Kinetic Theory of Spirals

cf. { L-B + K, '72; K, '71; P.D., et al '08 }

So far:

→ L-S hypothesis, spiral wave dynamics (basics)
→ propagation of spiral waves, range (prin range)
→ mechanisms of amplification ⟺ exploit ∈ energy (over-reflt)
→ role in disk energetics, angular momentum balance. (outward transport ⇒ relaxation of energy)

Prelude - ①

⇒ Resonances ⟺ How does wave interact with with Lindblad resonance, etc.??
→ bound principal Range, so
- emission absorption
- wave angular momentum deposition / acquisition

⇓

effect on

→ to understand wave - star transfer of angular momentum, etc. ⟺ kinetic theory.

→ Schematic OV:

- action-angle formulation most convenient

expect $J_1 \sim$ energy $\Rightarrow \theta_1 \sim$ epicycle
$J_2 \sim L_\varphi \Rightarrow \theta_2 \sim$ orbit angle.

- then, develop QLT in action - angle variables i.e. $\frac{\partial f}{\partial t} + \{f, H\} = 0$ $\qquad \frac{d\underline{\theta}}{dt} = \frac{\partial H}{\partial \underline{J}} = \underline{\omega}(\underline{J})$, $\quad \frac{d\underline{J}}{dt} = \frac{\partial H}{\partial \underline{\theta}}$

$$\Rightarrow \frac{\partial f}{\partial t} + \underline{\omega}\cdot\underline{\nabla}_\theta f + \frac{\partial \tilde{H}}{\partial \underline{\theta}}\cdot\underline{\nabla}_J f = 0$$

$\underline{\omega}\cdot\underline{\nabla}_\theta \to \omega_E \frac{\partial}{\partial \theta_E} + \omega_J \frac{\partial}{\partial \theta_J}$

$\hookrightarrow H = H_0 + \tilde{H}$

H_0: epicyclic; $\tilde{H}$: spiral mode perturbation

$$\therefore \frac{\partial \tilde{f}}{\partial t} + \underline{\omega}\cdot\underline{\nabla}_\theta \tilde{f} = \frac{\partial \tilde{H}}{\partial \underline{\theta}}\cdot\underline{\nabla}_J \langle f\rangle$$

$$\frac{\partial \langle f\rangle}{\partial t} = \frac{\partial}{\partial \underline{J}}\cdot\left\langle \frac{\partial \tilde{H}}{\partial \underline{\theta}}\tilde{f}\right\rangle$$

$\langle f\rangle = \langle f(J_E, J_h)\rangle$

$\ell \to$ mode # E (epicycle)
$m \to$ mode # circular orbit

$\frac{\partial \langle f\rangle}{\partial t} \Rightarrow \begin{cases}\text{absorption} \\ \text{emission}\end{cases}$

$$-i(\omega - \underline{p}\cdot\underline{\Omega})\tilde{f}_{p,\omega} = -\frac{\partial \tilde{H}}{\partial \underline{\theta}}\cdot\underline{\nabla}_J\langle f\rangle$$

$$\tilde{f}_{p,\omega} = \frac{-i}{(\omega - \underline{p}\cdot\underline{\Omega})}\frac{\partial \tilde{H}}{\partial \underline{\theta}}\cdot\underline{\nabla}_J\langle f\rangle$$

$$\Rightarrow \frac{\partial \langle f\rangle}{\partial t} = \frac{\partial}{\partial \underline{J}}\cdot\sum_p \frac{\partial \tilde{H}}{\partial \underline{\theta}}\pi\delta(\omega - \underline{p}\cdot\underline{\Omega})\frac{\partial \tilde{H}}{\partial \underline{\theta}}\cdot\frac{\partial \langle f\rangle}{\partial \underline{J}}$$

i.e.
$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial \underline{J}} \cdot \underline{\underline{D}} \cdot \frac{\partial \langle f \rangle}{\partial \underline{J}}$$

→ and can compute angular momentum scattering via moments

i.e.
$$\int dJ_E \int_{L_1}^{L_2} dJ_L \, J_L \frac{\partial \langle f \rangle}{\partial t} = \int dJ_E \int_{L_1}^{L_2} dJ_L \, J_L \frac{\partial}{\partial \underline{J}} \cdot \underline{\underline{D}} \cdot \frac{\partial \langle f \rangle}{\partial \underline{J}}$$

→ Wave - star angular momentum exchange will occur at resonances, i.e.

$$\omega = \underline{\ell} \cdot \underline{\Omega} = \underbrace{\ell \Omega_E}_{} + \underbrace{m \Omega_J}_{}$$
$$\sim \ell K + m \Omega(r))$$

$\ell = 1$ ⇒ Lindblad resonance for m-mode, etc.
etc.

→ sign ⇔ cross-term !

Thus, need:

→ formulate action - angle variables
→ derive Vlasov, QL equation
→ understand angular momentum transfer.

① Formulation

recall, from beginning:

$$E = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \underset{\text{mean}}{\bar{\Phi}} + \underset{\text{wave}}{\tilde{\phi}} \qquad (m=1)$$

$$d/dt\,(R^2\dot{\phi}) = -\frac{\partial\tilde{\phi}}{\partial\phi}$$

$$h = R^2\dot{\phi}$$

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial}{\partial R}\left(\bar{\Phi} + \tilde{\phi}\right)$$

u.p.o., $\tilde{\phi} = 0 \rightarrow \left.\begin{matrix} E \\ h \end{matrix}\right\}$ I.O.Ms for unperturbed motion

$$E = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2\right) + \bar{\Phi}$$

$$= \frac{1}{2}\left(\dot{R}^2 + h^2/R^2\right) + \bar{\Phi}$$

Now,
$$\frac{dR}{dt} = 2E - h^2/R^2 - 2\bar{\Phi}$$

$$= 2(E - \bar{\Phi}) - h^2/R^2$$

seek identify cyclic motion → angle variables

$$t - t_0 = \int^{R} \frac{1}{[\dot{R}]}\,dR \rightarrow \text{periodic epicyclic orbit}$$

$\left\{\begin{matrix}\text{epicycle} \\ \text{rotation}\end{matrix}\right.$

$$\boxed{\tilde{T} = \frac{2\pi}{\Omega_1} = \oint \frac{dR}{[\dot{R}]}}$$ epicyclic period!

$$\Rightarrow \boxed{\theta_1 = \theta_E = \Omega_1 \int^R \frac{dR}{[\dot{R}]}}$$ $\rightarrow$ phase (epicyclic angle variable)

now conjugate to θ_1 is action variable for epicyclic motion

i.e. $\sigma = J = \int p\, dq$

$$\Rightarrow \int \dot{R}\, dR$$

$$\therefore \boxed{J_1 = J_E = \frac{1}{2\pi} \oint [\dot{R}]\, dR}$$ $\left\{ J_1 = J_1(E, h) \right\}$

obviously: $\begin{cases} J_1 \to 0 & \text{for minimum energy circular orbit.} \\ J_1 \sim a_{epi}^2 \end{cases}$

$\therefore$ for E_j

$$J_1 = \frac{1}{2\pi} \oint [\dot{R}]\, dR \qquad \Omega_1 = \frac{1}{2\pi} \oint \frac{dR}{[\dot{R}]}$$

$$\theta_1 = \Omega_1 \int^R \frac{dR}{[\dot{R}]} \qquad T = \oint \frac{dR}{[\dot{R}]}$$

$$2E = \dot{R}^2 + h^2 + 2\Phi$$

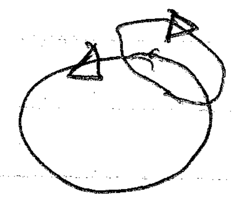
Now, for angular momentum variable:

$$\dot{\phi} = h/R^2$$

$$d\phi = \frac{h}{R^2}\,dt = \frac{h}{R^2}\,\frac{dR}{[\dot{R}]}$$

$$\boxed{\int_0^R \frac{h}{R^2}\,[\dot{R}]^{-1}\,dR = \phi - \phi_0}$$

i.e. epicycle affects both R, ϕ:



so ⟹ to isolate uniform change in ϕ

$$\Theta_2 = \phi - \int^R \left(hR^{-2} - \langle hR^{-2}\rangle\right)[\dot{R}]^{-1}\,dR$$

where:

$$\langle hR^{-2}\rangle = \frac{1}{2\pi}\,\Omega_1 \oint \left(hR^{-2}\right)[\dot{R}]^{-1}\,dK = \Omega_2$$

∴ Θ_2 implies $J_2 = h$

i.e. $\begin{cases} J_2 = J_L = h \Rightarrow \text{conserved angular momentum} \\ \Theta_2 \Rightarrow \text{above} \end{cases}$

$$\left.\begin{array}{l} J_1, \ \theta_1 \\ h = J_2, \ \theta_2 \end{array}\right\rangle \text{ action-angle variables}$$

Now then

$$\dot{R} = \left[2(E-\Phi) - h^2/R^2\right]^{1/2}$$

$\Rightarrow$

$$J_1 = J_1(E, h) = \frac{1}{2\pi}\oint [\dot{R}]\, dR$$

$$J_2 = h$$

$$H_{unperturbed} = E = E(J_1, J_2)$$

so

$$\boxed{\begin{aligned} \dot{J}_i &= -\frac{\partial H_u}{\partial \theta_i} = 0 \\ \dot{\theta}_i &= \frac{\partial H_u}{\partial J} = \Omega_i(J_1, J_2) \end{aligned}}$$

and

cyclic action-angle variables

Exercise: from

$$J_1 = J_1(E, h) = \frac{1}{2\pi}\oint [\dot{R}]\, dR$$

with

$$\dot{R} = \left[2(E-\Phi) - h^2/R^2\right]^{1/2}$$

show: $(\partial J_1/\partial E)_h = 1/\Omega_1$

i.e. $\partial H/\partial J_1 = \Omega_1$

and:

$(\partial J_1/\partial h)_E = -\Omega_2/\Omega_1$

using $(\partial E/\partial J_2)_{J_1} = -(\partial J_1/\partial h)_E / (\partial J/\partial E)_h$.

Special Limiting case:

~ nearly circular motion

~ $[\dot{R}]^2$ ~ quadratic deviation from circular
~ $(R-R_h)^2 \equiv R_1^2$

~ $\mathcal{E}(h)$, R_h as before circular

$\Rightarrow$ $\mu^2 = 3h^2/R_h^2 + \left.\dfrac{\partial^2\Phi}{\partial R^2}\right|_{R_h}$

$a^2 = 2(E-\mathcal{E}(h))/\mu_1^2$

and $[\dot{R}]^2 = \mu^2[a^2 - R_1^2]$

then,

$$\begin{cases} \Omega_1 = \kappa & J_1 = \tfrac{1}{2}\kappa a^2 = (E - E(h))/\kappa \\ \Omega_2 = \Omega_h & J_2 = h \end{cases}$$

$$\begin{cases} R_1 = R - R_h = a\sin\theta_1 \\ \phi = \theta_2 + 2(\Omega_h/\kappa)(a/R_0)\cos\theta_1 \end{cases}$$

② Now, with perturbation:

$$H_0 \rightarrow H_0 + \tilde{H}_1$$

$$\tilde{H}_1 = \tilde{\phi}$$

↳ wave gravitational potential,

so

$$\dot{J}_i = -\frac{\partial}{\partial\theta_i}(H_0 + \tilde{\phi})$$

$$= -\cancel{\frac{\partial H_0}{\partial\theta_0}} - \frac{\partial\tilde{\phi}}{\partial\theta_i}$$

$$= -\frac{\partial\tilde{\phi}}{\partial\theta_i}(J_i, \theta_i)$$

and

$$\dot{\theta}_i = \frac{\partial}{\partial J_i}(H_0 + \tilde{\phi})$$

$$= \Omega_i(J_0) + \frac{\partial \tilde{\phi}}{\partial J}$$

of course, take:

→ $\tilde{\phi} \sim e^{-i\omega t}$

→ turned on at $t = -\infty$

$$\tilde{\phi} = \sum_{l,m} \tilde{\phi}_{l,m}(J_0)\, e^{i(l\theta_1 + m\theta_2 - \omega t)}$$

Then for QL eqn

$$\frac{\partial f}{\partial t} + \Omega_1 \frac{\partial \tilde{f}}{\partial \theta_1} + \Omega_2 \frac{\partial \tilde{f}}{\partial \theta_2} = \left(\frac{\partial \tilde{\phi}}{\partial \theta_1}\frac{\partial}{\partial J_1} + \frac{\partial \tilde{\phi}}{\partial \theta_2}\frac{\partial}{\partial J_2} \right) \langle f \rangle$$

⟺ where ignore scattering correction to streaming

⟹

$$-i(\omega - l\Omega_1 - m\Omega_2)\tilde{f}_{l,m} = \tilde{\phi}_{l,m}\left(il\frac{\partial}{\partial J_1} + im\frac{\partial}{\partial J_2} \right)\langle f \rangle$$

$$\tilde{f}_{\ell,m} = \frac{-\tilde{\phi}_{\ell,m}\left(\ell \frac{\partial}{\partial J_1} + m \frac{\partial}{\partial J_2}\right)\langle f\rangle}{(\omega - \ell\Omega_1 - m\Omega_2)}$$

$$\cong \left(\frac{P}{(\omega - \ell\Omega_1 - m\Omega_2)} - i\pi\,\delta(\omega - \ell\Omega_1 - m\Omega_2)\right)\left(-\tilde{\phi}_{\ell,m}\right)$$

$$\left(\ell \frac{\partial}{\partial J_1} + m \frac{\partial}{\partial J_2}\right)\langle f\rangle$$

or, including non-resonant

$$= \frac{-\tilde{\phi}_{\ell,m}}{(\omega - \ell\Omega_1 - m\Omega_2 + i|\gamma|)}\left(\quad\right)$$

and so can write: (QL Eqn.)

$$\boxed{\frac{\partial\langle f\rangle}{\partial t} = \sum_{\ell,m}\left(\frac{\partial}{\partial J_1}\ell + \frac{\partial}{\partial J_2}m\right)|\tilde{\phi}_{\ell,m}|^2 \pi\,\delta(\omega - \ell\Omega_1 - m\Omega_2)\left(\ell\frac{\partial}{\partial J_1} + m\frac{\partial}{\partial J_2}\right)\langle f\rangle}$$

where considered only resonant diffusion, as spiral wave marginal

Then, for net change in angular momentum due resonant scattering: $(h = J_2)$

$$\langle h \rangle = \int dJ_2 \int_0^\infty dJ_1 \; h \, \langle f(J_1, J_2, t) \rangle$$

slow

so

$$\langle \dot{h} \rangle = \int dJ_2 \int_0^\infty dJ_1 \, h \, \frac{\partial}{\partial \underline{J}} \cdot \underline{\underline{D}} \cdot \frac{\partial \langle f \rangle}{\partial \underline{J}}$$

$$\langle \dot{h} \rangle = \int d\underline{J} \; h \, \partial \langle f \rangle / \partial t$$

$$= \int dJ_2 \int_0^\infty dJ_1 \; h \left\{ \frac{\partial}{\partial J_1} D_{1,1} \frac{\partial \langle f \rangle}{\partial J_1} + \frac{\partial}{\partial J_1} D_{1,2} \frac{\partial \langle f \rangle}{\partial J_2} + \frac{\partial}{\partial J_2} D_{2,1} \frac{\partial \langle f \rangle}{\partial J_1} + \frac{\partial}{\partial J_2} D_{2,2} \frac{\partial \langle f \rangle}{\partial J_2} \right\}$$

$$= -\int dJ_2 \int_0^\infty dJ_1 \left\{ D_{2,1} \frac{\partial \langle f \rangle}{\partial J_1} + D_{2,2} \frac{\partial \langle f \rangle}{\partial J_2} \right\}$$

$\Rightarrow$

$$\langle \dot{h} \rangle = -\int dJ_2 \int_0^\infty dJ_1 \sum_{\ell, m} m \left(\ell \frac{\partial \langle f \rangle}{\partial J_1} + m \frac{\partial \langle f \rangle}{\partial J_2} \right) *$$

$$\pi \, \delta(\omega - \ell \Omega_1 - m \Omega_2) \, |\tilde{\phi}_{\ell, m}|^2$$

Thus, have net rate of change of angular momentum due resonant interaction as:

$$\boxed{\langle \dot{h}_1 \rangle = -\int dJ_2 \int_0^{\infty} dJ_1 \sum_{\ell, m} m \left(\ell \frac{\partial \langle f \rangle}{\partial J_1} + m \frac{\partial \langle f \rangle}{\partial J_2} \right) * \pi \delta(\omega - \ell \Omega_1 - m \Omega_2) |\phi_{\ell, m}|^2}$$

Now, all "action" at resonances, so

→ resonance $\omega - \ell \Omega_1 - m \Omega_2 \cong 0$

$$\omega - \ell \kappa - m \Omega(r) \cong 0$$

$$\omega - m\Omega(r) = \ell \kappa$$

∴ $\ell = \pm 1$ → Lindblad resonances

$\ell = 0$ → co-rotation

$|\ell| > 1$ → h.o. Lindblad

→ resonant interaction spatially localized

$$\omega - \ell \Omega_1 - m \Omega_2(r) = 0.$$

→ the sign ??

$$\langle \dot{h} \rangle = -\int dJ_2 \int_0^{\infty} dJ_1 \sum_{l,m} \left(\overset{+}{m l}\, \overset{-}{\frac{\partial \langle f \rangle}{\partial J_1}} + \overset{+}{m^2}\, \overset{-}{\frac{\partial \langle f \rangle}{\partial J_2}} \right)$$

$$* \, \pi \, \delta(\omega - l\Omega_1 - m\Omega_2) \, |\tilde{\Phi}_{lm}|^2$$

Now;

→ $\partial\langle f \rangle / \partial J_1 < 0$

$J_1 \sim \kappa a^2$

so

$\langle f \rangle$ decreases with increasing epicyclic amplitude

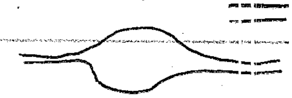
→ $\partial\langle f \rangle / \partial J_2 < 0$

$J_2 = h$

but → $VR \sim \Omega R^2$

$\sim R$.

→ $h \sim R$ but few stars at large R



relative size ?

⇒ $\partial\langle f \rangle / \partial J_2 < 0$

Now, for size: $J_1 \sim K a^2$
$J_2 \sim h \sim V_0 R$

$\Rightarrow J_2 \gg J_1$ so $\frac{\partial \langle f \rangle}{\partial J_2} \ll \frac{\partial \langle f \rangle}{\partial J_1}$, typically.

so

$$\langle \dot{h} \rangle \cong -\int dJ_2 \int_0^\infty dJ_1 \sum_{l,m} \left(m l \frac{\partial \langle f \rangle}{\partial J_1} \right) \pi \delta(\omega - l\Omega_1 - m\Omega_2) |\tilde{\phi}_{l,m}|^2$$

$$= \sum_m \sum_{l=1}^\infty -\int dJ_2 \int_0^\infty dJ_1 \; m \left(-|l| \frac{\partial \langle f \rangle}{\partial J_1} \pi \delta(\omega + \Omega_1 |l| - m\Omega_2) |\tilde{\phi}_{|l|,m}|^2 + l \frac{\partial \langle f \rangle}{\partial J_1} \pi \delta(\omega - l\Omega_1 - m\Omega_2) \times |\tilde{\phi}_{l,m}|^2 \right)$$

$$\boxed{\langle \dot{h} \rangle = \sum_m \sum_{l=1}^\infty -\int dJ_2 \int_0^\infty dJ_1 \; m \left\{ -|l| \frac{\partial \langle f \rangle}{\partial J_1} \pi \delta(\omega + |l| \Omega_1 - m\Omega_2) + l \frac{\partial \langle f \rangle}{\partial J_1} \pi \delta(\omega - l\Omega_1 - m\Omega_2) \right\} |\tilde{\phi}_{l,m}|^2}$$

$\rightarrow \sum_{\ell}$ = sum of h.o. Lindblad resonances

(see below) $\ell = \pm 1 \rightarrow$ primary

$\rightarrow \ \ell = -1 \Rightarrow -m\ell \frac{\partial \langle f \rangle}{\partial J_1}$

$\langle \dot{h} \rangle < 0 \Rightarrow$ angular momentum to wave

$\ell = +1 \Rightarrow -m\ell \frac{\partial \langle f \rangle}{\partial J}$

$\langle \dot{h} \rangle > 0 \Rightarrow$ angular momentum from wave.

n.b. What is going on?

$\rightarrow \ell$ resonances.

resonances at $\omega - m\Omega = \pm \ell \kappa$

$\ell = 1$; fundamental κ

$\ell = 3$: h.o. Lindblad

i.e. $\underbrace{\ }_{OLR_2}$ $\underbrace{\ }_{ILR_0}$ $\underbrace{\ }_{r_{co}}$ $\underbrace{\ }_{OLR_0}$ $\underbrace{\ }_{OLR_1}$

$\ell = 1$ Lindblad are inner bands around co-rotation

$\Leftrightarrow$ most relevant

$\rightarrow$

$$\langle \dot{h} \rangle = (\ell = -1 \text{ term}) + (\ell = +1 \text{ term})$$

evolution of particle angular momentum by resonant interaction

$$= \underbrace{m|\ell|}_{+} \underbrace{\frac{\partial \langle \Phi \rangle}{\partial J_1}}_{\ominus} \Big|_{-} - \underbrace{m\ell}_{+} \underbrace{\frac{\partial \langle \Phi \rangle}{\partial J_1}}_{\ominus} \Big|_{+}$$

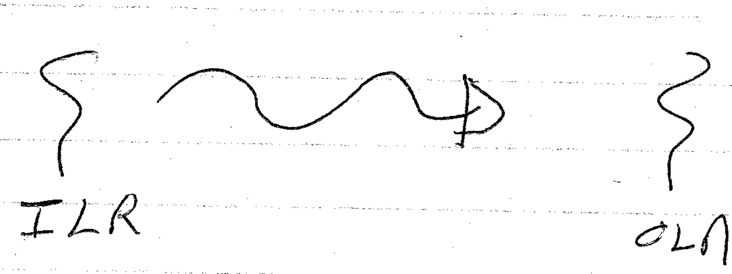
$\ominus \qquad \oplus$

$\Rightarrow$

at : r_{ILR} : stars give angular momentum to the spiral

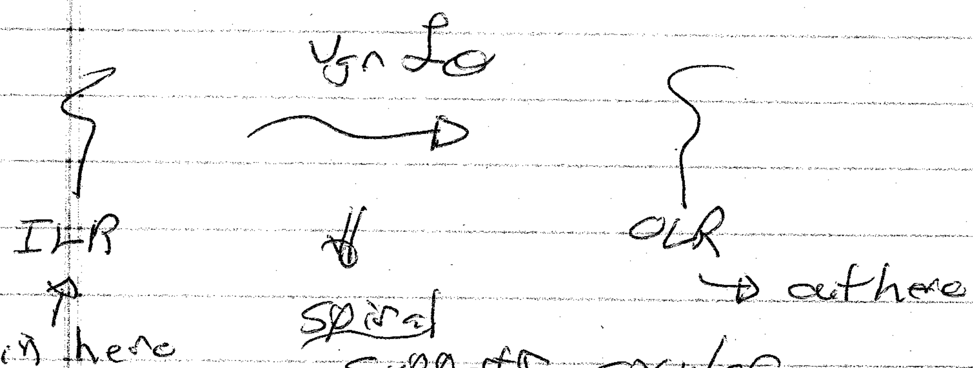
r_{OLR} : spiral returns angular momentum to the stars

$\rightarrow$ i.e. picture:



→ Wave transports angular momentum from r_{ILR} to r_{OLR}

→ angular momentum balance



spiral supports angular momentum transport.

→ Now, for energetics: (i.e. how accrete)

recall: $E_J = \frac{\dot{r}^2}{2} + \Phi + \tilde{\phi} - \frac{1}{2}|\Omega \times R|^2$ → Jacobi's Integral

here $\Omega = -\omega/m$ → frame in which spiral is steady ($\partial_t = 0$)

∴ in rotating frame:

$$E_J = \frac{1}{2}\left[v_r^2 + (v_\phi - \Omega R)^2\right] + \Phi + \tilde{\phi} - \frac{\Omega^2 R^2}{2}$$

$$= \frac{1}{2}\left[v_r^2 + v_\phi^2 - 2\Omega R v_\phi + \cancel{\frac{\Omega^2 R^2}{2}}\right] + \Phi + \tilde{\phi} - \cancel{\frac{\Omega^2 R^2}{2}}$$

$$E_J = \frac{1}{2}\left[v_r^2 + v_\phi^2\right] + \Phi + \tilde{\phi} - \Omega h$$

Now, $dE_J/dt = 0$

specific energy

$$E_J = G_T - \Omega h$$ specific angular momentum

$\Omega = +\omega/m$ = pattern freq

spec. energy loss

$\therefore$ $dG_T/dt = \Omega\, dh/dt$

specific angular momentum added.

$$dE/dt = \Omega\, dH/dt \qquad H = \int h \langle f \rangle$$

$\therefore$ rate of working of stars on wave
= Ω * (rate of angular momentum loss from wave)

Now,

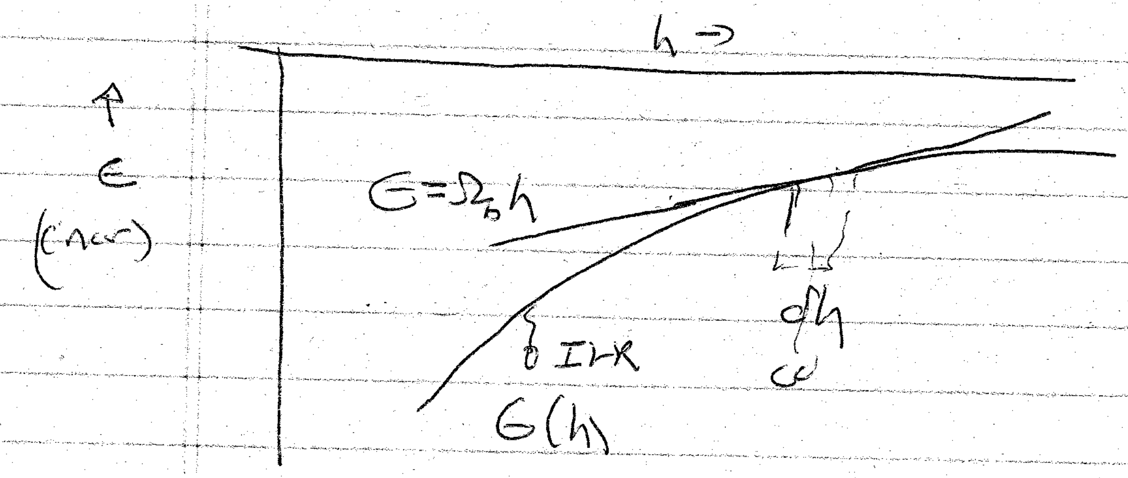
pattern

$\rightarrow$ $\Delta G_T = \Omega_p \Delta h \qquad (\Omega_p = +\omega/m)$

$\rightarrow$ $\epsilon(h) = \frac{1}{2} h^2/R_h^2 + \Phi(R_h)$

$$\frac{d\epsilon}{dH} = h/R_h^2 = \Omega(R_h)$$

∴ instructive to consider



at co-rotation slopes match → no change in energy

Now, $\frac{dE}{dh} = \Omega(R)$

$\underline{\underline{\text{so}}}$ $\Omega(R) = \Omega_p$ ⇒ co-rotation

so → if star δh at co-rotation ⇒ gains $\Omega_p \, \delta h$.

$\underline{\text{but}}$ $\frac{dE}{dh} = \Omega_p = \Omega(R)$ there

$\underline{\underline{\text{so}}}$ no change in stars vibrational energy

→ if star $\underline{\underline{\text{loses}}}$ δh at ILR,

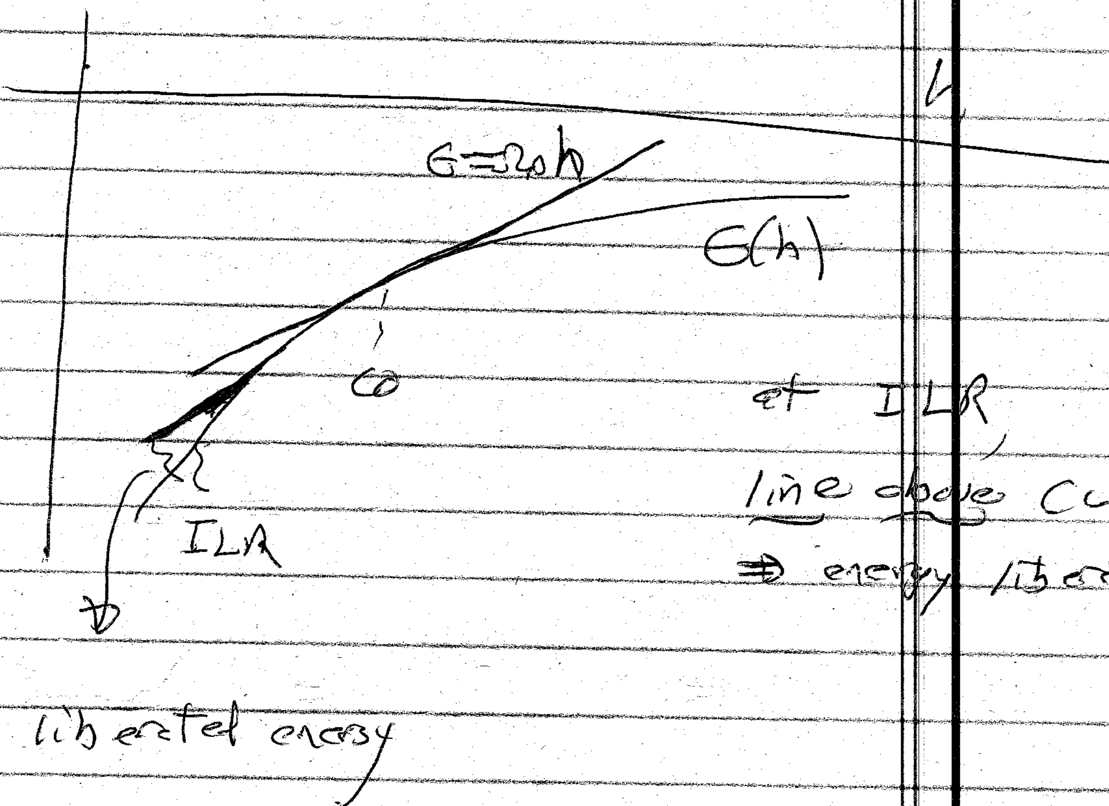
∴ loses $\Delta E = \Omega_p \delta h$

but $\Omega_p > \Omega(R)$ at ILR ⇒

star ... /radius with ...

less energy.

i.e.



at ILR

line above curve

⇒ energy liberated.