \tag{3.27}$$

$$\rho_t + (\rho v)_x = 0. \tag{3.28}$$

To lowest order in Δ and h, we would have

$$v = V(\rho), \qquad \rho_t + (\rho v)_x = 0,$$

which is just the kinematic theory. The differencing has been arranged so that the next order corrections leave the conservation equation (3.28) unchanged.

Equations 3.27 and 3.28 are identical with (3.2) and (3.3) if we take

$$\tau = \Delta, \qquad \nu = -\frac{1}{2}V'(\rho).$$

Since $V - c = -\rho V'(\rho)$, the stability criterion (3.7) may be written

$$2\rho^2|V'(\rho)|\Delta < 1,$$

or, equivalently,

$$2G'(h)\Delta < 1.$$

This is exactly the condition found in the car-following theories (Chandler, Herman, and Montroll, 1958; Kometani and Sasaki, 1958). Similarly the shock structures discussed earlier on the basis of (3.2) should be close to those discussed by Newell (1961) on the basis of (3.19).

An effect that cannot be covered by the continuum theory is the actual collision of cars. In a queue described by (3.19), this occurs if $s_{n-1} - s_n$ ever drops to the car length L. In the special case

$$\dot{s}_n(t+\Delta) = \alpha\{s_{n-1}(t) - s_n(t) - L\},$$

which can be solved by Laplace transforms, it may be shown that the criterion for avoiding collision is

$$\alpha\Delta < \frac{1}{e};$$

this is slightly more stringent than the stability criterion $2\alpha\Delta < 1$ found above. The analysis would take us too far afield and the reader is referred to the discussion of local stability in the paper by Herman, Montroll, Potts, and Rothery (1959).

3.2 Flood Waves

For flood waves, the "density" in the sense of the general theory presented in Chapter 2 is the cross-sectional area of the riverbed, $A(x,t)$, at position x along the river at time t. If the volume flow across the section is $q(x,t)$ per unit time, the conservation equation is

$$\frac{d}{dt}\int_{x_2}^{x_1} A(x,t)\,dx + q(x_1,t) - q(x_2,t) = 0,$$

or, in differentiated form,

$$\frac{\partial A}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{3.29}$$

Flow in a river is obviously so complicated that any flow model for the second relation between q and A must be extremely approximate, giving only qualitative effects and general order of magnitude results for propagation speeds, wave profiles, and so on. However, observations during slow changes in the river level may be used also to establish the dependence of depth and the area A on the flow q. These provide empirical curves for the function

$$q = Q(A,x) \tag{3.30}$$

in steady flows. This relation can be combined with (3.29) to give a first approximation for unsteady flows which vary slowly. Then $A(x,t)$ satisfies

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial A}\frac{\partial A}{\partial x} = -\frac{\partial Q}{\partial x}. \tag{3.31}$$

We have again the theory discussed in Chapter 2 with the propagation velocity

$$c = \frac{\partial Q}{\partial A} = \frac{1}{b}\frac{\partial Q}{\partial h}. \tag{3.32}$$

[The second form introduces the breadth b and depth h, and $dA = b\,dh$.] This is the Kleitz-Seddon formula for flood waves, apparently established first by Kleitz (1858, unpublished) and thoroughly discussed and used effectively by Seddon (1900).

Empirical relations for (3.30) can be viewed against simple theoretical models. The relation is an expression of the balance between the frictional force of the river bed and the gravitational force. In theoretical models, the frictional force is usually assumed to be proportional to v^2, where v is the average velocity

$$v = \frac{q}{A},$$

and also proportional to the wetted perimeter P of the cross-section at position x. This force is then expressed as $\rho_0 C_f P v^2$ per unit length of river, where ρ_0 is the density of water and C_f is a friction coefficient. The gravitational force is $\rho_0 g A \sin\alpha$ per unit length, where α is the angle of inclination of the surface of the river. Hence

$$v = \sqrt{\frac{A}{P}\frac{g\sin\alpha}{C_f}}\ ,$$
$$Q = vA = \sqrt{\frac{A^3}{P}\frac{g\sin\alpha}{C_f}}\ . \tag{3.33}$$

The wetted perimeter P is a function of A, and C_f may also be allowed to depend on A. For broad rivers P varies little with the depth of the river and may be taken to be constant. If C_f and α are also taken to be constants, (3.33) gives the Chezy law

$$v \propto A^{1/2}, \qquad Q \propto A^{3/2}.$$

Then the propagation velocity

$$c = \frac{d}{dA}(vA) = v + A\frac{dv}{dA} = \frac{3}{2}v.$$

More generally, P and C_f are functions of A, and power law dependences for these give $v \propto A^n$, $Q \propto A^{1+n}$ with other values for n. For example, a triangular cross-section gives $P \propto A^{1/2}$ and leads to $n = \frac{1}{4}$; Manning's law $C_f \propto A^{-1/3}$ leads to $n = \frac{2}{3}$. For all these power laws the propagation velocity is

$$c = (1+n)v.$$

As expected, flood waves move faster than the fluid but the propagation velocity may not be very much greater than the fluid velocity.

Seddon turns the calculation around and uses his observations of the propagation velocity to deduce the effective shape of the bed, that is, the dependence of P on A. This is a valuable idea in all kinematic wave problems: use observations of the propagation velocity c to infer the q-ρ relation.

If the dependence of Q on x is omitted, (3.31) reduces to

$$A_t + c(A)A_x = 0,$$

and the general solution may be taken from Chapter 2 with shocks fitted in

as discontinuities satisfying

$$U = \frac{q_2 - q_1}{A_2 - A_1}.$$

For the power laws suggested (and this is also borne out by observations), $c(A)$ is an increasing function of A: hence waves due to an increase in height break forward, and shocks carry an increase in height, $A_2 > A_1$.

Higher Order Effects.

As in the other examples discussed, a more accurate treatment of the relation between q and A than that expressed in (3.30) involves higher derivatives. In unsteady flow the frictional and gravitational forces do not balance exactly and their difference is proportional to the acceleration of the fluid: the difference between the slope of the water surface and the slope of the bottom also makes a contribution.

It will be valuable to express the equations in conservation form so that, when necessary, appropriate discontinuity conditions can also be deduced. For simplicity, we consider the case of a broad rectangular channel of constant inclination α and work with the depth h and mean velocity v as basic variables in place of A and q. The conservation of fluid for unit breadth can then be written

$$\frac{d}{dt}\int_{x_2}^{x_1} h\,dx + [hv]_{x_2}^{x_1} = 0, \tag{3.34}$$

and we need to add a more detailed formulation of the conservation of momentum. The appropriate equation in hydraulic theory is

$$\frac{d}{dt}\int_{x_2}^{x_1} hv\,dx + [hv^2]_{x_2}^{x_1} + \left[\frac{1}{2}gh^2\cos\alpha\right]_{x_2}^{x_1}$$
$$= \int_{x_2}^{x_1} gh\sin\alpha\,dx - \int_{x_2}^{x_1} C_f v^2\,dx. \tag{3.35}$$

Apart from the common factors ρ_0 (the constant density of water) and the breadth b which have been cancelled through, the five terms in this equation are, respectively, (1) the rate of increase of momentum in the section $x_2 < x < x_1$, (2) the net transport of momentum across x_1 and x_2, (3) the net total pressure force acting across x_1 and x_2, (4) the component of the gravitational force down the incline, and (5) the frictional effects of the bottom. The pressure term requires some comment perhaps. In hydraulic theory the dependence of the velocity on the coordinate y normal to the bed is averaged out to $v(x,t)$ and the fluid acceleration in the y direction is neglected. The latter assumption means that the pressure satisfies a hydrostatic law

$$\frac{\partial p}{\partial y} = -\rho_0 g \cos\alpha.$$

Hence

$$p - p_0 = (h-y)\rho_0 g \cos\alpha$$

and the total contribution of the perturbed pressure integrated over a cross section of the river is

$$b\int_0^h (p-p_0)\,dy = \frac{1}{2}h^2\rho_0 g b\cos\alpha;$$

this is the origin of the third term in (3.35).

Equations 3.34 and 3.35 are the two conservation equations for h and v. If h and v are assumed to be continuously differentiable, we may take the limit $x_1 - x_2 \to 0$ to obtain partial differential equations for h and v. It will be a minor saving in writing to introduce $g' = g\cos\alpha$ and the slope $S = \tan\alpha$. The equations for h and v are then

$$\begin{aligned} &h_t + (hv)_x = 0, \\ &(hv)_t + \left(hv^2 + \frac{1}{2}g'h^2\right)_x = g'hS - C_f v^2. \end{aligned} \tag{3.36}$$

We may also use the first equation to simplify the second and take the equivalent pair

$$\begin{aligned} &h_t + vh_x + hv_x = 0, \\ &v_t + vv_x + g'h_x = g'S - C_f\frac{v^2}{h}. \end{aligned} \tag{3.37}$$

The kinematic wave approximation to (3.37) neglects the left hand side of the second equation and takes

$$h_t + (hv)_x = 0, \qquad v = \left(\frac{g'S}{C_f}\right)^{1/2} h^{1/2}. \tag{3.38}$$

In this kinematic theory, discontinuous shocks must satisfy the shock condition

$$U = \frac{v_2h_2 - v_1h_1}{h_2 - h_1}. \tag{3.39}$$

Stability; Roll Waves.

We now consider the consequences of the additional terms in (3.37). For simplicity, S and C_f are assumed to be constant. As in the traffic flow problem, we look first at the linearized form of the equations for small perturbations about a constant state $v = v_0$, $h = h_0$, where

$$C_f\frac{v_0^2}{h_0} = g'S. \tag{3.40}$$

If we substitute

$$v = v_0 + w, \qquad h = h_0 + \eta$$

and neglect all but the first powers of w and η, we have

$$\begin{aligned} &\eta_t + v_0\eta_x + h_0 w_x = 0, \\ &w_t + v_0 w_x + g'\eta_x + g'S\left(\frac{2w}{v_0} - \frac{\eta}{h_0}\right) = 0. \end{aligned}$$

We may then eliminate w and write the single equation for η in the form

$$\left(\frac{\partial}{\partial t} + c_+\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_-\frac{\partial}{\partial x}\right)\eta + \frac{2g'S}{v_0}\left(\frac{\partial}{\partial x} + c_0\frac{\partial}{\partial x}\right)\eta = 0, \tag{3.41}$$

where

$$c_+ = v_0 + \sqrt{g'h_0}, \qquad c_- = v_0 - \sqrt{g'h_0}, \qquad c_0 = \frac{3v_0}{2}. \tag{3.42}$$

The equation is now the same as in the earlier discussion of (3.4) and (3.9), with appropriate changes in the expressions for c_+, c_-, and c_0. Accordingly, the stability condition is

$$c_- < c_0 < c_+, \tag{3.43}$$

and this also ensures that the lower order approximation

$$\frac{\partial \eta}{\partial t} + c_0\frac{\partial \eta}{\partial x} = 0 \tag{3.44}$$

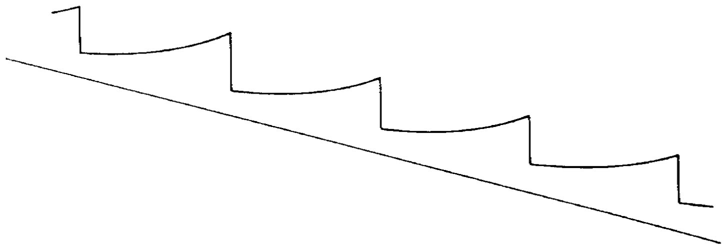

Fig. 3.7. Roll waves.

does not violate the characteristic condition. Equation 3.44 is the linearized version of (3.38), of course.

The stability conditions may also be written, using (3.42), as

$$v_0 < 2\sqrt{g' h_0}\,,$$

or, again, from (3.40) as

$$S < 4C_f.$$

For rivers, v_0 is usually much less than $\sqrt{g' h_0}$, but spillways from dams and other man-made conduits easily exceed the critical values. The resulting flow is not necessarily completely chaotic and without structure. In favorable circumstances, it takes the form of "roll waves," as shown in Fig. 3.7, with a periodic structure of discontinuous bores separated by smooth profiles. Early observational data and photographs of the phenomenon were obtained by Cornish in 1905 and are beautifully described in his classic book (Cornish, 1934), which summarizes his observations of waves in sand and water. The most specific data refer to a stone conduit in the Alps (the Grünnbach. Merligen) with a slope of 1 in 14. On an occasion when the mean depth was approximately 3 in., the mean flow velocity was estimated as 10 ft/sec and the whole roll wave pattern moved downstream at an average speed of 13.5 ft/sec. For these figures the Froude number $v_0/\sqrt{g' h_0}$ is 3.5, considerably in excess of the critical value of 2. These values would give $S/C_f = 12.5$ and lead to $C_f \approx 0.006$.

Jeffreys (1925) proposed the instability argument and noted that for smooth cement channels (for which he performed experiments) the friction coefficient is $C_f \approx 0.0025$; this value of C_f agrees with current values. For the latter value, uniform flows should become unstable when the slope S exceeds 1 in 100. Jeffreys found his own experiments on the production of roll waves inconclusive, but he felt that long channels with slopes considerably in excess of 1 in 100 were needed. Much later Dressler (1949)

took up the subject and showed how to construct nonlinear solutions of (3.36), with appropriate jump conditions, to describe the roll wave pattern. The details will be indicated after the question of steady profile waves for the stable case has been considered.

Monoclinal Flood Wave.

The structure of the shocks arising in the kinematic theory (3.38 and 3.39) is particularly important in the flood wave problem, since in reality the shock thickness is of the order of 50 miles! It is obtained as usual by searching for steady profile solutions in a more detailed description which, in this case, is provided by (3.37). We look for solutions with

$$h = h(X), \qquad v = v(X), \qquad X = x - Ut.$$

The equations may be written

$$(v - U)\frac{dv}{dX} + g'\frac{dh}{dX} = g'S - C_f\frac{v^2}{h}, \tag{3.45}$$

$$h(U - v) = B, \tag{3.46}$$

where the continuity equation has been integrated to (3.46) and B is the constant of integration. The uniform states (h_1, v_1) at $X = \infty$ satisfy

$$g'S - C_f\frac{v_1^2}{h_1} = g'S - C_f\frac{v_2^2}{h_2} = 0,$$

$$h_1(U - v_1) = h_2(U - v_2) = B.$$

If we express all flow quantities in terms of h_1 and h_2, we have

$$v_1^2 = \frac{S}{C_f}g'h_1, \qquad v_2^2 = \frac{S}{C_f}g'h_2, \tag{3.47}$$

$$B = \left(\frac{v_2 - v_1}{h_2 - h_1}\right)h_1h_2 = \left(\frac{g'S}{C_f}\right)^{1/2}\frac{h_1h_2}{h_1^{1/2} + h_2^{1/2}}. \tag{3.48}$$

$$U = \frac{v_2h_2 - v_1h_1}{h_2 - h_1} = \left(\frac{g'S}{C_f}\right)^{1/2}\frac{h_2^{3/2} - h_1^{3/2}}{h_2 - h_1}. \tag{3.49}$$

The last of these is exactly the shock condition governing discontinuities in the kinematic theory (3.39). This is the usual pattern and we expect the solutions of (3.45) and (3.46) to provide the structure of these kinematic shocks.

When v is eliminated from (3.45) and (3.46), the equation for $h(X)$ takes the form

$$\frac{dh}{dX} = -\frac{(B-Uh)^2 C_f - g'h^3 S}{g'h^3 - B^2}. \tag{3.50}$$

Since the numerator must vanish for $h=h_1$ and $h=h_2$, these two values must be roots of the cubic. Then the third root is

$$H = \frac{C_f}{S}\frac{B^2}{g'h_1h_2}$$

$$= \frac{h_1h_2}{(h_1^{1/2}+h_2^{1/2})^2}.$$

Since $H<h_1,h_2$, and the solution has h between h_1 and h_2, this third root $h=H$ is never a value taken in the solution considered.

Equation 3.50 may now be written

$$\frac{dh}{dX} = -S\frac{(h_2-h)(h-h_1)(h-H)}{h^3-B^2/g'}, \tag{3.51}$$

and the behavior of the solution depends critically on the sign of the denominator h^3-B^2/g' and its possible change of sign in the profile. From (3.46),

$$g'h^3 - B^2 = g'h^3 - (U-v)^2h^2$$

$$= h^2\{g'h-(U-v)^2\};$$

hence the sign is positive or negative depending on whether

$$U \lessgtr v + \sqrt{g'h}\ .$$

[From (3.48), $B>0$; therefore from (3.46), $U>v$ and U is always greater than $v-\sqrt{g'h}$.]

When $h_2 \to h_1$, we see from (3.49) that

$$U \to \frac{3}{2}\left(\frac{g'S}{C_f}\right)^{1/2} h_1^{1/2} = \frac{3}{2}v_1.$$

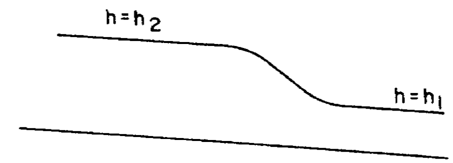

Fig. 3.8. Structure of the monoclinal flood wave.

In the stable case, $\frac{3}{2}v_1 < v_1 + \sqrt{g'h_1}$; thus for weak waves we start the integral curve of (3.51) from $h=h_1$, $X=\infty$, with the denominator in (3.51) positive. Accordingly, $h_X < 0$, h increases, $g'h^3 - B^2$ remains positive and we have a smooth profile as shown in Fig. 3.8. This is the so-called monoclinal flood wave. The fact that $h_2 > h_1$ is required for this profile agrees with the tendency of breaking of the kinematic waves, since this is a problem with $c'(h) > 0$. A smooth profile of this type will continue to hold for the range of shock velocities.

$$\frac{3v_1}{2} < U < v_1 + \sqrt{g'h_1}\ . \tag{3.52}$$

From (3.47) and (3.49) it is easily shown that this is the range

$$1 < \left(\frac{h_2}{h_1}\right)^{1/2} < \frac{1+\left\{1+4(S/C_f)^{1/2}\right\}^{1/2}}{2(S/C_f)^{1/2}}.$$

But (3.52) is the more significant form, in view of the physical interpretation of the velocities. The shock moves faster than the lower order waves but slower than the higher order waves in the flow ahead.

When

$$v_1 + \sqrt{g'h_1} < U < v_2 + \sqrt{g'h_2}\ ,$$

the denominator in (3.51) changes sign in the profile and the integral curve

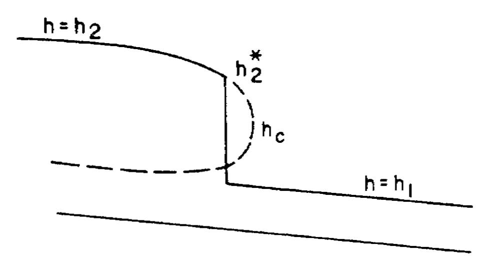

Fig. 3.9. Structure of the monoclinal flood wave with an inner discontinuity.

turns back on itself as in Fig. 3.9. A single-valued profile is recovered by fitting in a discontinuity as indicated. In contrast with the case of traffic flow, basic conservation equations in integrated form, (3.34) and (3.35), are known, and these apply whether the solution has discontinuities or not. If the same procedure developed in Section 2.3 is used on these,* the appropriate jump conditions at a discontinuity located at $x=s(t)$ are

$$-\dot{s}[h]+[hv]=0, \tag{3.53}$$

$$-\dot{s}[hv]+\left[hv^2+\frac{1}{2}g'h^2\right]=0. \tag{3.54}$$

It should be especially noted that the right hand side of (3.35) makes no contribution in the final limit $x_1 \to x_2$. These discontinuity conditions go along with the equations (3.36), just as (3.39) goes along with (3.38). One must be careful to pair correctly the equations and shock conditions at each level of description. In a change of the level of description, both the equations and shock conditions change in number. The discontinuities described by (3.53) and (3.54) are in reality the turbulent bores familiar in water wave theory as "hydraulic jumps" or breakers on a beach.

In the present context, the proposal is to fit a discontinuity satisfying (3.53) and (3.54) into the steady profile solution of (3.36); hence it will also have the velocity U. In view of (3.46) any discontinuity between branches of the profile [including the lines $h=h_1$ and $h=h_2$ as possible solutions of (3.51)] will automatically have $h(U-v)$ continuous to satisfy (3.53). The second requirement (3.54) determines where it should be placed. The condition (3.54) requires

$$hv(v-U)+\frac{1}{2}g'h^2$$

to be continuous. From (3.46), this can be modified into

$$\frac{B^2}{h}+\frac{1}{2}g'h^2$$

should be continuous. If the discontinuity is chosen to take the profile from $h=h_1$ to a point $h=h^*$ on the upper branch as shown in Fig. 3.9, the

*Here to complete the physical problem we are requiring results from the later mathematical development in Chapters 5 and 10, but it seems better to include them here, with a minimum of explanation, rather than delaying completion of the solution.

requirement is

$$\frac{B^2}{h^*}+\frac{1}{2}g'h^{*2}=\frac{B^2}{h_1}+\frac{1}{2}g'h_1^2,$$

that is,

$$\frac{h^*}{h_1}=\frac{\{1+8B^2/g'h_1^3\}^{1/2}-1}{2}.$$

This can be expressed in terms of h_1 and h_2 using (3.48). It can be verified that it meets the requirements (1) that $g'h^{*3}-B^2>0$ so that $h=h^*$ is on the upper branch, and (2) that $h^*<h_2$ provided $S<4C_f$.

The overall conclusion, then, is that the original discontinuity of the kinematic theory (3.38 and 3.39) is resolved into a smooth profile when viewed from the more detailed description (3.36), *provided (3.52) is satisfied.* For stronger shocks that violate (3.52), some discontinuity remains and corresponds to a shock discontinuity in the solution of (3.36). Further interpretations of the significance of (3.52) will be given (see Chapter 10) after the theory of characteristics and shocks for higher order systems has been developed in detail.

The roll wave patterns referred to earlier are obtained by piecing together smooth sections satisfying (3.50) with discontinuous bores satisfying (3.53) and (3.54). It may be shown that $g'h^3-B^2$ must change sign in the profile but the smooth parts are kept monotonic by demanding that the numerator of (3.50) also vanish at the critical depth. This requirement relates the two parameters B and U; one or the other may be kept as a basic parameter in the family of solutions and is determined by the total volume flow. For further details, reference should be made to Dressler's (1949) paper.

3.3 Glaciers

Nye (1960, 1963) has pointed out that these ideas on flood waves apply equally to the study of waves on glaciers and has developed the particular aspects that are most important there. He refers to Finsterwalder (1907) for the first studies of wave motion on glaciers and to independent formulations by Weertman (1958).

In view of the difficulties of collecting data on the flow curves for glaciers, due to both the inaccessibility and the slowness of the flow, more reliance is placed on semitheoretical derivations. These consider in more detail the shearing motion in two dimensional steady flow down a constant slope. Let $u(y)$ be the velocity of the layer at a distance y from the ground and let $\tau(y)$ be the shearing stress. For ice it seems to be appropriate to take the stress-strain relation as

$$\mu\frac{du}{dy}=\tau^n. \tag{3.55}$$

where $n\approx 3$ or 4. (Newtonian viscosity would be the case $n=1$.) In addition, ice slips in its bed according to the approximate law

$$\nu u(0)=\tau^m(0), \tag{3.56}$$

where $m\approx\frac{1}{2}(n+1)\approx 2$. On the layer between y and $y+\delta y$, the difference in shearing stress must balance the gravitational force. If α is the angle of the slope and ρ is the density of ice, we have

$$\delta\tau=-\rho\,\delta y\,g\sin\alpha.$$

That is,

$$\frac{d\tau}{dy}=-\rho g\sin\alpha. \tag{3.57}$$

Since τ vanishes at the surface $y=h$, the solution for τ is

$$\tau=(h-y)\rho g\sin\alpha.$$

Then, integrating (3.55) with boundary condition (3.56), we have

$$u(y)=\frac{(\rho g\sin\alpha)^m h^m}{\nu}+\frac{1}{\mu}\frac{(\rho g\sin\alpha)^n}{n+1}\left\{h^{n+1}-(h-y)^{n+1}\right\}. \tag{3.58}$$

The flow per unit breadth is

$$\begin{aligned}Q^*(h)&=\int_0^h u\,dy\\&=\frac{(\rho g\sin\alpha)^m h^{m+1}}{\nu}+\frac{(\rho g\sin\alpha)^n h^{n+2}}{n+2}.\end{aligned} \tag{3.59}$$

For order of magnitude purposes, one may take

$$Q^*(h)\propto h^N,$$

with N roughly in the range 3 to 5. The propagation speed is

$$c=\frac{dQ^*}{dh}=Nv$$

where v is the average velocity Q^*/h. Thus the waves move about three to

five times faster than the average flow velocity. Typical velocities are of the order of 10 to 100 meters per year.

Various problems can be solved using the results and ideas of Chapter 2. An interesting question considered by Nye is the effect of periodic accumulation and evaporation of the ice; depending on the period, this may refer either to seasonal or climatic changes. To do this a prescribed source term $f(x,t)$ is added to the continuity equation; that is, one takes

$$h_t+q_x^*=f(x,t), \qquad q^*=Q^*(h,x). \tag{3.60}$$

The consequences are determined from integration of the characteristic equations

$$\frac{dh}{dt}=f(x,t)-Q_x^*(h,x),$$

$$\frac{dx}{dt}=Q_h^*(h,x).$$

The main result is that parts of the glacier may be very sensitive, and relatively rapid local changes can be triggered by the source term.

3.4 Chemical Exchange Processes; Chromatography; Sedimentation in Rivers

The formulation of equations for exchange processes between a solid bed and a fluid flowing through it was given in Section 2.2. The exchange may involve particles or ions of some substance, or it may be heat exchange between the solid bed and the fluid. Another instance is sediment transport in rivers.

The equations coupling the density ρ_f in the fluid and the density ρ_s on the solid are

$$\frac{\partial}{\partial t}(\rho_f+\rho_s)+\frac{\partial}{\partial x}(V\rho_f)=0, \tag{3.61}$$

$$\frac{\partial\rho_s}{\partial t}=k_1(A-\rho_s)\rho_f-k_2\rho_s(B-\rho_f). \tag{3.62}$$

For relatively slow changes in the densities and relatively high reaction rates k_1, k_2, the second equation is taken in the approximate quasi-equilibrium form in which the $\partial\rho_s/\partial t$ is neglected and

$$\rho_s=R(\rho_f)=\frac{k_1A\rho_f}{k_2B+(k_1-k_2)\rho_f}. \tag{3.63}$$

When this relation is substituted into (3.61), we have

$$\frac{\partial \rho_f}{\partial t} + \frac{V}{1+R'(\rho_f)} \frac{\partial \rho_f}{\partial x} = 0. \tag{3.64}$$

Thus the density changes propagate downstream at the rate

$$c = \frac{V}{1+R'(\rho_f)}, \tag{3.65}$$

and

$$R'(\rho_f) = \frac{k_1 k_2 A B}{\{k_2 B + (k_1 - k_2)\rho_f\}^2}. \tag{3.66}$$

If the densities concerned are small this is approximately

$$c = \frac{k_2 B}{k_1 A + k_2 B} V. \tag{3.67}$$

The propagation speed depends on the reaction rates involved, being slower for substances with larger attraction toward the solid. If a mixture of substances is present in the fluid at the entrance of the column and the components have different reactions rates, they will travel down the column at different speeds. In this way the column can be used to separate the mixture into bands of the individual components. If they are also colored, a spectrum is formed. This is the basic process of chromatography. The nonlinear effects produce heavier concentrations at the beginning or end of a band depending on the sign of $c'(\rho_f)$. Of course, the nonlinear equations for a single component apply only after the separation has taken place.

The shock structure and other aspects can be studied from the full equations 3.61 and 3.62. It is remarkable in this case that the full equations can be transformed (exactly) into a linear equation. This was shown by Thomas (1944). First, a moving coordinate system

$$\tau = t - \frac{x}{V}, \qquad \sigma = \frac{x}{V}$$

is introduced; the equations then take the form

$$\begin{aligned} &\frac{\partial \rho_f}{\partial \sigma} + \frac{\partial \rho_s}{\partial \tau} = 0, \\ &\frac{\partial \rho_s}{\partial \tau} = \alpha\rho_f - \beta\rho_s - \gamma\rho_s\rho_f. \end{aligned} \tag{3.68}$$

The first equation is solved identically by

$$\rho_f = \psi_\tau, \qquad \rho_s = -\psi_\sigma, \tag{3.69}$$

and the second equation provides an equation for ψ:

$$\psi_{\sigma\tau} + \alpha\psi_\tau + \beta\psi_\sigma + \gamma\psi_\sigma\psi_\tau = 0. \tag{3.70}$$

If we now make the nonlinear transformation

$$\gamma\psi = \log\chi, \tag{3.71}$$

we deduce

$$\chi_{\sigma\tau} + \alpha\chi_\tau + \beta\chi_\sigma = 0; \tag{3.72}$$

the nonlinear transformation eliminates the nonlinear term. In terms of the original variables the transformation is

$$\rho_f = \frac{1}{\gamma}\frac{\chi_t}{\chi}, \qquad \rho_s = -\frac{\chi_t + V\chi_x}{\gamma\chi}, \tag{3.73}$$

$$\chi_{tt} + V\chi_{xt} + (\alpha+\beta)\chi_t + \beta V\chi_x = 0. \tag{3.74}$$

The linear equation can be solved in general by transform methods so that in this case the solutions of the approximate equation (3.64), including shocks when necessary, may be compared in detail with the exact solution. This has been investigated extensively by Goldstein (1953) and Goldstein and Murray (1959). The exact solution endorses the views and methods for including discontinuous shocks in solutions of (3.64). The details are not given here since Burgers' equation is simpler to deal with and provides the same endorsement. Some of the relevant analysis will appear in Chapter 10 in a different connection.

For exchange processes, $\alpha = k_1 A$ and $\beta = k_2 B$ are both positive. For these signs, the uniform state is always stable. This may be checked by considering perturbations in (3.74). We note that the lower order waves travel at a speed

$$c_0 = \frac{\beta V}{\alpha+\beta},$$

while the wave speeds given by the higher order terms are $c_- = 0$ and $c_+ = V$. Thus for $\alpha > 0$, $\beta > 0$, the stability criterion $c_- < c_0 < c_+$ is satisfied.

CHAPTER 4

Burgers' Equation

The simplest equation combining both nonlinear propagation effects and diffusive effects is Burgers' equation

$$c_t + cc_x = \nu c_{xx}. \tag{4.1}$$

In (2.28) we saw that this is an exact equation for waves described by

$$\rho_t + q_x = 0, \qquad q = Q(\rho) - \nu\rho_x, \tag{4.2}$$

in the case that $Q(\rho)$ is a quadratic function of ρ. In general, if the two effects are important in a problem, there is usually some way of extracting (4.1) either as a precise approximation or as a useful basis for rough estimates.

For a general $Q(\rho)$ in (4.2), for example, the equation may be written

$$c_t + cc_x = \nu c_{xx} - \nu c''(\rho)\rho_x^2, \tag{4.3}$$

where $c(\rho) = Q'(\rho)$ as usual. The ratio of $\nu c''(\rho)\rho_x^2$ to νc_{xx} is of the order of the amplitude of the disturbance, and we therefore expect that (4.1) is a good approximation for small amplitude. We are then assuming that omission of this particular small amplitude term does not produce accumulating errors (as $t \to \infty$, say) which eventually lead to nonuniform validity. We know, in contrast, that to linearize the left hand side by $c_t + c_0 c_x$, where c_0 is some constant unperturbed value, would be disastrous in this respect. But as a check, we may verify that in the shock structure solution (see Section 2.4), where the diffusion terms are greatest, the term $\nu c''(\rho)\rho_x^2$ remains of smaller order than νc_{xx} in the strength of the shock. This kind of argument can be made the basis of formal perturbation expansions in terms of appropriate precisely defined small parameters. On the other hand, the fact that the terms retained in (4.1) represent identifiable and important phenomena, whereas the term $\nu c''(\rho)\rho_x^2$ appears more as a mathematical nuisance, leads one to suggest (4.1) as a useful overall description even beyond the range of strict validity.

In a similar fashion, Burgers' equation is relevant in higher order systems such as (3.2)–(3.3), when nonlinear propagation is combined with diffusion. Of course it is limited to the stable range and to parts of the solution where the lower order waves are dominant. The appropriate form is easily recognized and again can usually be substantiated by more formal procedures. In the case of (3.2)–(3.3), we know from (3.6) that the effective diffusivity is $\nu^* = \nu - (v_0 - c_0)^2\tau$ and we would use (4.1) with this value. Indeed, (3.6) is the fully linearized Burgers' equation for this system.

Our general purpose now is to show that the exact solution of Burgers' equation endorses the ideas regarding shocks that were developed in Chapter 2. That is, we want to confirm that as $\nu \to 0$ (in appropriate dimensionless form) the solutions of (4.1) reduce to solutions of

$$c_t + cc_x = 0, \tag{4.4}$$

with discontinuous shocks which satisfy

$$U = \frac{1}{2}(c_1 + c_2), \qquad c_2 > U > c_1, \tag{4.5}$$

and the shocks are located at the positions determined in Section 2.8.

4.1 The Cole-Hopf Transformation

Independently, Cole (1951) and Hopf (1950) noted the remarkable result that (4.1) may be reduced to the linear heat equation by the nonlinear transformation

$$c = -2\nu\frac{\varphi_x}{\varphi}. \tag{4.6}$$

This is similar to Thomas' earlier transformation of the exchange equations described in Section 3.4. It is again convenient to do the transformation in two steps. First introduce

$$c = \psi_x,$$

so that (4.1) may be integrated to

$$\psi_t + \frac{1}{2}\psi_x^2 = \nu\psi_{xx}.$$

Then introduce

$$\psi = -2\nu\log\varphi$$

to obtain

$$\varphi_t = \nu \varphi_{xx}. \tag{4.7}$$

The nonlinear transformation just eliminates the nonlinear term. The general solution of the heat equation (4.7) is well known and can be handled by a variety of methods.

The basic problem considered in Chapter 2 is the initial value problem:

$$c = F(x) \qquad \text{at } t=0.$$

This transforms through (4.6) into the initial value problem

$$\varphi = \Phi(x) = \exp\left\{ -\frac{1}{2\nu}\int_0^x F(\eta)\,d\eta \right\}, \qquad t=0, \tag{4.8}$$

for the heat equation. The solution for φ is

$$\varphi = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \Phi(\eta)\exp\left\{ -\frac{(x-\eta)^2}{4\nu t}\right\} d\eta. \tag{4.9}$$

Therefore, from (4.6), the solution for c is

$$c(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G/2\nu}\,d\eta}{\displaystyle\int_{-\infty}^{\infty} e^{-G/2\nu}\,d\eta}, \tag{4.10}$$

where

$$G(\eta; x,t) = \int_0^\eta F(\eta')\,d\eta' + \frac{(x-\eta)^2}{2t}. \tag{4.11}$$

4.2 Behavior as $\nu \to 0$

The behavior of the exact solution (4.10) is now considered as $\nu \to 0$ while x, t and $F(x)$ are held fixed. [Strictly speaking this means we consider a family of solutions with $\nu = \epsilon\nu_0$ and take the limit as $\epsilon \to 0$, holding $\nu_0, x, t, F(x)$ fixed.] As $\nu \to 0$, the dominant contributions to the integrals in (4.10) come from the neighborhood of the stationary points of G. A stationary point is where

$$\frac{\partial G}{\partial \eta} = F(\eta) - \frac{x-\eta}{t} = 0. \tag{4.12}$$

Let $\eta = \xi(x,t)$ be such a point; that is, $\xi(x,t)$ is defined as a solution of

$$F(\xi) - \frac{(x-\xi)}{t} = 0. \tag{4.13}$$

The contribution from the neighborhood of a stationary point, $\eta = \xi$, in an integral

$$\int_{-\infty}^{\infty} g(\eta) e^{-G(\eta)/2\nu}\,d\eta,$$

is

$$g(\xi)\sqrt{\frac{4\pi\nu}{G''(\xi)}}\; e^{-G(\xi)/2\nu};$$

this is the standard formula of the method of steepest descents.

Suppose first that there is only one stationary point $\xi(x,t)$ which satisfies (4.13). Then

$$\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G/2\nu}\,d\eta \sim \frac{x-\xi}{t}\sqrt{\frac{4\pi\nu}{G''(\xi)}}\; e^{-G(\xi)/2\nu}, \tag{4.14}$$

$$\int_{-\infty}^{\infty} e^{-G/2\nu}\,d\eta \sim \sqrt{\frac{4\pi\nu}{G''(\xi)}}\; e^{-G(\xi)/2\nu}, \tag{4.15}$$

and in (4.10) we have

$$c \sim \frac{x-\xi}{t}, \tag{4.16}$$

where $\xi(x,t)$ is defined by (4.13). This asymptotic solution may be rewritten

$$\begin{aligned} c &= F(\xi) \\ x &= \xi + F(\xi)t. \end{aligned} \tag{4.17}$$

It is exactly the solution of (4.4) which was discussed in (2.5) and (2.6); the stationary point $\xi(x,t)$ becomes the characteristic variable.

However, we saw that in some cases (4.17) gives a multivalued solution after a sufficient time, and discontinuities must be introduced. Yet the solution (4.10) for Burgers' equation is clearly single-valued and continuous for all t. The explanation is that when this stage is reached there are two stationary points that satisfy (4.13), and the foregoing analysis of the asymptotic behavior requires modification. If the two stationary points are denoted by ξ_1 and ξ_2 with $\xi_1 > \xi_2$, there will be contributions as shown in (4.14) and (4.15) from both ξ_1 and ξ_2. Therefore the dominant behavior will be included if we take

$$c \sim \frac{\frac{x-\xi_1}{t}\{G''(\xi_1)\}^{-1/2}e^{-G(\xi_1)/2\nu} + \frac{x-\xi_2}{t}\{G''(\xi_2)\}^{-1/2}e^{-G(\xi_2)/2\nu}}{\{G''(\xi_1)\}^{-1/2}e^{-G(\xi_1)/2\nu} + \{G''(\xi_2)\}^{-1/2}e^{-G(\xi_2)/2\nu}}. \qquad (4.18)$$

When $G(\xi_1) \neq G(\xi_2)$, the accentuation by the small denominator ν in the exponents makes one or the other of the terms overwhelmingly large as $\nu \to 0$. If $G(\xi_1) < G(\xi_2)$, we have

$$c \sim \frac{x-\xi_1}{t};$$

if $G(\xi_1) > G(\xi_2)$,

$$c \sim \frac{x-\xi_2}{t}.$$

In each case (4.17) applies with either ξ_1 or ξ_2 for ξ. But the choice is now unambiguous. Both ξ_1 and ξ_2 are functions of (x,t); the criterion $G(\xi_1) \gtrless G(\xi_2)$ will determine the appropriate choice of ξ_1 or ξ_2 for given (x,t). The changeover from ξ_1 to ξ_2 will occur at those (x,t) for which

$$G(\xi_1) = G(\xi_2).$$

From (4.11), this is when

$$\int_0^{\xi_2} F(\eta')\,d\eta' + \frac{(x-\xi_2)^2}{2t} = \int_0^{\xi_1} F(\eta')\,d\eta' + \frac{(x-\xi_1)^2}{2t}. \qquad (4.19)$$

Since ξ_1 and ξ_2 both satisfy (4.13), the condition may be written

$$\frac{1}{2}\{F(\xi_1) + F(\xi_2)\}(\xi_1 - \xi_2) = \int_{\xi_2}^{\xi_1} F(\eta')\,d\eta'. \qquad (4.20)$$

This is exactly the shock determination obtained in (2.45). The changeover in the choice of terms in (4.18) leads to the discontinuity in $c(x,t)$ in the limit $\nu \to 0$. All the details of Section 2.8 can be confirmed similarly. We conclude that solutions of Burgers' equation approach those described by (4.4) and (4.5) as $\nu \to 0$.

In reality ν is fixed, but it is relatively small and we expect that the limit solution for $\nu \to 0$ will often be a good approximation. For this argument, since ν is a dimensional quantity, we have to introduce a

nondimensional measure of ν by comparing it with some other quantity of the same dimension. This is not hard to do. In the single hump problem, for example, where $F(x)$ is as shown in Fig. 2.9, we may introduce the parameter

$$A = \int_{-\infty}^{\infty} \{F(x) - c_0\}\,dx. \qquad (4.21)$$

The dimensions of A and ν are both L^2/T, so that

$$R = \frac{A}{2\nu} \qquad (4.22)$$

is a dimensionless number, and by "ν small" we mean $R \gg 1$. If the length of the hump is L, the number R measures the ratio of the nonlinear term $(c-c_0)c_x$ to the diffusion term νc_{xx}, *in those regions where the x scale for the derivatives is L*. (Inside shocks, for example, the x scale is of smaller order.) It will be convenient to refer to R as the Reynolds number, following the practice in viscous flow.

Even with the meaning of "small ν" decided, there are distinctions between the limit solution $\nu \to 0$ and the solution for fixed small ν. As we saw in (2.26), the shock thickness tends to infinity if the strength tends to zero. Therefore for fixed R, even if it is large, any solution that includes shock formation or a shock decaying as $t \to \infty$ will not always be well approximated by the discontinuity theory in these regions. As regards a shock formation region, the precise details are not usually important; one just wants a good estimate of *where* it forms, without details of the profile, and this is provided by the discontinuity theory. The effects of diffusion on decaying shocks as $t \to \infty$ is of more interest. We will explore these questions through typical examples in the following sections.

4.3 Shock Structure

The shock structure for (4.1) satisfies

$$-Uc_X + cc_X = \nu c_{XX}, \qquad X = x - Ut.$$

Hence

$$\frac{1}{2}c^2 - Uc + C = \nu c_X.$$

If $c \to c_1, c_2$ as $X \to \pm\infty$,

$$U = \frac{1}{2}(c_1 + c_2), \qquad C = \frac{1}{2}c_1c_2.$$

and the equation may be written

$$(c-c_1)(c_2-c) = -2\nu c_X.$$

The solution is

$$\frac{X}{\nu} = \frac{2}{c_2-c_1}\log\frac{c_2-c}{c-c_1};$$

this agrees with (2.25), since $c=2\alpha\rho+\beta$ for quadratic $Q(\rho)$. Solving for c, we have

$$c = c_1 + \frac{c_2-c_1}{1+\exp\dfrac{c_2-c_1}{2\nu}(x-Ut)}, \qquad U=\frac{c_1+c_2}{2}. \tag{4.23}$$

One can study how an initial step diffuses into this steady profile by taking

$$F(x) = \begin{cases} c_1, & x>0, \\ c_2>c_1, & x<0, \end{cases}$$

in (4.10)–(4.11). The solution may be put in the form

$$c = c_1 + \frac{c_2-c_1}{1+h\exp\dfrac{c_2-c_1}{2\nu}(x-Ut)}, \qquad U=\frac{c_1+c_2}{2}. \tag{4.24}$$

where

$$h = \frac{\displaystyle\int_{-(x-c_1t)/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2}d\zeta}{\displaystyle\int_{(x-c_2t)/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2}d\zeta}. \tag{4.25}$$

For fixed x/t in the range $c_1<x/t<c_2$, $h\to 1$ as $t\to\infty$, and the solution approaches (4.23).

4.4 Single Hump

A special solution with a single hump may be obtained by taking

$$F(x) = c_0 + A\delta(x) \tag{4.26}$$

as the initial condition. The parameter A agrees with (4.21) and the

Reynolds number is $R=A/2\nu$. The constant c_0 may be omitted without loss of generality, since the substitution

$$c = c_0+\tilde{c}, \qquad x = c_0t+\tilde{x} \tag{4.27}$$

in Burgers' equation reduces it to

$$\tilde{c}_t + \tilde{c}\tilde{c}_{\tilde{x}} = \nu\tilde{c}_{\tilde{x}\tilde{x}} \tag{4.28}$$

Thus omission of c_0 is equivalent to viewing the solution from a frame of reference moving with velocity c_0. Accordingly we consider only

$$F(x) = A\delta(x). \tag{4.29}$$

The lower limit in the integral in (4.11) is arbitrary since it cancels out in (4.10). Therefore we may choose it to be $0+$ and include the δ function for $\eta<0$ but not for $\eta>0$. Then

$$G = \begin{cases} \dfrac{(x-\eta)^2}{2t}, & \eta>0, \\[2ex] \dfrac{(x-\eta)^2}{2t} - A, & \eta<0. \end{cases}$$

The integrals in the numerator of (4.10) may be evaluated and those in the denominator written in terms of the complementary error function. The result is

$$c(x,t) = \sqrt{\frac{\nu}{t}}\,\frac{(e^R-1)e^{-x^2/4\nu t}}{\sqrt{\pi}+(e^R-1)\displaystyle\int_{x/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2}d\zeta}, \qquad R=\frac{A}{2\nu}. \tag{4.30}$$

The similarity form of the solution, that is,

$$c = \sqrt{\frac{\nu}{t}}\, f\left(\frac{x}{\sqrt{\nu t}}; \frac{A}{\nu}\right),$$

could have been predicted by dimensional arguments. The only dimensional parameters in the problem, A and ν, both have dimensions L^2/T; there is no separate length and time with which to scale x and t separately.

As $R\to 0$ we would expect the diffusion to dominate over the nonlinearity. For $R\ll 1$ the denominator in (4.30) is $\sqrt{\pi}+O(R)$, uniformly in

x, t, ν; hence c may be approximated by

$$c(x,t) = \sqrt{\frac{\nu}{\pi t}}\, R e^{-x^2/4\nu t}$$

$$= \frac{A}{\sqrt{4\pi\nu t}} e^{-x^2/4\nu t}. \tag{4.31}$$

This is the source solution of the heat equation $c_t = \nu c_{xx}$, so the expectation is verified.

To discuss the behavior for large R it is convenient to introduce the similarity variable $z = x/\sqrt{2At}$ and to write (4.30) as

$$c = \sqrt{\frac{2A}{t}}\; g(z,R),$$

$$g(z,R) = \frac{(e^R - 1)}{2\sqrt{R}} \frac{e^{-z^2 R}}{\sqrt{\pi} + (e^R - 1)\int_{z\sqrt{R}}^{\infty} e^{-\zeta^2} d\zeta}, \tag{4.32}$$

$$z = \frac{x}{\sqrt{2At}}.$$

We now discuss the behavior of g as $R \to \infty$ for different ranges of z. In all cases, $e^R - 1$ may be approximated by e^R and we may use

$$g \sim \frac{1}{2\sqrt{R}} \frac{e^{R(1-z^2)}}{\sqrt{\pi} + e^R \int_{z\sqrt{R}}^{\infty} e^{-\zeta^2} d\zeta}. \tag{4.33}$$

If $z < 0$, the integral tends to

$$\int_{-\infty}^{\infty} e^{-\zeta^2} d\zeta = \sqrt{\pi}\,;$$

therefore $g \to 0$ at least like $1/\sqrt{R}$. If $z > 0$, the integral becomes small and we use the asymptotic expansion

$$\int_{\eta}^{\infty} e^{-\zeta^2} d\zeta \sim \frac{e^{-\eta^2}}{2\eta} \qquad \text{as } \eta \to \infty.$$

Therefore

$$g \sim \frac{z}{1 + 2z\sqrt{\pi R}\, e^{R(z^2-1)}}, \qquad z > 0, \quad R \to \infty. \tag{4.34}$$

If $0 < z < 1$, we have

$$g \sim z, \qquad 0 < z < 1, \quad R \to \infty, \tag{4.35}$$

whereas if $z > 1$, $g \to 0$ as $R \to \infty$. Thus $g \to 0$ except in $0 < z < 1$, and in that range $g \sim z$. In the original variables, the result reads

$$c \sim \begin{cases} \dfrac{x}{t} & \text{in } 0 < x < \sqrt{2At}\,, \\ 0 & \text{outside.} \end{cases}$$

This is the appropriate solution of (4.4) with a shock at $x = \sqrt{2At}$. The shock velocity is $U = \sqrt{A/2t}$, and c jumps from zero to $\sqrt{2A/t}$, so the shock condition (4.5) is satisfied.

The same expression (with $c_0 = 0$ to fit our assumption here) appeared in (2.52) for the ultimate behavior of the solution of (4.4) for a general single hump. That was asymptotic in a different sense; it was the behavior as $t \to \infty$ within the description provided by (4.4). For a δ function initial condition, it is valid immediately.

The shock is located at $z = 1$, and for large but finite R (4.34) shows a rapid transition from exponentially small values in $z > 1$ to $g \sim z$ in $z < 1$. In the transition layer $z \approx 1$, (4.34) may be approximated by

$$g \approx \frac{1}{1 + 2\sqrt{\pi R}\, e^{2R(z-1)}}. \tag{4.36}$$

In the original variables this would give

$$c \approx \sqrt{\frac{2A}{t}} \frac{1}{1 + \exp\left\{ \dfrac{1}{2\nu}\sqrt{\dfrac{2A}{t}}\,(x - \sqrt{2At}\,) + \frac{1}{2}\log\dfrac{2\pi A}{\nu} \right\}}. \tag{4.37}$$

It agrees with the shock profile (4.23), with $c_2 - c_1 = \sqrt{2A/t}$ and the shock located at $x = \sqrt{2At}$ to first order. From (4.36), the transition layer is of thickness $O(R^{-1})$ around $z = 1$.

There is another (weaker) transition layer at $z = 0$ to smooth out the discontinuity in derivative between $g \sim 0$ in $z < 0$ to $g \sim z$ in $0 < z < 1$. It is clear from (4.33) that this transition layer occurs for

$$z = O(R^{-1/2}),$$

and for these values (4.33) may be approximated by

$$g \approx \frac{e^{-Rz^2}}{2\sqrt{R}\int_{z\sqrt{R}}^{\infty} e^{-\zeta^2}\,d\zeta}. \tag{4.38}$$

In the original variables we have

$$c \approx \sqrt{\frac{\nu}{t}}\;\frac{e^{-x^2/4\nu t}}{\int_{x/\sqrt{4\nu t}}^{\infty} e^{-\zeta^2}\,d\zeta}. \tag{4.39}$$

The form of the solution for large R is shown in Fig. 4.1, where $g(z)=c\sqrt{t/2A}$ is plotted against z. As $R\to\infty$ the shock layer becomes a discontinuity in c and the transition layer at $x=0$ becomes a discontinuity in c_x. In the scaled variables g and z, the profile is independent of t. Therefore if the value of R provided by the initial condition is large, the shock remains relatively thin and the discontinuity theory of (4.4) is a good approximation *for all* t. This is true even though the shock strength is proportional to $\sqrt{2A/t}$ and tends to zero as $t\to\infty$.

A significant point in this connection is that the area under the profile remains constant even with diffusion included, since

$$\frac{d}{dt}\int_{-\infty}^{\infty} c\,dx = \left[\nu c_x - \frac{1}{2}c^2\right]_{-\infty}^{\infty} = 0.$$

Hence the "effective" Reynolds number defined as

$$\frac{1}{2\nu}\int_{-\infty}^{\infty} c\,dx$$

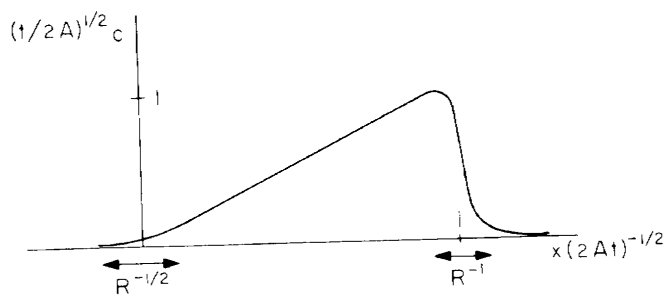

Fig. 4.1. Triangular wave solution of Burgers' equation.

remains constant for all t. The next example will show that the more usual situation is for diffusion to take over ultimately in the final decay, and that the single hump is exceptional in this respect.

4.5 *N* Wave

The final examples we consider are more easily derived by choosing appropriate solutions for φ to satisfy the heat equation (4.7) and then substituting in (4.6) to obtain c. As a rough qualitative guide to the appropriate choice, the profile for c will be something like φ_x. Thus for the single hump we could have taken the solution of φ corresponding to an initial step function. To obtain an N wave for c, we choose the source solution of the heat equation for φ:

$$\varphi = 1 + \sqrt{\frac{a}{t}}\,e^{-x^2/4\nu t}. \tag{4.40}$$

From (4.6), the corresponding solution for c is

$$c = -\frac{2\nu\varphi_x}{\varphi} = \frac{x}{t}\,\frac{\sqrt{a/t}\,e^{-x^2/4\nu t}}{1+\sqrt{a/t}\,e^{-x^2/4\nu t}}. \tag{4.41}$$

Since φ has a δ function behavior as $t\to 0$, this is a little hard to interpret as an initial value problem on c. However, for any $t>0$ it has the form shown in Fig. 4.2, with a positive and negative phase, and we may take the profile at any $t=t_0>0$ to be the initial profile. It should be typical of all N wave solutions.

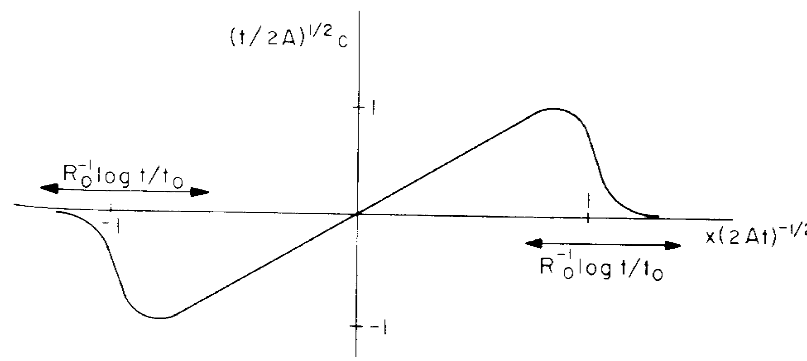

Fig. 4.2. *N* wave solution of Burgers' equation.

The area under the positive phase of the profile is

$$\int_0^\infty c\,dx = -2\nu[\log\varphi]_0^\infty$$

$$= 2\nu \log\left(1+\sqrt{\frac{a}{t}}\right). \tag{4.42}$$

The magnitude of the area of the negative phase is the same. Thus in marked contrast with the previous case, the area of the positive phase tends to zero as $t\to\infty$. If the value of (4.42) at the initial time t_0 is denoted by A, we may introduce a Reynolds number

$$R_0 = \frac{A}{2\nu} = \log\left(1+\sqrt{\frac{a}{t_0}}\right). \tag{4.43}$$

But as time goes on the *effective* Reynolds number will be

$$R(t) = \frac{1}{2\nu}\int_0^\infty c\,dx = \log\left(1+\sqrt{\frac{a}{t}}\right), \tag{4.44}$$

and this tends to zero as $t\to\infty$. If $R_0 \gg 1$, we may expect the "inviscid theory" of (4.4)–(4.5) to be a good approximation for some time, but as $t\to\infty$, $R(t)\to 0$ and the diffusion term will eventually become dominant. This is different from the previous example in which the effective Reynolds number defined in the same way remains equal to the initial Reynolds number. We now verify the details.

In terms of R_0 and t_0, $a = t_0(e^{R_0}-1)^2$; hence (4.41) may be written

$$c = \frac{x}{t}\left\{1+\sqrt{\frac{t}{t_0}}\,\frac{e^{x^2/4\nu t}}{e^{R_0}-1}\right\}^{-1}. \tag{4.45}$$

For $R_0 \gg 1$ (corresponding to $t_0 \ll a$), it may be approximated by

$$c \sim \frac{x}{t}\left\{1+\sqrt{\frac{t}{t_0}}\,e^{(x^2/2At-1)R_0}\right\}^{-1} \tag{4.46}$$

for all x and t. Now for fixed t and $R_0\to\infty$,

$$c \sim \begin{cases} \dfrac{x}{t}, & -\sqrt{2At} < x < \sqrt{2At}, \\ 0, & |x| > \sqrt{2At}. \end{cases}$$

This is exactly the inviscid solution. However, for any fixed a and ν we see directly from (4.41) [and it may be verified also from (4.46)] that

$$c \sim \frac{x}{t}\sqrt{\frac{a}{t}}\,e^{-x^2/4\nu t} \qquad \text{as } t\to\infty. \tag{4.47}$$

This is the dipole solution of the heat equation. The diffusion dominates the nonlinear term in the final decay. It should be remembered, though, that this final period of decay is for extremely large times; the inviscid theory is adequate for most of the interesting range.

4.6 Periodic Wave

A periodic solution may be obtained by taking for φ a distribution of heat sources spaced a distance λ apart. Then

$$\varphi = (4\pi\nu t)^{-1/2}\sum_{n=-\infty}^{\infty}\exp\left\{-\frac{(x-n\lambda)^2}{4\nu t}\right\}, \tag{4.48}$$

$$c = -2\nu\frac{\varphi_x}{\varphi} = \frac{\sum_{-\infty}^{\infty}\{(x-n\lambda)/t\}\exp\{-(x-n\lambda)^2/4\nu t\}}{\sum_{-\infty}^{\infty}\exp\{-(x-n\lambda)^2/4\nu t\}}. \tag{4.49}$$

When $\lambda^2/4\nu t \gg 1$, the exponential with the minimum value of $(x-n\lambda)^2/4\nu t$ will dominate over all the others. Therefore the term with $n=m$ will dominate for $(m-\frac{1}{2})\lambda < x < (m+\frac{1}{2})\lambda$, and (4.49) is approximately

$$c \sim \frac{x-m\lambda}{t}, \qquad \left(m-\frac{1}{2}\right)\lambda < x < \left(m+\frac{1}{2}\right)\lambda.$$

This is a sawtooth wave with a periodic set of shocks a distance λ apart, and c jumps from $-\lambda/2t$ to $\lambda/2t$ at each shock. The result agrees with (2.56).

To study the final decay, $\lambda^2/4\nu t \ll 1$, we may use an alternative form of the solution. The expression (4.48) is periodic in x, and in the interval $-\lambda/2 < x < \lambda/2$,

$$\varphi \to \delta(x) \qquad \text{as } t\to 0.$$

The initial condition can be expanded in a Fourier series as

$$\Phi(x)=\frac{1}{\lambda}\left\{1+2\sum_{1}^{\infty}\cos\frac{2\pi nx}{\lambda}\right\},$$

and the corresponding solution of the heat equation for φ is

$$\varphi=\frac{1}{\lambda}\left\{1+2\sum_{1}^{\infty}\exp\left(-\frac{4\pi^2n^2}{\lambda^2}\nu t\right)\cos\frac{2\pi nx}{\lambda}\right\}. \tag{4.50}$$

It may be verified directly that this is the Fourier series of (4.48). In this form

$$c=-\frac{2\nu\varphi_x}{\varphi}=\frac{\dfrac{8\pi\nu}{\lambda}\displaystyle\sum_{1}^{\infty}n\exp\left(-\frac{4\pi^2n^2}{\lambda^2}\nu t\right)\sin\frac{2\pi nx}{\lambda}}{1+2\displaystyle\sum_{1}^{\infty}\exp\left(-\frac{4\pi^2n^2}{\lambda^2}\nu t\right)\cos\frac{2\pi nx}{\lambda}}. \tag{4.51}$$

When $\nu t/\lambda^2\gg 1$, the term with $n=1$ dominates the series and we have

$$c\sim\frac{8\pi\nu}{\lambda}\exp\left(-\frac{4\pi^2\nu t}{\lambda^2}\right)\sin\frac{2\pi x}{\lambda}. \tag{4.52}$$

This is a solution of $c_t=\nu c_{xx}$, and again the diffusion dominates in the ultimate decay.

4.7 Confluence of Shocks

When a shock overtakes another shock, they merge into a single shock of increased strength as described for the inviscid solution ($\nu\to 0$) on the F curve in Fig 2.16. It is possible to give a simple solution of Burgers' equation that describes this process for arbitrary ν.

The solution for a single shock is given in (4.23) and the corresponding expression for φ may be written in the form

$$\varphi=f_1+f_2, \qquad f_j=\exp\left(-\frac{c_jx}{2\nu}+\frac{c_j^2t}{4\nu}-b_j\right). \tag{4.53}$$

In (4.23), the parameters b_1, b_2 which locate the initial position of the shock are taken to be zero. The expressions f_1, f_2 are clearly solutions of

the heat equation (4.7). The expression for c is

$$c=-\frac{2\nu\varphi_x}{\varphi}=\frac{c_1f_1+c_2f_2}{f_1+f_2}. \tag{4.54}$$

For $c_2>c_1$, f_1 dominates as $x\to+\infty$ and we have $c\to c_1$; f_2 dominates as $x\to-\infty$ to give $c\to c_2$. The center of the shock is where $f_1=f_2$, that is, $x=\frac{1}{2}(c_1+c_2)t$.

Now since any f_j is a solution of the heat equation, we may clearly add further terms in (4.53) and generate more general solutions of Burgers' equation. Such solutions represent interacting shocks. We consider the case

$$\varphi=f_1+f_2+f_3, \qquad b_1=b_2=0, \qquad b_3=\frac{c_3-c_2}{2\nu}$$
$$c=\frac{c_1f_1+c_2f_2+c_3f_3}{f_1+f_2+f_3}, \qquad c_3>c_2>c_1>0. \tag{4.55}$$

If ν is reasonably small, we can recognize shock transitions between the states c_1, c_2, c_3 by noting in which regions the corresponding f dominates. At $t=0$, f_1 dominates in $0<x$, f_2 in $-1<x<0$, f_3 in $x<-1$. Thus we have a shock transition from c_1 to c_2 centered at $x=0$, and one from c_2 to c_3 centered at $x=-1$. For $t>0$, the regions in which $c\simeq c_1$, $c\simeq c_2$, $c\simeq c_3$ can be found in the same way and the result is shown in Fig 4.3. For early times the transition from c_1 to c_2 occurs where $f_1=f_2$ on

$$x=\frac{c_1+c_2}{2}t; \tag{4.56}$$

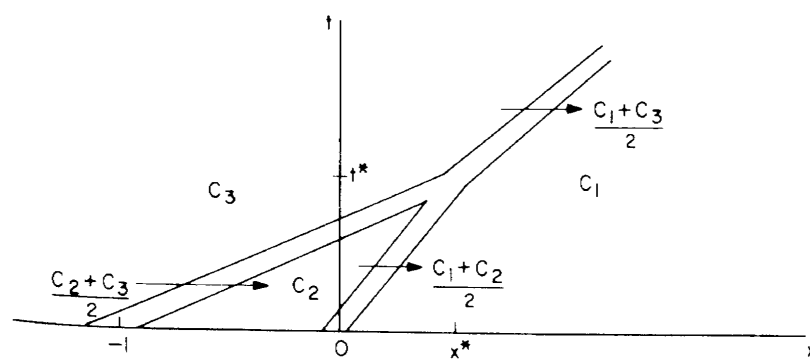

Fig. 4.3. Merging shocks.

the transition from c_2 to c_3 occurs for $f_2 = f_3$ on

$$x = \frac{c_2 + c_3}{2} t - 1. \tag{4.57}$$

Since $\frac{1}{2}(c_2 + c_3) > \frac{1}{2}(c_1 + c_2)$, the second shock overtakes the first at the point (x^*, t^*) determined by (4.56) and (4.57). At this point

$$f_1 = f_2 = f_3.$$

For $t > t^*$, there is no longer any region where f_2 dominates and the continuing solution describes a single shock transition between c_1 and c_3, moving with velocity $\frac{1}{2}(c_1 + c_3)$ on the path

$$x - x^* = \frac{c_1 + c_3}{2}(t - t^*) \tag{4.58}$$

determined by $f_1 = f_3$.