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DYNAMICS OF FLAT GALAXIES. I

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ABSTRACT

The linearized Vlasov equation that governs the oscillations of a flat collisionless stellar disk is integrated in action-angle variables, and the Poisson equation is solved exactly by use of a logarithmic-spiral representation for the surface densities and potentials. An integral equation for the potential of unstable modes is derived, as well as small-amplitude conservation theorems for angular momentum and energy. A necessary and sufficient condition for the stability of axisymmetric disturbances is derived.

I. INTRODUCTION

In this and following papers we shall explore the possibility that the spiral pattern, or structure, seen in flat rotating galaxies is a slowly growing density wave associated with noncircular motions along the galactic plane.

The suggestion that the patterns are density waves is old and was first explored by Bertil Lindblad. His emphasis was mainly on kinematics and less on collective effects on a large scale, though many of the kinematical effects he discovered can still be seen in the collective modes. Most of Lindblad's work predated plasma physics, the techniques of which gave a new impetus to the density-wave theories.

The importance of collective effects in our Galaxy was first clearly pointed out by Toomre (1964). He showed that in the disk the stellar motions are sufficiently coherent to make it almost vulnerable to collapse. He also pointed out that the scale on which this would occur is quite large, roughly the circumference of a typical epicycle (6–8 kpc in the solar neighborhood).

It is customary to assume that the spiral patterns represent a small deviation from a stationary, axisymmetric state. The spiral pattern can then be looked for among the possible small-amplitude perturbations. Even if the perturbation turns out to be not so small, an understanding of the dynamics close to equilibrium should provide us with some qualitative, if not quantitative, information.

A comprehensive WKBJ-like method for solving the linearized Vlasov (or collisionless Boltzmann) and Poisson equations needed to calculate the density response of a galaxy to tightly wound spiral fields of force has been developed by Lin and his students (Lin and Shu 1964; Lin 1966; Shu 1968). They have applied their method in the search for a self-consistent gravitational explanation for the tightly wound pattern that is thought to be seen in the H I distribution of our Galaxy (Lin, Yuan, and Shu 1969). The WKBJ method is well adapted to response calculations. However, their treatment of the self-consistent waves is incomplete since the origin and the evolution of these waves depend on the behavior of the disk at special radii (the edge, resonance points, and/or center) where the WKBJ method is inapplicable. The importance of these radii, as well as the possibility that the WKBJ pattern results from some yet undiscovered forcing mechanism, has been discussed by Shu (1970*a*, *b*). Lin (1970) has suggested that gravitational turbulence at the outer edge might be the driving mechanism. A further clarification of this suggestion is needed.

A desirable feature of the WKBJ waves is their mathematical simplicity; their physical relevance to the "grand design" of a spiral galaxy is less transparent. The response to a disturbance of a flat galaxy made of stars in circular orbits grows inversely

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as the geometrical scale of the disturbance. The presence of random motions actually reverses this trend. Along an eccentric orbit a spatially corrugated force field will appear to be rapidly varying in time. The orbital deflection caused by such a field will be small as well as complicated, and the resulting density change computed by adding the deflections of many stars will be smaller still. This sharp reduction of the response gives rise to a characteristic scale, namely, that scale associated with the disturbance of a given pattern speed for which the response is a maximum. Since a real galaxy is an inhomogeneous system, a precise expression for this scale is difficult to obtain, but for disturbances whose pattern speeds are comparable to revolution rates, it is again of the order of a typical epicycle circumference. For a galaxy such as ours the scale is large—about half the galactic radius. The response is a measure of the gravitational coupling between the stars, and therefore we would expect the characteristic scale to be the scale on which the collective effects are most pronounced. For two-armed disturbances a more precise measure of coupling is the shift of the pattern speed from the kinematical value $\Omega - \frac{1}{2}\kappa$ which obtains in the absence of coupling. The short-wave WKBJ disturbances can be characterized by this criterion as weakly coupled kinematic waves. These considerations suggest that the underlying cause of the "grand design" of spiral galaxies is a loosely wound two-armed pattern associated with the characteristic length scale. For such disturbances the WKBJ method is not applicable.

Preliminary calculations (Kalnajs 1970) using a theory based on epicyclic orbits (Kalnajs 1965) suggest that a large-scale unstable mode can exist in a galaxy similar to ours. The density wave itself resembles a bar; however, the pattern revealed by the subsystem of objects with the lowest eccentricities is a relatively tightly wound spiral and has the largest density contrast. It is this subsystem which includes the very bright young objects and gas that we actually observe. If we examine the velocity field of the low-eccentricity subsystem, we observe that when the amplitude of the mode reaches the point where the density contrast in the subsystem becomes large, the noncircular motions become large enough to distort severely the density pattern of the gas that would be obtained on the assumption of circular motion. An observer situated in the plane of the galaxy would no longer observe a monotonic change of radial velocity along his line of sight in the second and third quadrants. In certain directions the relatively small noncircular motions may vary rapidly enough along his line of sight to produce maxima and minima. Each maximum and minimum not only gives rise to a distance ambiguity but also contributes a peak to the H I line profile. It is possible to obtain as many as three such peaks in the line profile from a single spiral-arm crossing. Such spurious "arms" due to velocity crowding arise also in the first and fourth quadrants.

We can already see from this relatively simple calculation that the observable features of a barlike density wave can mimic a tightly wrapped pattern, and that the observational evidence must therefore be sifted with care. The problem may be further complicated by nonlinear effects. For even with growth rates measured in 10^9 years, the time the mode spends in the linear regime at an observable amplitude is short compared with the lifetime of the galaxy. A further discussion of the observable aspects of barlike modes will be given in a later paper.

Here we present a formulation of the small-amplitude oscillations around an axisymmetric stationary equilibrium. The novel feature that allows us to say something useful while remaining rather general is the use of coordinates that explicitly incorporate all the qualitative features of the equilibrium. The perturbed disk behaves as an infinite set of coupled harmonic oscillators, which, however, are unlike bedsprings in one important respect: the energies of some oscillators can be negative and the system can be unstable. The coupling is through the disturbed force field. If the latter is assumed to be given, the equations of motion of the oscillators can be integrated explicitly and the disturbed distribution can be found. Requiring it to be self-consistent with the force fields leads to a well-behaved integral equation to be satisfied by the perturbed poten-

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tial. The equation has at most a countable set of unstable modes. The system of coupled harmonic oscillators has an infinite set of constants of motion, the simplest two of which we identify as angular momentum and energy. We choose a representation for the potentials and surface densities in which the Poisson equation can be solved exactly and explicitly. Finally, we discuss the nature and stability of axisymmetric modes.

II. EQUATIONS OF MOTION

We shall restrict our attention to dynamical effects which involve motions along the galactic plane. Since the characteristic scale associated with the collective phenomena is comparable to the size of the galaxy, the thickness of the disk can be neglected. The corrections introduced by the thickness are small in this case (Vandervoort 1970).

In the absence of collisions the dynamics of the stars can be described by the single-particle distribution function $f(r, \theta, v_r, v_\theta, t)$. It satisfies the collisionless Boltzmann or Vlasov equation

$$\frac{\partial f}{\partial t} + [f, H] = 0 , \tag{1}$$

where H is the Hamiltonian, $\frac{1}{2}(v_r^2 + v_\theta^2) + V(r, \theta, t)$, and the bracketed term denotes the Poisson bracket

$$[f, g] \equiv \Sigma_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) . \tag{2}$$

The motion takes place in a self-consistent field. Therefore, the potential V is related to the distribution function by Poisson's equation

$$\nabla^2 V(r, \theta, z) = 4\pi G \delta(z) \int\int f(r, \theta, v_r, v_\theta) dv_r dv_\theta . \tag{3}$$

The Hamiltonian of the unperturbed galaxy is time independent and has rotational symmetry around the z-axis. These two symmetries imply the conservation of energy E and angular momentum J for each star. Because the motion of stars is in the plane $z = 0$, the orbits have one more important property: they are doubly periodic.

The equilibrium distribution, according to Jeans's theorem, will be a function of the two isolating integrals E and J.

We shall discuss perturbations of the equilibrium in two stages: (*a*) the kinematical evolution, which ignores any force field produced by the perturbation; and (*b*) dynamical evolution, which takes the first-order effects of the perturbed force field into account. We feel that a review of the kinematical aspect is necessary, for all too often an incomplete discussion of it is used to show that spiral structure cannot persist in differentially rotating systems.

III. KINEMATIC DENSITY WAVES

It is quite natural that the existence of differential or shearing motions in a flat galaxy should bring to mind the dissolving patterns formed by milk being stirred into a cup of tea. Pursuing the analogy, we might conclude that shearing motions would in a few revolutions wind up and thus dissolve any structure in the galaxy. However, this argument overlooks one essential difference between a galaxy and a cup of tea: the fluid elements in the latter are constrained by viscous forces to move in circular orbits, whereas stars in the galaxy not only can rotate around the center but can also execute oscillations in the radial direction.

Shear in a teacup is caused by the different rates of angular rotation of fluid elements at different radii. The same mechanism would act in a galaxy if the radial oscillations of the stars were not present. It is true that the radial velocities are generally an order of magnitude smaller than the circular velocities. But it is important to realize that shear phenomena are characterized by a spread in frequencies and that the radial and

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angular periods of orbits are comparable. The resulting shear is quite complicated since linear combinations of the frequencies are observed at a fixed point in the galaxy. It is possible to create smooth disturbances which show little shear, as well as features that appear to have retrograde motion and to shear out into leading structures. Such structures must clearly be waves, and their description is somewhat less intuitive than shear in a teacup.

The detailed description of the kinematics as well as dynamics can be considerably simplified if we use coordinates in phase space that incorporate the double periodicity of the orbits in the plane of an axisymmetric galaxy. We introduce the action-angle variables (J_1, J_2, w_1, w_2) in place of the usual cylindrical coordinates $(r, \theta, v_r, v_\theta)$. Instead of the customary definition (Goldstein 1950), we shall divide the action variables (and multiply the angles) by 2π:

$$J_1 \equiv \frac{1}{2\pi} \oint p_r dr = \frac{1}{2\pi} \oint \sqrt{\left[2E - 2V(r) - \frac{J^2}{r^2}\right]} dr \,, \tag{4a}$$

$$J_2 \equiv \frac{1}{2\pi} \oint p_\theta d\theta = J \,, \tag{4b}$$

where the J_i are functions of E and J and are thus constants of motion. The Hamiltonian is a function of the J_i alone, and the equations of motion for the w_i are

$$\dot{w}_j = \frac{\partial H_0(J_1, J_2)}{\partial J_j} \equiv \Omega_j(J_1, J_2) = \text{const.} \tag{5}$$

With our normalization of the J_i, the angles w_i change by 2π in one period of oscillation, and therefore the Ω_i are angular frequencies rather than inverse oscillation periods.

If the orbits are nearly circular, or $J_1/J_2 \ll 1$, they can be represented by epicycles (Lindblad 1959). For such orbits, Ω_1 corresponds to the epicyclic frequency κ and Ω_2 is the angular rotation rate Ω.

Any single-valued function g in phase space must be periodic in the angle variables. More explicitly, g must have a Fourier series expansion

$$g(J, w_i) = \frac{1}{4\pi^2} \sum_{l\, m=-\infty}^{\infty} g_{lm}(J) \exp\left[i(lw_1 + mw_2)\right] \,, \tag{6}$$

where

$$g_{lm}(J) = \int_0^{2\pi}\int_0^{2\pi} g(J, w_i) \exp\left[-i(lw_1 + mw_2)\right] dw_1 dw_2 \,. \tag{7}$$

If $g(J_i, w_i, 0)$ represents a disturbance of the galaxy at $t = 0$, the subsequent kinematical evolution is a convection of g along the orbits. But since we choose to view g at a fixed point in phase space, the value it has there at some later time t can be found only if we know where that point was at $t = 0$. Integrating the equations of motion (5) backward in time, we find that it came from $(J_i, w_i - \Omega_i t)$, or that

$$\begin{aligned} g(J, w_i, t) &= g(J, w_i - \Omega_i t, 0) \\ &= \frac{1}{4\pi^2} \Sigma_{l,m} g_{lm}(J) \exp\left[il(w_1 - \Omega_1 t) + im(w_2 - \Omega_2 t)\right] \,. \end{aligned} \tag{8}$$

The only part of the disturbance that does not evolve is the g_{00} component in the expansion (8). It is clear that the series expansion (6) of the equilibrium state will contain only the $l = m = 0$ term.

The Fourier series expansion and the evolution of g described by equation (8) show

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that the galaxy can be viewed as an infinite collection of harmonic oscillators. The amplitude and phase of the oscillator labeled by (J_1, J_2, l, m) is the Fourier coefficient $g_{lm}(J)$, and its frequency is $(l\Omega_1 + m\Omega_2)$. The projection of an oscillator on the galactic plane is a density wave which rotates with the angular velocity $(l/m)\Omega_1 + \Omega_2$; m is the angular periodicity of the wave, whereas l indicates the radial structure.

A single oscillator is a particular case of a dispersion orbit as defined by Lindblad (1958), although sums such as

$$d_{00}(J) + \Sigma_{l/m=g} d_{lm}(J) \exp[i(lw_1 + mw_2)] \tag{9}$$

with fixed g satisfy the definition as well. Lindblad gives a detailed prescription for obtaining some of the simplest dispersion orbits of type (9) in case $g = 1, \frac{1}{2}, \ldots, 1/n$.

A Fourier analysis of a smooth disturbance will in general reveal that a continuum of oscillators has been excited, which in turn implies that the density (or any other macroscopic property) associated with the disturbance consists of a continuum of rotating waves. The evolution of the density will be characterized by a transformation or shearing of the initially smooth pattern into consecutively smaller scales. The dissolution can be traced to the interference or phase mixing of the various harmonic components. The characteristic time for this process is roughly the inverse of the spread of the phase velocities over the region under consideration. The only permanent visible effect after a long time will be that associated with the g_{00} term.

An examination of the range of frequencies associated with a typical flat galaxy shows that, with two exceptions, the mixing process for most disturbances will have a time scale of one revolution. The exceptions are the $l = -1$, $m = 2$ terms. Lindblad (1958) noted that the linear combination $\kappa - 2\Omega$ remains approximately constant over a large portion of a galaxy and, because of this, bisymmetrical disturbances could persist for many revolutions.

The relative constancy of $\kappa - 2\Omega$ is a consequence of the mass distribution in flat systems, and appears to be the explanation for the prevalence of bisymmetrical shapes observed in these systems. The requirement that the galaxy be stable against axisymmetric disturbances (Toomre 1964) limits the effects that a given force field can produce, particularly on the shortest scales. The reduction in effectiveness has to be compensated by a longer time over which it can act in order to produce a given response. Therefore, the persistent nature of the bisymmetrical disturbances makes them the likeliest self-consistent disturbances.

An example which contradicts intuition is any one-armed structure formed by $l = -1$, $m = 1$ oscillators. Its rotation is retrograde, and it shears out into a leading structure.

IV. POISSON'S EQUATION

The potential $V(r, \theta)$ from a surface density $S(r, \theta)$ is obtained by Poisson's integral

$$V(r, \theta) = -G \int_0^\infty r' dr' \int_0^{2\pi} \frac{d\theta' S(r', \theta')}{\sqrt{[r'^2 + r^2 - 2r'r \cos(\theta' - \theta)]}} . \tag{10}$$

It is a linear integral equation which has a number of complete sets of eigenfunctions, depending on the range of r over which S is assumed to be nonzero (Snow 1952). The integral operator clearly commutes with rotations, so that the angular part of the eigenfunctions will be $\exp(im\theta)$. If the domain of r is (0, 1), the radial part of the eigenfunctions can be Legendre polynomials (Hunter 1963) or Bessel functions (Yabushita 1966), whereas if it is $(0, \infty)$ we may use Bessel functions (Toomre 1963) or logarithmic spirals (Kalnajs 1965). Here we will introduce the latter.

A well-known, but seldom exploited, symmetry of expression (10) is the scale invariance of the operator that connects V and rS, i.e., it is invariant if $r \to r\lambda$. The change of

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variable $r = e^u$ turns a scale change into a translation of the u variable. Substituting $r = e^u$ into equation (10), we see that

$$V(e^u, \theta) = -G \int_{-\infty}^{\infty} du' \int_0^{2\pi} d\theta' \frac{[e^{u'} S(e^{u'}, \theta)] e^{(u'-u)/2}}{\sqrt{[2 \cosh (u' - u) + 2 \cos (\theta' - \theta)]}}, \quad (11)$$

which shows the translation invariance of the kernel in the (u, θ)-plane. A more symmetrical kernel is obtained if the reduced potential $r^{1/2}V$ and reduced surface density $r^{3/2}S$ are introduced in place of V and rS (Snow 1952). The kernel that connects the latter quantities is invariant also to inversions in a sphere (Kelvin's transformation).

The eigenfunctions of a translation-invariant operator are exponentials exp $[i(m\theta + \alpha u)]$, or logarithmic spirals in the (r, θ)-space. The reduced potential that corresponds to a reduced density exp $[i(m\theta + \alpha u)]$ can be obtained from expression (11) by integration. If we call this constant of proportionality $-2\pi GK(\alpha, m)$, then

$$K(\alpha, m) \equiv \frac{1}{2} \frac{\Gamma[(m + 1/2 + i\alpha)/2]\Gamma[(m + 1/2 - i\alpha)/2]}{\Gamma[(m + 3/2 + i\alpha)/2]\Gamma[(m + 3/2 - i\alpha)/2]}. \quad (12)$$

The quantity K is real and positive. From its definition we may deduce the recurrence relation with respect to m,

$$K(\alpha, m)K(\alpha, m + 1) = [(m + \tfrac{1}{2})^2 + \alpha^2]^{-1}. \quad (13)$$

If $\alpha^2 + m^2$ is large, we may use Stirling's formula to evaluate the gamma functions and show that $K(\alpha, m) \approx (m^2 + \alpha^2)^{-1/2}$. The accuracy of the asymptotic result can be inferred from Table 1. The slow $|\alpha|^{-1}$ decay of K reflects the logarithmic singularity of the kernel in equation (10) at $r = r'$.

To calculate the reduced potential from a reduced surface density, we first express it as a superposition of logarithmic spirals (Fourier series integral),

$$e^{3u/2}S(e^u, \theta) = \frac{1}{4\pi^2} \Sigma_m \int_{-\infty}^{\infty} d\alpha A_m(\alpha) \exp [i(m\theta + \alpha u)]. \quad (14)$$

TABLE 1

THE FUNCTION $K(\alpha, m)$

α	$K(\alpha, 0)$	$K(\alpha, 1)$	$K(\alpha, 2)$	$(\alpha^2+4)^{-1/2}$
0.0......	4.376879	0.913893	0.486320	0.500000
0.2......	3.823712	0.901814	0.484226	0.497519
0.4......	2.810230	0.867909	0.478089	0.490290
0.6......	2.003811	0.818113	0.468324	0.478913
0.8......	1.479225	0.759584	0.455540	0.464238
1.0......	1.145176	0.698582	0.440452	0.447214
1.4......	0.773840	0.584731	0.406220	0.409616
2.0......	0.519350	0.453055	0.353158	0.353553
3.0......	0.338438	0.319432	0.278272	0.277350
4.0......	0.252052	0.244150	0.224430	0.223607
5.0......	0.201030	0.197005	0.186276	0.185695
6.0......	0.167257	0.164933	0.158512	0.158114
8.0......	0.125247	0.124268	0.121466	0.121268
10.0......	0.100126	0.099625	0.098167	0.098058
12.0......	0.083406	0.083116	0.082265	0.082199
16.0......	0.062531	0.062408	0.062046	0.062017
20.0......	0.050016	0.049953	0.049767	0.049752

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In this logarithmic-spiral representation the potential operator is a multiplication operator. The reduced potential is

$$e^{u/2}V(e^u, \theta) = -\frac{1}{4\pi^2}\Sigma_m \int_{-\infty}^{\infty} d\alpha 2\pi G K(\alpha, m) A_m(\alpha) \exp\left[i(m\theta + \alpha u)\right], \quad (15)$$

and the potential energy from the pair is

$$\int_{-\infty}^{\infty} du \int_0^{2\pi} d\theta e^{u/2} V(e^u, \theta) e^{3u/2} S(e^u, \theta) = -\frac{1}{4\pi^2}\Sigma_m \int_{-\infty}^{\infty} d\alpha 2\pi G K(\alpha, m) |A_m(\alpha)|^2 . \quad (16)$$

The equality is a consequence of Plancherel's theorem.

We shall find it necessary to place some restrictions on the admissible density perturbations. A convenient, yet sufficiently wide, class of perturbations are those that have finite potential energies. Equation (16) shows that the potential energy is negative definite. If we divide it by $-2\pi G$, we obtain a positive-definite quadratic form which we will use as a norm in our function space $\mathfrak{P}$. $\mathfrak{P}$ becomes a Hilbert space if we define the inner product between two surface densities to be the interaction potential divided by $-2\pi G$. That is, if s_i are surface densities and h_i the corresponding potentials, then the inner product is defined as

$$(s_i, h_j) \equiv -\frac{1}{2\pi G}\int\int r^{3/2} s_i^* r^{1/2} h_j \frac{dr}{r} d\theta = (h_j, s_i)^* . \quad (17)$$

The expression for the inner product becomes more symmetrical in the logarithmic-spiral representation. Denote the Fourier amplitudes of $r^{3/2}s_1$ and $r^{3/2}s_2$ by $A_m(\alpha)$ and $B_m(\alpha)$, respectively. Then

$$(s_1, h_2) = \frac{1}{4\pi^2}\Sigma_m \int d\alpha A_m^*(\alpha) K(\alpha, m) B_m(\alpha) d\alpha . \quad (18)$$

This expression also demonstrates the self-adjointness of the potential operator.

The choice of the density (or the corresponding potential) as the field variable is convenient because of its immediate physical significance, but it results in the unsymmetrical expression (17) for the inner product, and an unsymmetrical linear-response kernel. The expression (18) of the inner product in the logarithmic-spiral representation suggests a possible way to redefine our function space which would lead to a symmetrical expression for the inner product as well as the response kernel. We may absorb the $K(\alpha, m)$ term by multiplying all Fourier transforms of the reduced densities by $K^{1/2}(\alpha, m)$. This multiplication may be viewed as an isometric mapping of $\mathfrak{P}$ onto a new space $\mathfrak{S}$: $r^{3/2}S(r, \theta) \to C_m(\alpha)$ defined by

$$C_m(\alpha) \equiv K^{1/2}(\alpha, m)\int\int r^{3/2}S(r, \theta) \exp\left(-i\alpha \ln r - im\theta\right) \frac{dr}{r} d\theta . \quad (19)$$

The inner product between two functions $C_m(\alpha)$ and $D_m(\alpha)$ in $\mathfrak{S}$ is

$$[C_m(\alpha), D_m(\alpha)] \equiv \frac{1}{4\pi^2}\Sigma_m \int C_m^*(\alpha) D_m(\alpha) d\alpha . \quad (20)$$

$\mathfrak{S}$ is also known as an $L_2(-\infty, \infty)$ function space, and is separable.

Besides the simple relation between potentials and surface densities there are two other advantages of the logarithmic-spiral representation. (*a*) Danver (1942) pointed out that the observed shapes of spiral galaxies can be represented by segments of loga-

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rithmic spirals. This suggests that the functional forms of the observed shapes should be simple functions of α. (*b*) Any classification of functions into leading or trailing classes has to be invariant under the spiral groups, i.e., rotations and radial expansions. If we assume that a galaxy rotates counterclockwise, then a logarithmic spiral will be leading or trailing according to whether $\alpha/m < 0$ or $\alpha/m > 0$. This suggests that we may decompose the Fourier amplitudes into leading and trailing parts. The two parts are orthogonal, and for complicated functions the potential energy densities in this representation could be used as classification criteria. The densities are identical for anti-spiral disturbances. A precise definition of leading and trailing is needed in order to demonstrate the leading-trailing asymmetry that is introduced by resonances (Kalnajs 1965).

V. DYNAMICS OF PERTURBED DISKS

To study small-amplitude disturbances we must take into account the effect of the force field they produce on the equilibrium disk. If we write the disturbed distribution function as $F + f$, then the Hamiltonian will be changed to $H_0 + h$, where h is the potential caused by the excess surface density $\int f d^2 v$. Vlasov's equation becomes

$$\frac{\partial F}{\partial t} + [F, H_0] + \frac{\partial f}{\partial t} + [f, H_0] + [F, h] + [f, h] = 0 . \qquad (21)$$

The first two terms are zero since F is the equilibrium distribution. The last term is quadratically small, and we linearize by omitting it.

The linearized terms expressed in action-angle variables have a simple form

$$\frac{\partial f}{\partial t} + \Sigma_i \frac{\partial f}{\partial w_i}\frac{\partial H_0}{\partial J_i} - \Sigma_i \frac{\partial F}{\partial J_i}\frac{\partial h}{\partial w_i} = 0 . \qquad (22)$$

If we expand f and h in Fourier series using equations (6) and (7), and substitute the results into equation (22), we find

$$\frac{1}{4\pi^2}\Sigma_{l,m}\Big[\frac{\partial f_{lm}}{\partial t} + i(l\Omega_1 + m\Omega_2) f_{lm} - i\Big(l\frac{\partial F}{\partial J_1} + m\frac{\partial F}{\partial J_2}\Big) h_{lm}\Big] \exp\,[i(lw_1 + mw_2)] = 0 . \qquad (23)$$

Since the exponentials are linearly independent, equation (23) is satisfied by each term in the sum. These equations for the Fourier coefficients are identical with those for an infinite set of coupled linear oscillators. The coupling is through h, which depends linearly on f. We have one equation for each l, m, J_1, and J_2.

If we know the time dependence of the perturbing potential h, it is a simple matter to integrate each harmonic-oscillator equation. For definiteness we will assume that the perturbation is created at some finite time, say $t = 0$. The solution for $t > 0$ is

$$f_{lm}(t) = i\Big(l\frac{\partial F}{\partial J_1} + m\frac{\partial F}{\partial J_2}\Big)\int_0^t \exp\,[i(l\Omega_1 + m\Omega_2)(t - t')] h_{lm}(t')dt' + f_{lm}(0)\exp\,[-i(l\Omega_1 + m\Omega_2)t] . \qquad (24)$$

The response consists of two parts: a forced component proportional to h, and the initial transient. The latter evolves kinematically.

If $|h(t)|$ does not grow faster than $Ae^{\eta t}$, $\eta > 0$, then it can be represented as a Fourier

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integral

$$h(t) = \frac{1}{2\pi} \int_{-\infty-i\eta}^{\infty-i\eta} \tilde{h}(\omega) \exp(i\omega t) d\omega , \tag{25}$$

and the transform $\tilde{h}(\omega)$ has an analytic continuation in $\mathrm{Im}(\omega) < -\eta$. We may also express the transient term as a Fourier integral

$$f_{lm}(0) \exp[-i(l\Omega_1 + m\Omega_2)]t = \frac{1}{2\pi} \int_{-\infty-i\eta}^{\infty-i\eta} \frac{f_{lm}(0) \exp(i\omega t) d\omega}{i(l\Omega_1 + m\Omega_2 + \omega)} . \tag{26}$$

If we introduce expressions (25) and (26) into equation (24), we may evaluate the t' integral and obtain a Fourier integral representation of $f_{lm}(t)$, with the transform

$$\tilde{f}_{lm}(\omega) = \frac{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}{l\Omega_1 + m\Omega_2 + \omega} \tilde{h}_{lm}(\omega) + \frac{f_{lm}(0)}{i(l\Omega_1 + m\Omega_2 + \omega)} . \tag{27}$$

Equation (24) or (27), which expresses the perturbed distribution function in terms of the perturbing potential, remains valid in the more general case where h consists of the internal contribution arising from f as well as any externally imposed field. The latter causes little difficulty since it is a known function. We may combine it with the initial transient and denote the sum by $\tilde{q}_{lm}(\omega)$. The difficult part of the problem is the simultaneous determination of the internal potential h, and f.

Having calculated f from an assumed h, we integrate it over all velocities and obtain the perturbed density, from which we calculate h with the help of Poisson's equation. These steps lead to a linear inhomogeneous operator equation of the form

$$\tilde{h}(\omega) = \tilde{R}(\omega)\tilde{h}(\omega) + \tilde{p}(\omega) . \tag{28}$$

We shall call the linear operator $\tilde{R}$ the *response*, and the inhomogeneous term $\tilde{p}(\omega)$ arising from $\tilde{q}$, the *driving* term. The nature of the dynamical evolution of a small disturbance is completely determined by the operator $\tilde{R}$.

We shall restrict our discussion to equilibrium distributions that are nonzero only over a closed finite range of the action variables and that have continuous first derivatives. Similarly, we shall admit perturbations that have finite potential energies. These restrictions suffice to make $\tilde{R}$ a compact (or completely continuous) operator. The properties of compact operators are in most respects similar to finite-dimensional matrix operators. Because of this relative simplicity of $\tilde{R}$ we shall be able to deduce a number of concrete results about the response of thin disks.

Since we do not possess explicit transformations between cylindrical coordinates and action-angle variables, the route from the above prescription for obtaining $\tilde{R}$ to an explicit integral equation representation will be devious. An intermediate step involves the introduction of a bilinear functional, which we justify because it displays only the features of $\tilde{R}$ and not the peculiarities associated with a particular basis.

The inner product between the density response of the disk due to a potential h, arising from a member of $\mathfrak{P}$, and any other member is a bilinear functional on $\mathfrak{P}$ which defines the response operator. Let $\tilde{h}(J, w, \omega)$ and $\tilde{g}(J, w, \omega)$ be the potentials associated with two surface densities in $\mathfrak{P}$. The angle dependences of $\tilde{h}$ and $\tilde{g}$ may be expanded in Fourier series (6), and the corresponding coefficients of the perturbed distribution function $\tilde{f}(J, w, \omega)$ caused by $\tilde{h}$ are determined by equation (27). The perturbed surface density is obtained by integrating $\tilde{f}$ over all velocities. The surface density multiplied by $-\tilde{g}^*/2\pi G$ and integrated over the spatial coordinates yields the desired inner product. But since the two integrations span all phase space and $\tilde{g}^*$ is constant over the

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velocity part, we may evaluate the integral in action-angle variables. The angle integrations can be done explicitly, which leaves us with

$$-\frac{1}{2\pi G}\frac{1}{4\pi^2}\Sigma_{l,m}\int\int dJ_1 dJ_2 \tilde{g}^*{}_{lm}(J)\tilde{f}_{lm}(J)$$

$$= -\frac{1}{2\pi G}\frac{1}{4\pi^2}\Sigma_{l,m}\int\int dJ_1 dJ_2 \frac{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}{l\Omega_1 + m\Omega_2 + \omega}\tilde{g}^*{}_{lm}(J)\tilde{h}_{lm}(J)\,. \quad (29)$$

The above bilinear functional defined on $\mathfrak{P}$ is represented by a kernel of the traditional type in the Hilbert space $\mathfrak{S}$ which was obtained from $\mathfrak{P}$ by the isometric transformation (19). To obtain the transformed functional on $\mathfrak{S}$, we express $\tilde{h}$ and $\tilde{g}$ in terms of their images in $\mathfrak{S}$ and substitute these expressions in equation (29).

The reduced potentials $r^{1/2}\tilde{h}$ and $r^{1/2}\tilde{g}$ are related to their corresponding density images $C_m(\alpha, \omega)$ and $D_m(\alpha, \omega)$ in $\mathfrak{S}$ by the transformation

$$r^{1/2}\tilde{h} = -\frac{G}{2\pi}\Sigma_m\int C_m(\alpha, \omega)K^{1/2}(\alpha, m)\exp(i\alpha \ln r + im\theta)d\alpha \quad (30a)$$

and

$$r^{1/2}\tilde{g} = -\frac{G}{2\pi}\Sigma_{m'}\int D_{m'}(\alpha', \omega)K^{1/2}(\alpha', m')\exp(i\alpha' \ln r + im'\theta)d\alpha'\,. \quad (30b)$$

The transformations between r, θ and the action-angle variables are of the form

$$r = \Theta_1(J_1, J_2, w_1)\,, \quad \theta = w_2 + \Theta_2(J_1, J_2, w_1)\,, \quad (31)$$

where the Θ_i are periodic functions of w_1. We insert these expressions in the right-hand side of expression (30a) and expand the angle dependences of $\tilde{h}$ in Fourier series (6). Because of the above relation (31) between θ and w_2 the Fourier expansion of $e^{im\theta}$ in w_2 will have only one term $e^{im\Theta_2}e^{imw_2}$, and the Fourier coefficients of $\tilde{h}$ will be

$$\tilde{h}_{lm}(J) = -G\int C_m(\alpha)K^{1/2}(\alpha, m)e_l(\alpha, m, J)\,d\alpha, \quad (32)$$

where

$$e_l(\alpha, m, J) \equiv \int_0^{2\pi} \exp[(i\alpha - \tfrac{1}{2})\ln r + im(\theta - w_2) - ilw_1]dw_1\,. \quad (33)$$

There is a corresponding expression for the $\tilde{g}^*{}_{lm}$.

If we substitute equation (32) and the corresponding $\tilde{g}^*{}_{lm}$ into the right-hand side of equation (29), we obtain the transformed functional

$$-\frac{G}{8\pi^2}\Sigma_{l,m}\int\int dJ_1 dJ_2 \frac{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}{l\Omega_1 + m\Omega_2 + \omega}\int\int e^*{}_l(\alpha', m)$$
$$\times\, e_l(\alpha, m)K^{1/2}(\alpha', m)K^{1/2}(\alpha, m)C_m(\alpha, \omega)D^*{}_m(\alpha', \omega)d\alpha d\alpha'\,. \quad (34)$$

The left-hand side of equation (29) is the inner product between the density corresponding to $\tilde{g}$ and the density response. Denote the image of the latter in $\mathfrak{S}$ by $E_m(\alpha, \omega)$; then the inner product computed in $\mathfrak{S}$ is

$$\frac{1}{4\pi^2}\Sigma_m\int d\alpha' D^*{}_m(\alpha', \omega)E_m(\alpha', \omega) \quad (35)$$

and is equal to equation (34) by definition for *all* D_m in $\mathfrak{S}$. For this to be true we must have

$$E_m(\alpha', \omega) = -\frac{G}{2\pi}\Sigma_l\int d\alpha C_m(\alpha, \omega)\Big[\int\int dJ_1 dJ_2 \frac{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}{l\Omega_1 + m\Omega_2 + \omega}$$
$$\times\, e^*{}_l(\alpha', m, J)e_l(\alpha, m, J)K^{1/2}(\alpha', m)K^{1/2}(\alpha, m)\Big]\,. \quad (36)$$

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The bracketed term is the desired kernel, and equation (36) is an integral equation representation of the response operator.

The kernel is bounded over the whole (α, α')-plane, and decays at least as fast as $K^{1/2}(\alpha, m)K^{1/2}(\alpha', m) \sim |\alpha\alpha'|^{-1/2}$. The boundedness can be inferred from the convergence of the sum

$$\Sigma_l |e_l(\alpha)e^*_l(\alpha')| \leq [\Sigma_l |e_l(\alpha)|^2 \Sigma_{l'} |e_{l'}(\alpha')|^2]^{1/2} = 2\pi \int_0^{2\pi} \frac{dw_1}{r} . \quad (37)$$

The above bound is integrable. This can be best seen if we transform back to cylindrical coordinates.

If we were to modify $K(\alpha, m)$ by incorporating a small thickness correction, we could state without further comment that the kernel is also quadratically integrable over the (α, α')-plane. Even without such a correction it is quadratically integrable, but to show it we must examine the asymptotic properties of the oscillatory functions $e_l(\alpha, m, J)$ and their integrals over J_i. A more detailed discussion of this point will be found in the Appendix.

The quadratic integrability of the kernel implies that it is completely continuous (Riesz and Nagy 1955) or compact.

The above properties are true for each ω in $\text{Im}(\omega) < 0$. More properly, we should speak of an analytic family of compact operators (Kato 1966). That the operators are analytic follows from the fact that for every allowable g and h the bilinear form (29) is an analytic function of ω. Moreover the forms can be analytically continued into $\text{Im}(\omega) > 0$, and their singularities lie on the real axis. As $\text{Im}(\omega) \to -\infty$, the norm of the operator goes to zero. The bilinear form (29) certainly suggests this, but the result depends on the behavior of the $e_l(\alpha)$ for large l (see Appendix).

With these facts we are in a position to state that the initial-value problem, as well as the forced problem, have unique solutions. The argument is simple: we choose η in equation (25) so large that the bound of $\tilde{R}$ along the line $\text{Im}(\omega) = -\eta$ is less than 1. Then the resolvent $(I - \tilde{R})^{-1}$ is represented by its convergent Neumann series $I + R + R^2 + \ldots$, and equation (28) may be solved to obtain

$$\tilde{h}(\omega) = (I - \tilde{R})^{-1}\tilde{p}(\omega) , \quad (38)$$

where I is the identity operator. From $\tilde{h}(\omega)$ we may obtain $\tilde{f}(\omega)$ and determine their time development with the help of equation (24).

If the driving term is analytic in $\text{Im}(\omega) < 0$—and this will be the case if it arises from an initial perturbation—we may use the fact that the resolvent is analytic along $\text{Im}(\omega) = -\eta$ and attempt to continue the solution analytically up to $\text{Im}(\omega) = 0$. Because $\tilde{R}(\omega)$ is compact, such a continuation is possible. $\tilde{R}(\omega)$ can have only a finite number of eigenvalues equal to 1 in any compact subset of $\text{Im}(\omega) < 0$, and only at these *singular values* do we find that the resolvent does not exist. A continuation of the resolvent around a singular ω shows that it is an isolated singularity. Therefore, we can move the contour of integration in equation (24) from $\text{Im}(\omega) = -\eta$ to $\text{Im}(\omega) = 0$ as long as we include the contributions from the singular values. The resultant time development of the initial perturbation can then be expressed as

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t)[I - \tilde{R}(\omega)]^{-1}\tilde{p}(\omega)d\omega$$

$$+ \frac{1}{2\pi} \Sigma_j \exp(i\omega_j t) \int_{C_j} \exp[i(\omega - \omega_j)t][I - \tilde{R}(\omega)]^{-1}\tilde{p}(\omega)d\omega . \quad (39)$$

The $\{\omega_j\}$ are the singular values of ω, and C_j is a small positively oriented circle around ω_j. As in the case of finite-dimensional analytic matrix operators, the integral of the

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resolvent around ω_j is a projection with a finite range. Thus each C_j integral can contribute only a finite number of linearly independent functions. The particular combination is determined by the driving term $\tilde{p}(\omega)$. The time dependence of a C_j integral will be a polynomial of finite degree in t.

The ω_j which give rise to multiple functions are termed *exceptional points*. The source of the multiplicity is the existence of a group of eigenvalues of $\tilde{R}(\omega)$ in the neighborhood of ω_j that converge to 1 as $\omega \to \omega_j$. This degeneracy appears to be accidental.[1]

Whether ω_j is exceptional or not, at least one of the linearly independent functions associated with it will be a solution of the homogeneous equation

$$\tilde{h}(\omega_j) = \tilde{R}(\omega_j)\tilde{h}(\omega_j) . \tag{40}$$

We shall call such eigensolutions *modes*. The significance of the modes is clear from the form of the solution as expressed by equation (39). In the absence of degeneracy, they comprise all the exponentially growing solutions. We would expect the fastest growing modes or their nonlinear counterparts ultimately to dominate the appearance of the galactic disk.

Because the singularities of $\tilde{R}(\omega)$ and $\tilde{p}(\omega)$ lie on $\text{Im}(\omega) = 0$, it is difficult to say anything specific about the nature of the integral contribution in equation (39). The integral describes all the transient phenomena as well as disturbances that grow more slowly than any exponential. The possibility of further simplification of the integral depends on the analytical properties of the integrand. In particular, the generally complicated time behavior of the integral may have discrete oscillatory components which arise from real singular points of $\tilde{R}(\omega)$. Lynden-Bell and Ostriker (1967) have argued that in a gaseous disk the nondegenerate discrete oscillatory modes cannot show any spiral structure. Shu (1970*a*) has argued that in certain cases a stellar disk might violate the *antispiral theorem* of Lynden-Bell and Ostriker. All three authors base their conclusion on the form their equations take along $\text{Im}(\omega) = 0$. We would like to point out that all antispiral theorems arise from a common symmetry property of the equations of motion. If we reverse the direction of time *and* at the same time turn the galaxy over ($\theta \to -\theta$), the Vlasov and Poisson equations as well as the equilibrium distribution remain unchanged, but any initial disturbance is transformed into its mirror image. The symmetry transformation allows us to generate new modes. A single growing mode is transformed into its mirror image, which rotates in the same sense but decays in time. In particular, if we can excite a single oscillatory mode, we can excite its mirror image. If the mirror image differs from the original only by a rotation around the z-axis through a fixed angle, we do not obtain a new mode. In this case we can orient the coordinate system so that the mirror image actually equals the original and therefore the mode is *antispiral*. If the mirror image is distinct from the original, we have two oscillatory modes which rotate at the same rate. Hence we conclude that *in the absence of degeneracy all discrete oscillatory modes are antispiral.*

The above symmetry properties are contained in equation (39) and can be made explicit by further analytic continuation of the integrand in $\text{Im}(\omega) > 0$. It is possible to deform the integral along the real axis into a long thin loop encircling the real axis plus a discrete sum of terms arising from the singularities of the continued resolvent at the frequencies $\{\omega^*_j\}$. The details of this continuation and further discussion of discrete oscillatory modes will be found in a future paper.

VI. CONSERVATION THEOREMS

Angular momentum and energy are rigorous constants of motion in any isolated system, and therefore the difference between the perturbed and unperturbed states of the

[1] It can be removed by any small change in the equilibrium model that makes the degenerate eigenvalue $\neq 1$, or splits the degeneracy. We can also remove it by the artificial device of placing a $\lambda \neq 1$ in front of $\tilde{R}(\omega)$ and taking the limit $\lambda \to 1$ after the ω integration. The removal of degeneracy makes the discussion simpler, but it has little or no effect on numerical calculations.

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system of these quantities will be conserved also. Writing out the time derivatives of the differences, one notes that they appear to be conserved only to first order in the perturbation parameter since there is an uncertainty about the quadratic terms discarded in the linearization of equation (21) which governs the evolution of f. This is the order to which the differences in energy and angular momentum can in general be conserved if linearization is involved, unless the unperturbed state is time independent and axisymmetric. Then the differences are indeed quadratic in the disturbances, and will be conserved to that order even if the disturbance is calculated accurately only to first order. Here we shall start with the linearized equation and show that it has an infinite set of integrals, the first two of which we will identify as angular momentum and energy.

If the time development of f is governed by a linear equation of the form

$$A\frac{\partial f}{\partial t} = iBf\,, \tag{41}$$

where A, B are self-adjoint operators and A is time independent, then the inner product $\langle f, Af\rangle$ is conserved. The inner product here is defined between two functions in phase space as

$$\langle f, g\rangle \equiv \tfrac{1}{2}\int f^* g d\mu \equiv \langle g, f\rangle^*\,, \tag{42}$$

and $d\mu$ is a volume element in phase space. The conservation can be seen easily. We compute the time derivative and use the time independence and self-adjointness of A:

$$\frac{d}{dt}\langle f, Af\rangle = \left\langle\frac{\partial f}{\partial t}, Af\right\rangle + \left\langle f, A\frac{\partial f}{\partial t}\right\rangle = \left\langle A\frac{\partial f}{\partial t}, f\right\rangle + \left\langle f, A\frac{\partial f}{\partial t}\right\rangle. \tag{43}$$

Eliminating the time derivatives via equation (41) leaves

$$\frac{d}{dt}\langle f, Af\rangle = \langle iBf, f\rangle + \langle f, iBf\rangle = -i\langle f, Bf\rangle^* + i\langle f, Bf\rangle\,. \tag{44}$$

Because B is self-adjoint also,

$$\langle f, Bf\rangle^* = \langle Bf, f\rangle = \langle f, Bf\rangle = \text{real}\,, \tag{45}$$

and therefore the right-hand side of equation (44) is zero.

We note that if another self-adjoint and time-independent operator C commutes with A and B, then CA and CB have the same properties as A and B. Therefore, $\langle f, CAf\rangle$ is conserved. Similarly, if A^{-1} exists and if B is now time independent as well, then successive multiplication of equation (41) by BA^{-1} will also preserve the necessary properties to ensure conservation of $\langle f, Bf\rangle$, $\langle f, BA^{-1}Bf\rangle, \ldots,$ since such symmetric products of self-adjoint operators are self-adjoint. With the exception of the first two, the usefulness of the integrals decreases in proportion to the difficulty of computing the products $BA^{-1}B$, $BA^{-1}BA^{-1}B, \ldots.$

The linearized part of equation (21) is not in the form of equation (34). The first two terms are, but the third involves a product of two noncommuting self-adjoint operators, which is not self-adjoint. This is fortunate, for otherwise it would follow that $\langle f, f\rangle$ is conserved, and that the galaxy is stable. Clearly the kinematical evolution, which neglects the third term, is stable. It can be seen from equation (8) or (23) that each amplitude $|f_{lm}|$ is constant in time.

The linearized equation (23) can be manipulated into the form (41). Since this equation is satisfied for each l, m, we may divide each member by $l(\partial F/\partial J_1) + m(\partial F/\partial J_2)$ to obtain

$$\frac{(\partial/\partial t)f_{lm}}{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)} = \frac{-i(l\Omega_1 + m\Omega_2)}{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}f_{lm} + ih_{lm}\,. \tag{46}$$

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Here h should be viewed as the result of the self-adjoint potential operator acting on f. In each l, m subspace of phase space our operator A is a diagonal matrix with real elements $[l(\partial F/\partial J_1) + m(\partial F/\partial J_2)]^{-1}$, while on the J_1, J_2 part it is multiplication by a real function, and hence is self-adjoint. The right-hand side of equation (46) has a similar object plus the potential operator. The latter is, of course, self-adjoint. All three operators are time independent. The operator $L = i(\partial/\partial w_2)$, which is the same as the infinitesimal generator of rotations around the z-axis $i(\partial/\partial\theta)$, commutes with the above operators and in this representation is also a diagonal matrix, with the elements $-m$.

For the first conserved quantity we choose $\langle f, LAf\rangle$, and obtain

$$\frac{1}{8\pi^2}\frac{\partial}{\partial t}\Sigma_l \int\int dJ_1 dJ_2 \frac{-m|f_{lm}|^2}{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)} = 0\,. \tag{47}$$

This we shall identify as angular momentum.[2]

The second conserved quantity, which we shall identify as energy, is $\langle f, Bf\rangle$, or

$$\frac{1}{8\pi^2}\frac{\partial}{\partial t}\Sigma_l \int\int dJ_1 dJ_2\left[-\frac{l\Omega_1 + m\Omega_2}{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}|f_{lm}|^2 + f^*{}_{lm}h_{lm}\right] = 0\,. \tag{48}$$

The other integrals do not have such explicit forms and will not be examined here.

The axisymmetric form of equation (48) was derived and identified by Lynden-Bell (1966). To identify equations (47) and (48) we shall suppose that the change from the unperturbed to the perturbed state is brought about by an external potential $h_e(t)$, which acts only for a finite time.

In the expression for energy we must distinguish between the contribution to the potential energy arising from self-gravitation and that due to the external field. We write the total Hamiltonian as before, $H_0 + h$, and consider V_0, h to be the sums of the external and self-contributions $V_e + V_s$, $h_e + h_s$, respectively. The total energy becomes

$$\mathfrak{E} = \int (F + f)(H_0 + h - \tfrac{1}{2}V_s - \tfrac{1}{2}h_s)d\mu\,, \tag{49}$$

and its time derivative is

$$\begin{aligned}\frac{d\mathfrak{E}}{dt} &= \int\left\{H_0\frac{\partial f}{\partial t} + \frac{\partial}{\partial t}(\tfrac{1}{2}h_s f) + \frac{\partial}{\partial t}[h_e(F + f)] + \frac{\partial}{\partial t}(\tfrac{1}{2}h_s F - \tfrac{1}{2}f V_s)\right\}d\mu \\ &= \int \frac{\partial h_e}{\partial t}(F + f)d\mu\,.\end{aligned} \tag{50}$$

The last derivative in the first integral vanishes identically upon integration because of the self-adjointness of the potential operator. If we subtract the third term from both sides, we obtain

$$\frac{d\mathfrak{E}'}{dt} = \int H_0\frac{\partial f}{\partial t}d\mu + \frac{d}{dt}\tfrac{1}{2}\int h_s f d\mu = -\int\frac{\partial f}{\partial t}h_e d\mu\,. \tag{51}$$

We shall call $\mathfrak{E}'$, which differs from $\mathfrak{E}$ only in the presence of an external field h_e, the energy. Similarly, the change in angular momentum is

$$\frac{dJ}{dt} = \frac{d}{dt}\int J(F + f)d\mu = \int J\frac{\partial f}{\partial t}d\mu = -\int (F + f)\frac{\partial h_e}{\partial\theta}d\mu\,. \tag{52}$$

[2] The choice of LA rather than A makes the conserved quantity the angular momentum in case we choose to sum over m. We know, in fact, that each term in the m sum is conserved separately since it arises from an eigenfunction.

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The contribution from $F(\partial h_e/\partial\theta)$ vanishes upon integration over θ.

In both cases we have a linear term $\partial f/\partial t$ which we can convert into a quadratic one with the help of equation (21). We then obtain an expression of the type

$$-\int\int\int\int G(J_1, J_2)\{[f, H_0] + [F, h] + [f, h]\}dJ_1 dJ_2 dw_1 dw_2 \,. \quad (53)$$

Since H_0 and F depend only on the J_i, the first two bracketed terms will give rise to terms of the type $R(J_1, J_2)$ $(\partial g/\partial w_i)$ which vanish upon integration over w_i. The last bracketed term is quadratic in the disturbance, and upon an integration by parts becomes

$$\int\int\int\int \Sigma_i \frac{\partial G}{\partial J_i}\frac{\partial f}{\partial w_i} h dJ_1 dJ_2 dw_1 dw_2 \,. \quad (54)$$

This exact result will remain correct to second order if we write h in terms of f, using the linearized equation (23).

In the first case we have $G = H_0$. Expanding h and f in Fourier series (6), using definition (5), then eliminating h and integrating over the w_i, we obtain

$$-\frac{1}{4\pi^2}\Sigma_{l,m}\int\int dJ_1 dJ_2 \frac{l\Omega_1 + m\Omega_2}{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}\left[f_{-l-m}\frac{\partial}{\partial t}f_{lm} + i(l\Omega_1 + m\Omega_2)f_{-l-m}f_{lm}\right] . \quad (55)$$

On summation, only the symmetrized part of the first term remains; the second is antisymmetric in l, m and vanishes. Since f is real, $f_{-l-m} = f^*{}_{lm}$, and equation (55) reduces to

$$-\frac{1}{8\pi^2}\Sigma_{l,m}\int\int dJ_1 dJ_2 \frac{l\Omega_1 + m\Omega_2}{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}\frac{\partial}{\partial t}|f_{lm}|^2 \,. \quad (56)$$

To this we must add the remainder of equation (51), which, evaluated in action-angle variables, becomes

$$\frac{1}{8\pi^2}\Sigma_{l,m}\int\int dJ_1 dJ_2 f^*{}_{lm}h_{lm} \,. \quad (57)$$

We can restrict the m sum to include only m or $-m$, for their contributions are the same. This is clearly true of equation (56) and is also true of equation (57), but to see it we must go back to the potential equations (12) and (16), or note that the potential energy associated with a surface distribution is the same as that with its mirror image.

In the angular-momentum case, $G = J_2$ and the result differs from equation (55) only in that the numerator of the first factor is $-m$ rather than $l\Omega_1 + m\Omega_2$.

We have shown that the two exact integrals (47) and (48) of the linearized equation are approximate integrals of the exact nonlinear equation. The neglected terms are of cubic order in f. The integrals are indefinite functionals of the perturbed distribution f and therefore do not imply the stability of the system.

The one difference between equations (47) and (48) and the above expressions is that the former are defined for complex functions as well as real ones. The possibility of representing the perturbed quantity by the real part of a complex function is a convenient consequence of linearization. If we used the complex function instead of its real part in equation (47) or (48), the values of these constants would differ from the angular momentum and energy by some constant factors. But because integrations over angles are involved, one half of the real parts of these constants will equal the angular momentum and energy.

The change in energy and angular momentum caused by an external field can be calculated by equations (51) and (52). These relations can also be derived from the

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linearized equations directly. The equation of motion (41) is changed by the presence of the external field h_e which appears as the inhomogeneous term ih_e on the right-hand side. Then instead of being zero, the right-hand side of equation (47) is

$$-\left\langle \frac{\partial h_e}{\partial \theta}, f \right\rangle - \left\langle f, \frac{\partial h_e}{\partial \theta} \right\rangle, \tag{58}$$

and that of equation (48) is

$$-\left\langle h_e, \frac{\partial f}{\partial t} \right\rangle - \left\langle \frac{\partial f}{\partial t}, h_e \right\rangle. \tag{59}$$

The two conservation laws become essentially one if we excite an unstable mode. The potential associated with such a mode will be of the form Re[h exp($i\omega t$)] and the Fourier expansion (6) of h will have only one nonzero m-term. We use equation (27) to eliminate the distribution function f_{lm} from both (47) and (48). In the first case we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\,|\exp(i\omega t)|^2 \frac{-m}{8\pi^2} \Sigma_l \int\int dJ_1 dJ_2 \frac{l(\partial F/\partial J_1) + m(\partial F/\partial J_2)}{|l\Omega_1 + m\Omega_2 + \omega|^2}\,|h_{lm}|^2 = 0\,. \tag{60}$$

The energy equation is identical save for the factor Re(ω) in place of $-m$. But Re($-\omega/m$) $\equiv \Omega_p$ is the pattern speed; hence for such a mode we have

$$\text{energy} = \Omega_p \times (\text{angular momentum})\,. \tag{61}$$

A necessary condition for a mode to be unstable is that both the energy and angular momentum should vanish, for otherwise both will grow as $|\exp(i\omega t)|^2$.

For a growing mode, the conservation of angular momentum is equivalent to the requirement that the imaginary part of the response vanish. This can be seen if we set $g = h$ in equation (29) and compare the imaginary part of the right-hand side with equation (60). The imaginary part of the left-hand side of (29) vanishes because the potential operator is self-adjoint.

VII. STABILITY OF AXISYMMETRIC MODES

The two conservation theorems are constraints on the possible evolution of perturbations. Since they involve the structure of the equilibrium distribution, it is natural to ask whether they can be used to characterize stable distributions. The answer is: probably not.

The energy integral does rule out overstable axisymmetric modes whenever the equilibrium distribution is a decreasing function of the radial action, or $\partial F/\partial J_1 < 0$. The restriction that ω^2 be real makes the response operator self-adjoint, and its bound becomes a continuous nonincreasing function of $-\omega^2$. Because of this we can infer a necessary and sufficient condition for stability from the value of the bound at $\omega^2 = 0$.

To demonstrate the absence of overstability we first note that because the radial oscillation of a particle is symmetric about the turning points ($\dot{r} = 0$), we have

$$\begin{aligned} h_{l0} &= \int_0^{2\pi} dw_2 \int_0^{2\pi} h(J_1, J_2, w_1) \exp(ilw_1) dw_1 \\ &= \int_0^{2\pi} dw_2 \int_0^{2\pi} h(J_1, J_2, w_1) \exp(-ilw_1) dw_1 = h_{-l0} \end{aligned} \tag{62}$$

if we choose $w_1 = 0$ to be one of the turning points. The energy conservation (61) for the $m = 0$ modes can then be stated in the following way:

$$\text{Re}(\omega)\ \text{Im}(\omega) \frac{1}{8\pi^2} \sum_{l=1}^{\infty} \int\int dJ_1 dJ_2 l \frac{\partial F}{\partial J_1}\,|h_{l0}|^2 \left(\frac{1}{|l\Omega_1 - \omega|^2} - \frac{1}{|l\Omega_1 + \omega|^2}\right) = 0\,. \tag{63}$$

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If $\partial F/\partial J_1 < 0$, this expression can vanish only if $\mathrm{Re}(\omega)\mathrm{Im}(\omega) = 0$. Therefore, any exponentially growing mode must have $\mathrm{Re}(\omega) = 0$. The result was shown to be true locally by Julian (1969).

To investigate the exponentially growing modes we set $\omega = -i\sigma$, $\sigma > 0$, and write the potential of the mode as $h(J_1, J_2, w_1)e^{\sigma t}$. For such a mode the quadratic form (29) is real and with the help of equation (62) can be written as

$$\beta(\sigma) = -\frac{1}{2\pi G}\frac{1}{4\pi^2}\sum_{l=1}^{\infty}\int\int dJ_1 dJ_2 \frac{2l^2\Omega_1}{l^2\Omega_1^2 + \sigma^2}\frac{\partial F}{\partial J_1}|h_{l0}|^2 \geq 0\,. \tag{64}$$

Let s be the surface density associated with h. Then if we let s range over all axisymmetric densities in $\mathfrak{P}$, subject to the condition $(s, h) = 1$, the maximum value obtained by equation (64) is the bound of the response operator. Clearly $\beta_{\max}(\sigma)$ is a continuous nonincreasing function of σ, and tends to zero as $\sigma \to \infty$. Therefore, if $\beta_{\max}(0) < 1$, we can have no exponentially growing modes. The condition is also necessary, for if $\beta_{\max}(0) > 1$ we can find at least one exponentially growing mode. This follows from the fact that the response operator defined by the integral equation (36) is symmetric along $\omega^2 < 0$ and is by definition bounded by $\beta_{\max}(\sigma)$. A classical result on symmetric compact operators states that the function which maximizes $\beta(\sigma)$ is an eigenvector and $\beta_{\max}(\sigma)$ the corresponding eigenvalue. Since $\beta_{\max}(0) > 1$ and goes continuously to zero as $\sigma \to \infty$, there exists a σ for which $\beta_{\max} = 1$, and a corresponding eigenvector in $\mathfrak{S}$. The image of this vector in $\mathfrak{P}$ is the unstable mode.

The stability criterion $\beta_{\max}(0) < 1$ has a simple form. If we let $\sigma^2 \to 0$ in equation (64), we obtain

$$\beta = -\frac{1}{2\pi G}\frac{1}{4\pi^2}\sum_{l=1}^{\infty}\int\int dJ_1 dJ_2 \frac{\partial F}{\partial J_1}\frac{2}{\Omega_1}|h_{l0}|^2\,; \tag{65}$$

but since

$$\frac{1}{4\pi^2}\sum_{l=-\infty}^{\infty}|h_{l0}|^2 = \int_0^{2\pi} dw_2 \int_0^{2\pi}|h|^2 dw_1\,, \tag{66}$$

$$\beta = -\frac{1}{2\pi G}\int\int\int\int dJ_1 dJ_2 dw_1 dw_2 \frac{\partial F}{\partial J_1}\frac{1}{\Omega_1}\left(|h|^2 - \frac{1}{4\pi^2}|h_{00}|^2\right). \tag{67}$$

The expression in parentheses is the square of the fluctuating part of the potential as seen by a star. The constant part plays no role in dynamics.

If we write the stability criterion in the form $1 - \beta_{\max}(0) > 0$, it becomes identical with the positive-definiteness criterion of the Hartree-Fock exchange operator (Lynden-Bell 1969). Despite our choice of action-angle variables we have made no explicit use of adiabatic invariance in our arguments. The adiabaticity of h follows as a corollary.

The stability criterion based on equation (67) is an extension of Toomre's local result (Toomre 1964). The calculation of the upper bound of the response is clearly a variational matter, and Toomre's local result should be viewed as such also. Hence it is apt to be more relevant than it appears from the context of its derivation.

To complete the discussion of axisymmetric modes, we should examine the possibility of discrete stable oscillations. In a finite galaxy we may assume that the radial frequencies Ω_1 will have a lower bound Ω_*. The above arguments show that there will exist a stable axisymmetric mode whenever the margin of stability $\beta_* = 1 - \beta_{\max}(0)$ is sufficiently small. We see from equation (64) that $\beta_{\max}(\sigma)$ increases as σ^2 becomes negative, and if it reaches 1 for some $\omega < \Omega_*$ then there will be a stable mode with that frequency. The existence of such slow primeval oscillation in a galaxy such as ours would imply a rather small β_*. A rough estimate, based on equation (64), suggests that β_* is not likely to exceed 0.2.

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VIII. DISCUSSION

The development presented here has been intentionally on the formal side, for we wished to emphasize the form that the equations take. In the future calculations it will be necessary to introduce some approximations such as epicyclic orbits, Gaussian velocity distributions, etc. Had these approximations been introduced at the outset, the question of how they affect the results and whether they introduce any new effects could not be answered. These approximations will not introduce qualitative differences, and even quantitatively they are no worse than our ignorance about the unperturbed state.

The expression for the kernel of the response operator (eq. [36]) in the epicyclic approximation was derived by Kalnajs (1965). The epicyclic orbits were introduced at the outset, and because the fact that they do not represent an incompressible flow in phase space was overlooked, a spurious term appears in that epicyclic response operator. That spurious contribution is small (of order J_1/J_2), and has been ignored in all discussions based on that equation.

The dynamics of cold disks differs qualitatively as well as quantitatively from that of disks with random velocities. The response of a cold disk is unbounded, which leads to instabilities that grow arbitrarily fast (Hunter 1969). Some further information about such disturbances can be obtained from the two conservation theorems which remain valid for the cold disks.

The results of this paper apply also to the tidal-perturbation problem, and to a discussion of dynamical friction and statistical density fluctuations (Gilbert 1968). In both cases one seeks the density change produced by an orbiting point mass. Preliminary calculations have indicated that it is not sufficient to calculate the tidal effect as the sum of individual orbital displacements caused by the imposed potential, as the density change due to the displacements contributes a potential which in general is comparable in magnitude to the imposed potential. Such a significant polarization indicates that collective effects are important in flat galaxies.

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APPENDIX

A detailed discussion of the rate of decay of the kernel is beyond the scope of this paper. We will see that with epicyclic orbits the decay is sufficiently rapid that the kernel is well approximated in a finite region of the (α, α')-plane by only a finite number of terms in the l-sum of equation (36).

Without an explicit representation of the orbits we have to rely on asymptotic estimates of the behavior of the $e_l(\alpha, m, J)$. We note the following relevant facts that are shared by a wide class of orbits. From equation (33) we see that whenever α or l is large, $e_l(\alpha, m, J)$ is defined by a rapidly oscillating integral. If $l \gg \alpha$, we may integrate by parts twice and show that these terms decay at least as fast as l^{-2}. When $\alpha \gg l$, the leading contributions to the integral will come from points where the phase of the exponential is stationary. For most orbits these contributions will be of order $|\alpha|^{-1/2}$.

Using these estimates, we can show that along any line α (or α') = constant, the l-sum decays as $|\alpha'|^{-1/2}$ (or $|\alpha|^{-1/2}$). Only along the diagonal $\alpha = \alpha'$ does it remain constant (see eq. [37]). The decay is further enhanced by the integration over the J_i. However, the above estimates already suffice to ensure that the kernel is quadratically integrable.

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