Spiral Modes / Spiral Structure

→ Fascination with, and beauty of, spiral waves is obvious and natural.

→ Here we discuss spiral modes in detail, in a sequence of:

- what? – what is a spiral mode?
  – how is it described?

- how do spiral wave patterns form? sustain self? – wave propagation dynamics → resonance ranges?
  – how waves amplify?
  – energy balance?

- why? – what role do spiral waves play in disk evolution/dynamics? – energetics?
  – relaxation?

Proceed from what? → why? → how?

→ What are spiral waves?

- Spiral waves are stable / quasi-marginal $m \neq 0$ Jeans modes of a differentially rotating disk

- spirals are the $m \neq 0$, stable counter-part of Toomre modes.

- why bother? → spiral structure!

Spiral density waves are the most likely mechanism for the origin of spiral structure in disk galaxies, etc.

Lin-Shu Hypothesis:

Origin of spiral structure is density wave propagating thru a differentially rotating disk!

→ wave / density → compression (bright arms → star formation etc.)

→ wave propagates in disk, i.e. ω, m, k_r
so ω / k_θ → azimuthal phase velocity
ω / k_r → radial phase velocity

→ wind up / "tight winding"

→ due differential rotation, refraction ⇒

$$dk_r/dt = -\frac{\partial}{\partial r}\left(\omega + k_\theta V_\theta\right)$$

$$\cong -k_\theta V_\theta'$$

$$k_r = k_r^{(0)} - k_\theta V_\theta' t$$ ⇒ k_r increases with time

thus, spiral pitch $= \frac{k_r}{k_\theta} \sim \frac{r\Delta\theta}{\Delta r} = V_\theta' t$

i.e. $\frac{r\,d\theta}{dr} \cong V_\theta' t$

pitch, increasing with time!

∴ $r\,d\theta/dr \sim k_r/k_\theta \sim V_\theta' t$

⇒ natural to work in "tightly wound" limit of $k_r \gg k_\theta$

⇒ allows local (radially) theory, avoids global mode problems, via WKB $\left(|1/k_r r| \ll 1\right)$

→ spiral structure

$$\phi(r) = i(m\theta + g(r,t))$$

spiral phase ↓ ; phase fctn ↓

i.e.

$$\tilde{\Gamma} \sim e^{i\phi} \Rightarrow \tilde{\Gamma} \; e^{i(m\theta + g(r,t))}$$

$$k_r = \frac{\partial g}{\partial r} \qquad d\phi = m\,d\theta + \frac{\partial g}{\partial r}dr = 0$$

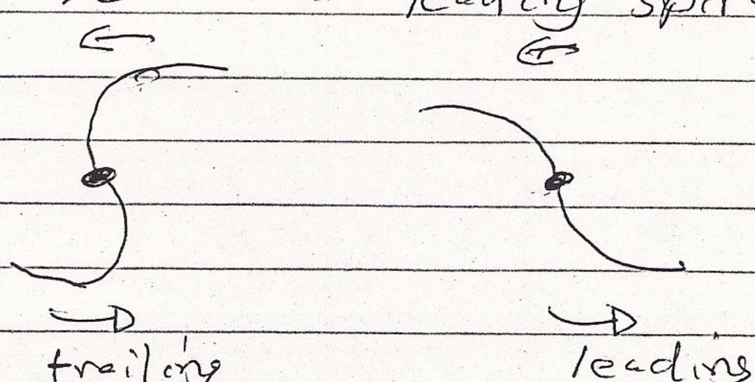
$$\omega = -\frac{\partial g}{\partial t}$$

$$d\phi = 0 \Rightarrow \frac{d\theta}{dr} = -\frac{\partial g/\partial r}{m} = \frac{-k_r}{m}$$

$d\theta/dr < 0$ → trailing spiral

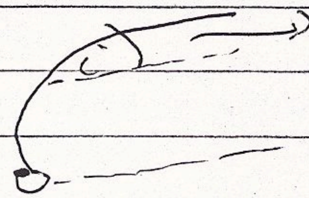
$d\theta/dr > 0$ → leading spiral

i.e.

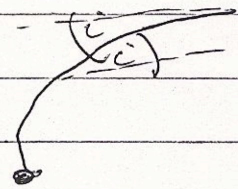


$$\frac{d\theta}{dr} < 0 \; \left(\begin{array}{l}\text{out in } r\\ \to \text{back in } \theta\end{array}\right)$$

Spiral-ology



→ pitch $\measuredangle$



$$\cot i = \frac{r\,\Delta\phi}{\Delta r} = r\,d\phi/dr$$

so if $\phi(r,t) = \phi_0 + \Omega(r)\,t$

$\cot i = r\,\Omega'\,t \Rightarrow$ "wind up"

i.e. pitch $\uparrow$ with time

note : $k_r k_\theta = m k_r / r$ → different for trailing, leading

obviously linked to

~~angular momentum transport~~

N.B. Tight winding points toward local limit.

⇒ Spiral Waves – Tight winding Limit

a la Toomre calculation, i.e. gas dynamics

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot (\sigma \underline{v}) = 0$$

$$\sigma_0 \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p - \sigma \nabla \phi$$

$$\nabla^2 \phi = 4\pi G\, \sigma \delta(z)$$

⇒

$$-i(\omega - m\Omega)\tilde{\sigma}_{\underline{k}} + \sigma_0 \left(i k_r \tilde{v}_{r_{\underline{k}}} + i k_\theta \tilde{v}_{\theta_{\underline{k}}} \right) = 0$$

→ enthalpy

$$-i(\omega - m\Omega)\tilde{v}_{r_{\underline{k}}} - 2\Omega \tilde{v}_{\theta_{\underline{k}}} = -i k_r (\tilde{\phi} + \tilde{h})_{\underline{k}}$$

$$-i(\omega - m\Omega)\tilde{v}_{\theta_{\underline{k}}} + \frac{\tilde{v}_{r_{\underline{k}}}}{r} \frac{\partial}{\partial r}(r^2 \Omega) = -i k_\theta (\tilde{\phi} + \tilde{h})_{\underline{k}}$$

$\therefore \quad \bar{\omega} = \omega - m\Omega$

$$\bar{\omega}\,\tilde{v}_r - 2i\Omega\,\tilde{v}_\theta = k_r(\tilde{\phi}+\tilde{h})$$

$$\frac{i}{r}\frac{\partial}{\partial r}(r^2\Omega)\,\tilde{v}_r + \bar{\omega}\,\tilde{v}_\theta = k_\theta(\tilde{\phi}+\tilde{h})$$

so

$$\tilde{v}_r = \left[\bar{\omega}^2 - \kappa^2\right]^{-1}\left\{\left(\bar{\omega}k_r - 2ik_\theta\Omega\right)(\tilde{\phi}+\tilde{h})\right\}$$

$$\tilde{v}_\theta = \left[\bar{\omega}^2 - \kappa^2\right]^{-1}\left\{\left(\bar{\omega}k_\theta - k_r\frac{i}{r}\frac{\partial}{\partial r}(r^2\Omega)\right)(\tilde{\phi}+\tilde{h})\right\}$$

obviously: $\tilde{v}_r, \tilde{v}_\theta \uparrow$ at

$\bar{\omega}^2 = \kappa^2 \Rightarrow \omega = m\Omega \pm \kappa$

Lindblad resonance (i.e. r_{LR})

at Lindblad resonance, locally Doppler shifted frequency matches $\pm$ epicyclic frequency

$\Rightarrow$ spiral pattern pushes continually on star.

⇒ Response to external forcing / torque strong at Lindblad resonance.

⇒ LBR ⟺ wave emission / absorption sites

Now, in tight winding limit: $(k_r \gg k_\theta)$

$$\tilde{\sigma}/\sigma_0 = k_r \tilde{v}_r / \bar{\omega}$$

$$\tilde{v}_r = \left(\bar{\omega}^2 - \kappa^2\right)^{-1} \left(\bar{\omega} k_r\right) \left(\tilde{\phi} + \tilde{h}\right)$$

$$\tilde{v}_\theta = \left(\bar{\omega}^2 - \kappa^2\right)^{-1} \left(-i \frac{k_r}{r} \frac{\partial}{\partial r}\left(r^2 \Omega\right)\right) \left(\tilde{\phi} + \tilde{h}\right)$$

and, as before:

$$\nabla^2 \tilde{\phi} = 4\pi G \sigma_0 \frac{\tilde{\sigma}}{\sigma_0} \delta(z)$$

$$\left(\frac{\partial^2}{\partial z^2} - k_r^2\right) \tilde{\phi} = 4\pi G \sigma_0 \frac{\tilde{\sigma}}{\sigma_0} \delta(z)$$

$(k_r \gg k_\theta)$

$$\Rightarrow \quad -2|k_r| \tilde{\phi}_u = 4\pi G \sigma_0 \frac{\tilde{\sigma}_u}{\sigma_0}$$

$$\tilde{\phi}_u = \frac{-2\pi G \sigma_0}{|k_r|} \left(\tilde{\sigma}_u / \sigma_0\right)$$

and $$\tilde{\phi} + \tilde{h} = \left(-\frac{2\pi G \sigma_0}{|k_r|} + c_s^2 \right) \tilde{\sigma}/\sigma_0$$

$$\tilde{\sigma}/\sigma_0 = k_r \tilde{v}_r/\bar{\omega} = \frac{k_r}{\bar{\omega}} \left(\frac{\bar{\omega} k_r (\tilde{\phi} + \tilde{h})}{\bar{\omega}^2 - \kappa^2} \right)$$

$$= \frac{k_r^2}{\bar{\omega}^2 - \kappa^2} \left(-\frac{2\pi G \sigma_0}{|k_r|} + c_s^2 \right) \tilde{\sigma}/\sigma_0$$

$\Rightarrow$

$$\boxed{\bar{\omega}^2 = \kappa^2 - 2\pi G \sigma_0 |k_r| + k_r^2 c_s^2}$$

Lin-Shu Dispersion relation

$\Rightarrow$ given tight winding, no surprise

Toomre + $\omega \to \omega - m\Omega$ recovers Lin-Shu

$\Rightarrow$ issue of excitation open. Must address how spirals excited, maintained,

$\Rightarrow$ in stellar dynamics:

$$\bar{\omega}^2 = \kappa^2 - 2\pi G \sigma_0 |k_r| \underbrace{\mathcal{F}}_{\text{reduction factor}}(\bar{\nu}, k_r \sigma/\kappa)$$

Aside: Note can write Lin-Shu dispersion relation as:

$$\bar{\omega}^2 = \kappa^2 - 2\pi G|k|\sigma_0 + k^2 c_s^2$$

$$= \underbrace{\kappa^2}_{\substack{\text{kinematic} \\ \text{(single particle)}}} + \text{collective}$$

$$\omega = \underbrace{m\Omega}_{\substack{\text{kinematic vibration} \\ \text{frequency}}} \pm \kappa \qquad (\to \text{Lindblad})$$

now, in reduction factor: $\bar{\omega}/\kappa = \bar{\nu}$

Note: Simplest to address reduction factor via pressure or enthalpy, i.e. calculate dispersion via $\tilde{P}_1$

$$\nabla^2\phi = 4\pi G\sigma_0 \tilde{\sigma}/\sigma_0 \Rightarrow -2|k_r|\hat{\phi}_1 = 4\pi G\sigma_0(\tilde{\sigma}_1/\sigma_0)$$

$$-i\bar{\omega}\tilde{\sigma}_1/\sigma_0 + ik_r\tilde{v}_{r_1} = 0$$

$$-i\bar{\omega}\tilde{v}_{\theta_1} + \frac{\tilde{v}_{r_1}}{r}\frac{d}{dr}(r^2\Omega) = 0$$

$$-i\bar{\omega}\tilde{v}_{r_1} = -ik_r(\hat{\phi}_1 + \hat{p}_1) + 2i\Omega\tilde{v}_{\theta_1}$$

preserves continuity + momentum balance of Jeans Eqns.

$\Rightarrow$

$$(\bar{\omega}^2 - \kappa^2)\left(\frac{1}{k_r}\right)\frac{\tilde{\sigma}_1}{\sigma_0} = k_r(\hat{\phi}_1 + \hat{p}_1)$$

$$\hat{\phi}_1 = \frac{-2\pi G\sigma_0}{|k_r|}(\tilde{\sigma}_1/\sigma_0)$$

$$\hat{p}_1 = \frac{\delta\hat{p}_1}{\delta(\tilde{\sigma}_1/\sigma_0)}\frac{\tilde{\sigma}_1}{\sigma_0}$$

$$\therefore (\bar{\omega}^2 - \kappa^2)\frac{\tilde{\sigma}_1}{\sigma_0} = k_r^2\left(\frac{-2\pi G\sigma_0}{|k_r|} + \frac{\delta\hat{p}_1}{\delta(\tilde{\sigma}/\sigma_0)_1}\right)\frac{\tilde{\sigma}_1}{\sigma_0}$$

$$\Rightarrow \quad \bar{\omega}^2 = K^2 - 2\pi \sigma_0 G |k_r| + k_r^2 \left[\frac{\delta \hat{p}_{u}}{\delta(\tilde{\sigma}/\sigma_0)_{u}} \right]$$

now $\delta \hat{p} / \delta(\tilde{\sigma}/\sigma_0) = \frac{\delta}{\delta(\tilde{\sigma}/\sigma_0)} \left[c_s^2 (\tilde{\sigma}/\sigma_0) \right]$

$$= c_s^2$$

$$\bar{\omega}^2 = K^2 - 2\pi G \sigma_0 |k_r| + k_r^2 c_s^2 \quad \checkmark$$

$$\delta \hat{p} = \int d^3 v \, \delta \hat{f} \, v^2$$

in Vlasov,

$$\delta p = \left(\int d^3 v \, \frac{\delta \hat{f}}{\delta \phi} v^2 \right) \delta \phi$$

<u>but</u> $\delta \phi = - \frac{2\pi G \sigma_0}{|k_r|} (\tilde{\sigma}/\sigma_0)_{u}$

$$\delta p = \int d^3 v \left(\frac{\delta \hat{f}}{\delta \phi} v^2 \right) \left(- \frac{2\pi G \sigma_0}{|k_r|} (\tilde{\sigma}/\sigma_0)_{u} \right)$$

$$\rightarrow \quad \bar{\omega}^2 - K^2 = -2\pi G |k_r| \sigma_0 \left[1 + \int d^3 v \left(v^2 \frac{\delta f_{u}}{\delta \phi_{u}} \right) \right]$$

This isolates kinetic reduction factor:

$$\mathcal{F} = 1 + \int d^3v \; v^2 \, \delta f_{\underline{k}} / \delta \phi_{\underline{k}}$$

Point is that

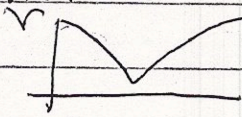
$$\delta f_{\underline{k}} = \underbrace{R(\overline{\omega}, k, v)}_{\text{response function}} \delta \hat{\phi}_{\underline{k}}$$

$$\boxed{\begin{aligned} \overline{\omega}^2 &= \kappa^2 - 2\pi G |k_r| \Sigma_0 \, \mathcal{F}(\overline{\nu}, k_r \sigma / \kappa) \\ \mathcal{F} &= 1 + \int d^3v \; v^2 R(\overline{\omega}, k_r, v) \end{aligned}}$$

then can write from prior:

$$\left\{ \begin{aligned} \mathcal{F} &\doteq \frac{2}{x} (1 - \overline{\nu}^2) e^{-x} \sum_{n=1}^{\infty} \frac{n^2 I_n(x)}{n^2 - \nu^2} \\ \overline{\nu} &= \overline{\omega} / \kappa \\ x &= k_r \sigma / \kappa \end{aligned} \right.$$

Point : ⇒ Structure of $\Im$ differs from gas dynamics.



↳ high k_r asymptote different

→ $k_r \gg k_\theta$ → only radial epicyclic scale significant. Can take

$$\langle f \rangle = \frac{1}{\sqrt{\pi}\, v_r} \exp\left[-v_r^2/\sigma^2\right] \delta\left(v_\phi - v_\phi(r)\right)$$

→ synergy between wind-up (i.e. $k_r = k_r^{(0)} - k_\theta V_\theta' t$) and $k_r \sigma / \kappa > 1$ cut-off ⇒ short wave somewhat debatable relevance, i.e. beware gas ⇒ dynamics transcription

Now, proceeding in gas dynamics equations, seek: → patterns (i.e. $k(\omega)$, as $\omega \leftrightarrow$ forcing)
→ resonances/barriers

$$\bar{\omega}^2 = \kappa^2 - 2\pi G \sigma_0 |k| + k^2 c_s^2$$

$$|k|^2 - \frac{2\pi G \sigma_0}{c_s^2} |k| + \frac{\kappa^2}{c_s^2}\left(1 - \frac{\bar{\omega}^2}{\kappa^2}\right) = 0$$

$\nu^2 = \bar{\omega}^2/\kappa^2$ $\qquad$ $\nu^2 = 1$ ⇒ $\omega = m\Omega \pm \kappa$
(Lindblad resonance)

$\Rightarrow$

$$|k_r| = \frac{\pi G\sigma_0}{c_s^2}\left[1 \pm \left(1 - \frac{4\kappa^2}{c_s^2}\left(\frac{c_s^2}{2\pi G\sigma_0}\right)^2(1-\nu^2)\right)^{1/2}\right]$$

$$\frac{\cancel{4}\kappa^2 c_s^{\cancel{4}}}{\cancel{4}\cancel{c_s^2}\pi^2 G^2\sigma_0^2} = Q^2 \equiv \kappa c_s/\pi G\sigma_0$$

Toomre stability factor

$$k_r = \frac{\pi G\sigma_0}{c_s^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]$$

$$k_T = \kappa_0^2/2\pi G\sigma_0$$

$$\boxed{|k|/k_T = \frac{2}{Q^2}\left[1 \pm \left(1 - Q^2(1-\nu^2)\right)^{1/2}\right]}$$

$\rightarrow |k_r(\omega)|$ form of Lin-Shu dispersion relation

$\rightarrow$ $+$ $\rightarrow$ short wave (more sensitive to gas dynamic nature of wave (i.e. sound) at high k)

$-$ $\rightarrow$ long wave

$\rightarrow$ yields $|k|$, $k > 0 \rightarrow$ trailing

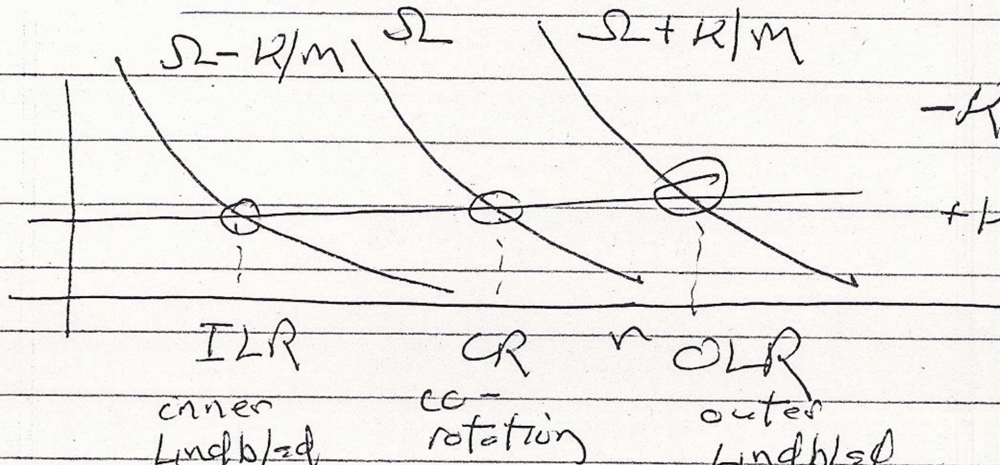
$k < 0 \rightarrow$ leading

related structures:

- resonances $\omega = m\Omega \pm \kappa$ $\qquad V_c \sim$ const.
  $\nu = \pm 1$

$$= m\Omega_0 \frac{r_0}{r} \pm \kappa$$

$$\Rightarrow r_{res} = m\Omega_0 r_0 / (\omega \mp \kappa)$$



(or $\pm$)
$-\kappa \to$ smaller r $\to$ ILR
$+\kappa \to$ larger r $\to$ OLR

$\Rightarrow$ Lindblad resonances $\Leftrightarrow$ $k_L = 0$

$\Rightarrow$ defines Principal Range

$$r_{ILR} < r < r_{OLR}$$

(long) wave propagates in range inbetween 2 Lindblad resonances.

∴ – long spiral wave propagates on $r_{ILR} < r < r_{OLR}$

– waves excited / absorbed at Lindblad resonances ⇒ tidal force is likely mechanism (c.f. inter-galactic torque). Reason for importance of resonance is intergalactic torque varies on scale $\ell \gg k_r^{-1}$ ⇒ spatially broad forcing ⇒ resonance sets location of effect.

N.B.: Confirmed on Voyager data on Saturn rings (i.e. Moons are "external forcing").

– barriers

$$\frac{|k|}{k_T} = \frac{2}{Q^2}\left\{1 \pm \left[1 - Q^2(1-\nu^2)\right]^{1/2}\right\}$$

so if $Q(r)$ s/t $\frac{1}{Q^2} < (1-\nu^2)$

⇒ $k_I \neq 0$, so evanescent wave.

⇝ defines Q barrier

⇝ waves damp in radius

now: $1/Q^2 < 1 - \nu^2$

$$\nu^2 < 1 - 1/Q^2$$

so for $Q \sim 1 + \epsilon$ $\quad \nu^2 < \epsilon$

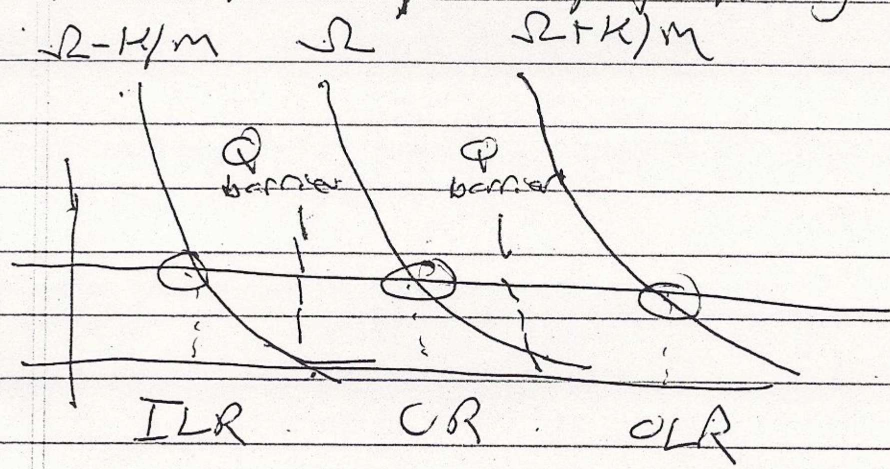
at co-rotation <u>i.e.</u> $\nu \to 0$

$\therefore \to$ Q - barrier surrounds co-rotation

$\to Q(r) \Rightarrow$ barrier may be related to bulge

$\to$ k_{Long}, k_{Short} co-desce at Q barrier.

so now have Principal Range:



$\omega = m\Omega \pm \kappa \quad \to LR$

$\frac{1}{Q^2} \lesssim 1 - \nu^2 \quad \to$ Q barrier region

∴ can see two useful limits:

$Q \gg 1 \Rightarrow \quad 1/Q^2 \leq 1-\nu^2$

$r_Q \to r_{LBR}$

$\Rightarrow$ Q barrier at $\nu = \pm 1$
Lindblad resonances

$\Rightarrow$ Principal Range excluded.

i.e. 'Wisp' galaxies - disks with low σ and $Q \gg 1$ do not have spirals! Exception to 'rule' of disk $\to$ spirals which proves rule of spirals $\leftrightarrow$ principal range.

$Q \gtrsim 1 \Rightarrow$ narrow barrier about co-rotation

- waves (long) everywhere except ring at $\omega \sim m\Omega$

- evanescent waves can tunnel / interact thru Q barrier

$\Rightarrow$ possibility of feedback amplification mechanisms.

⇒ long, short waves can refract into one another.

So seek understand dynamics of spiral wave propagation on Principal Range, with resonances, barriers.

∴ come to:

- spiral wave
  - energy
  - momentum
  - action
- wave amplification mechanisms
  - WASER
  - SWING
- spiral wave kinetics
- excitation / absorption

Role of Spirals in Disk Dynamics.

Why? → See Lynden-Bell & Pringle → '74
" " + Kalnajs → '72

① What is the state/profile of rotation which minimizes rotational kinetic energy?

{n.b. why a consideration?

Leading to:

② What must happen to angular momentum to achieve this state? – Process/dynamically?

Theorem: For a given density distribution, and fixed total angular momentum, the minimum energy profile is that of uniform rotation (Ω = const)

{a la' Rayleigh! but different

To Prove:

1) $K_{Rot} = \frac{1}{2}\int \rho V_\phi^2 d^3x$

$= \frac{1}{2}\int V_\phi^2 dM$ → rotational kinetic energy

2) $L_\phi = \int R V_\phi \, dm$ → angular momentum

3) $I = \int dm R^2$ → moment of inertia

then $\quad K_{rot} * I$

$$= \int dm \frac{V_\phi^2}{2} \int dm\, R^2 \geq \left(\int dm\, V_\phi R\right)^2 / 2$$

$$\geq L_\phi^2 / 2$$

by Schwarz Inequality

$$\Rightarrow \quad \boxed{K_{rot} \geq L_\phi^2 / 2I}$$

equality for uniform/solid body rotation

i.e. $V_\phi \sim \Omega_{const} R$ equality holds

for solid body:

$$V_\phi = \underbrace{\left(\frac{L_\phi}{I}\right)}_{\Omega_{eff}} R$$

i.e. $$\int dm\, V_\phi^2 = \frac{L_\phi}{I} \int dm\, R \cancel{V_\phi}$$

$$\Rightarrow \quad V_\phi = \frac{L_\phi}{I} R \ldots$$

Thus, minimum energy profile is state of uniform rotation $\Leftrightarrow$ i.e. solid body

$$V_\phi = \Omega_{eff} R.$$

aside: For given L_ϕ, can always find the minimum energy state by:

$$\rightarrow K_{Rot} = L_\phi^2 / 2I \quad \rho(r) \rightarrow I \text{ known}$$

$$\Omega_{eff} = L_\phi / I$$

and $\delta K_{Rot} / \delta \rho(r) = 0$ $\Rightarrow$ minimize w/r density distributions

since $\rho(r)$ determines I

i.e. vary ρ to minimize K_{rot}, assuming solid body.

Then:

1) to approach minimum energy, system should approach uniform rotation. $\rightarrow$ $V_\phi(R)$ evolution

2) energy minimization $\Rightarrow$ angular momentum transport (spirals, turbulent viscosity)

3) energy minimization via transport $\Rightarrow$ energy shed / converted to fluctuations and dissipated, i.e.

- spirals $\Leftrightarrow$ Lindblad resonance, co-rotation resonance } wave absorption, Landau damping, etc.

$\Rightarrow$ heating!

- turbulence $\Leftrightarrow$ viscous heating.

$\Rightarrow$ will increase entropy / probability!

$\therefore$ {suggests that galaxy seeks to <u>lower</u> its energy by (transporting / re-distributing) angular momentum.

$\Rightarrow$ as $L_\phi \sim \Omega R^2 \sim R$
($V_\phi \sim \Omega R \sim$ const)

$\Rightarrow$ galaxy should transport L_ϕ <u>outward</u>, to approach $L_\phi \sim R^2$
(also allows <u>accretion</u>!)

to see:

$h = R V_\phi \rightarrow$ specific angular momentum (i.e. of star/particle)

$\mu(h)\,dh \rightarrow$ mass with specific angular momentum in $(h, h+dh)$

mass density, as fctn h

$$M(h) = \int_0^h dh\, \mu(h) \rightarrow \text{total mass (in } h' < h)$$

$\Rightarrow$

$$K_{Rot} = \int dm\, V_\phi^2/2 = \int dh\, \mu(h)\, \frac{h^2}{2R}$$

$$\boxed{K_{Rot} = \int dh\, \mu(h)\, h^2/2R^2}$$

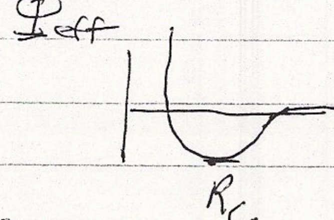
Also, for given star / particle:

$$\epsilon = \frac{1}{2}\left(V_r^2 + V_z^2 + h^2/R^2\right) + \Phi$$

→ specific energy, in thin disk

then: ϵ minimal for:

- circular orbit, with
  $V_r = V_z = 0$
  $z = 0$



- R_c s/t
  $$-\frac{h^2}{R_c^3} + \left.\frac{d\Phi}{dR_c}\right|_{z=0} = 0$$

in terms h (angular momentum):

$$\left.\epsilon(h)\right|_{minimal} = h^2/2R_c^2 + \Phi(R_c, 0) \rightarrow$$

energy for circular orbit of $\begin{cases} \text{specific angular momentum } h \\ \text{circular } R_c. \end{cases}$

$$\left.\frac{d\epsilon(h)}{dh}\right|_{R_c} = h/R_c^2 = \Omega(R_c)$$

Now consider 2 stars ("rings" - à la Rayleigh); but don't conserve each ℓ, only total ℓ.

$\Rightarrow$ h_1 with $\epsilon(h_1)$ any h_1 and m_1
h_2 $\epsilon(h_2)$ h_2 m_2.

$\therefore$ then ask how alter system such that
$\rightarrow$ reduce energy
$\rightarrow$ conserve L_ϕ ?

$$E = m_1\,\epsilon(h_1) + m_2\,\epsilon(h_2)$$

$$L_\phi = m_1 h_1 + m_2 h_2$$

seek: $dL_\phi = 0$

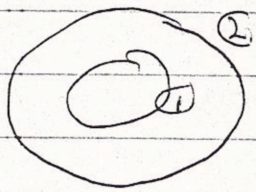
$dE < 0$

$$dL_\phi = 0 \Rightarrow m_1\,dh_1 + m_2\,dh_2 = 0$$

$$dE = m_1\,\epsilon_1'(h_1)\,dh_1 + m_2\,\epsilon_2'(h_2)\,dh_2$$

Note: In familiar Rayleigh argument, angular momentum of each ring is conserved

i.e.



① → r_1, L_1, M

② → r_2, L_2, M

$$2M\Delta E = \frac{L_2^2}{r_1^2} - \frac{L_2^2}{r_2^2} + \frac{L_1^2}{r_2^2} - \frac{L_1^2}{r_1^2}$$

$$= L_2^2\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) - L_1^2\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

$\therefore$ $\Delta E > 0 \Leftrightarrow L_2 > L_1$

$< 0 \Leftrightarrow L_1 < L_2$

Point: Lynden-Bell argument conserves total angular momentum of 2 rings, not each individually. Allows ① → ② transfer.

so $dE = m_1\, dh_1 \left(\epsilon'(h_1) - \epsilon_2'(h_2) \right)$

$= m_1\, dh_1 \left(h_1/R_{c_1}^2 - h_2/R_{c_2}^2 \right)$

$$\boxed{dE = m_1\, dh_1 \left(\Omega(R_{c_1}) - \Omega(R_{c_2}) \right)}$$

if: $R_{c_1} < R_{c_2}$

$\Omega \sim 1/R \Rightarrow$ angular velocity decreases outward

$\Rightarrow \Omega(R_{c_1}) - \Omega(R_{c_2}) > 0$

$\therefore\ dE < 0 \Rightarrow dh_1 < 0$

i.e. for $d\Omega/dr < 0 \rightarrow dh < 0$; obviously $dh = 0$ for $\Omega' = 0$.

$\Rightarrow$ Inner particle, higher $\Omega \Rightarrow$ loses angular momentum ($dh_1 < 0$)

Outer particle, lower $\Omega \rightarrow$ gains angular momentum ($dh_2 > 0$)

$dh_1 < 0$, $dh_2 > 0$ $\Rightarrow$ angular momentum must be re-distributed outward to large R, to reduce E for $\Omega' < 0$.

∴ have shown:

→ minimum energy lowered — energy relaxes — *if* angular momentum flows outward.

→ consistent with accretion
i.e. mass inflow → E_{grav} lowered
needs angular momentum outflow

i.e. how?
- galaxy → spirals (waves) → wave transport
- thin disk → turbulence (eddys → incoherent) → viscous transport

→ For angular momentum to flow outward, spiral waves must generate correct sign angular momentum flux

∴ For relaxation of energy ⇔ trailing spiral required!

i.e. $\Pi_{r\phi} = \sigma(r)\, r \langle \tilde{v}_r \tilde{v}_\phi \rangle$

now, recall:

$\bar{\omega} = \omega - m\Omega$

enthalpy

$$\tilde{v}_{r_u} = (\bar{\omega}^2 - \kappa^2)^{-1} \left\{ (\bar{\omega} k_r - 2i k_\theta \Omega)(\tilde{\phi} + \tilde{h})_u \right\}$$

$$\tilde{v}_{\phi_u} = (\bar{\omega}^2 - \kappa^2)^{-1} \left\{ \left(\bar{\omega} k_\theta - k_r \frac{i}{r} \frac{\partial}{\partial r}(r^2\Omega) \right) (\tilde{\phi} + \tilde{h})_u \right\}$$

then for $\bar{\omega}$ purely real,

$$\langle \tilde{v}_r \tilde{v}_\phi \rangle = \sum_u \frac{1}{(\bar{\omega}^2 - \kappa_0^2)^2} \left[\bar{\omega}^2 k_r k_\theta - \kappa^2 k_r k_\theta \right] \left| (\tilde{\phi} + \tilde{h})_u \right|^2$$

$$\langle \tilde{v}_r \tilde{v}_\phi \rangle = k_r k_\theta \left(\frac{\bar{\omega}^2 - \kappa^2}{(\bar{\omega}^2 - \kappa^2)^2} \right) \left| (\tilde{\phi} + \tilde{h})_u \right|^2$$

$$= k_r k_\theta \left(\frac{1}{(\bar{\omega}^2 - \kappa^2)} \right) \left| (\tilde{\phi} + \tilde{h})_u \right|^2$$

$\Rightarrow \;\rightarrow k_r k_\theta$

$\Rightarrow \;\rightarrow$ large at Lindblad resonance

$\Rightarrow$ strong coupling

now, $\tilde{\phi} \sim e^{im\theta} e^{if}$

$$m\,d\theta + \frac{df}{dr} dr = m\,d\theta + k_r dr = 0$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{-k_r}{m}$$

$$\sigma_0 r \langle \tilde{v}_r \tilde{v}_\varphi \rangle = \sigma_0 m k r \frac{(\bar{\omega}^2 - \kappa^2)}{(\bar{\omega}^2 - \kappa^2)^2} |(\tilde{\varphi} + \tilde{h}_1)|^2$$

$$= \sigma_0 m k r \left[\frac{(\bar{\omega}^2 - \kappa^2)}{(\bar{\omega}^2 - \kappa^2)^2}\right] |\tilde{\varphi}|^2 \left(1 + \frac{k_r^2 c_s^2}{\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2}\right)^2$$

$$= \sigma_0 m k r \frac{(\bar{\omega}^2 - \kappa^2)}{(\bar{\omega}^2 - \kappa^2)^2} |\tilde{\varphi}|^2 \left(\frac{\bar{\omega}^2 - \kappa^2}{\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2}\right)^2$$

$$= \frac{\sigma_0 m k r \, 2(\bar{\omega}^2 - \kappa^2)}{(\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2)^2} \frac{|\tilde{\varphi}_4|^2}{2}$$

$$\bar{\omega}^2 = \kappa^2 - 2\pi G \sigma_0 |k| + k^2 c_s^2$$

$$\bar{\omega}^2 - \kappa^2 = -2\pi G \sigma_0 |k| + k^2 c_s^2$$

$$\sigma_0 r \langle \tilde{v}_r \tilde{v}_\varphi \rangle = \sigma_0 m k r \frac{(-2\pi G \sigma_0 |k| + k^2 c_s^2)}{(\bar{\omega}^2 - \kappa^2 - k_r^2 c_s^2)^2} |\varphi_4|^2$$

so $\quad d\theta/dr < 0 \Rightarrow$ trailing spiral

$\Rightarrow k_r m > 0$

$\Rightarrow \langle \tilde{v}_r \tilde{v}_\phi \rangle > 0$

$\Rightarrow$ trailing spiral $\Leftrightarrow \Pi_{r\phi} > 0$ (outward angular momentum transport)

$\Rightarrow$ acts to reduce energy

$\Leftrightarrow$ (allows / furthers) accretion.

Note: Can see alternatively, via:

$$\underline{f}_g = +\rho\,\underline{g} = -\rho\,\underline{\nabla}\phi$$

$$\nabla^2\phi = 4\pi G\rho$$

$$\underline{f}_g = -\frac{\nabla^2\phi}{4\pi G}\,\underline{\nabla}\phi$$

mean $\downarrow$ $\quad$ wave $\tilde{\phi}$ $\downarrow$

$$\langle \underline{f}_g \rangle = -\frac{\langle \nabla^2\tilde{\phi}\,\underline{\nabla}\tilde{\phi}\rangle}{4\pi G}$$

on slab

$$= -\left\langle \left(\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{\partial^2\tilde{\phi}}{\partial y^2}\right)\left(\frac{\partial\tilde{\phi}}{\partial x}\hat{x} + \frac{\partial\tilde{\phi}}{\partial y}\hat{y}\right)\right\rangle$$

$$= -\left\langle \left(\frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial\tilde{\phi}}{\partial x}\right)^2\right] + \frac{\partial}{\partial y}\left(\left[\left(\frac{\partial\tilde{\phi}}{\partial y}\right)\left(\frac{\partial\tilde{\phi}}{\partial x}\right)\right] - \left(\frac{\partial\tilde{\phi}}{\partial y}\right)\frac{\partial^2\tilde{\phi}}{\partial y\,\partial x}\right)\right)\hat{x}\right\rangle$$

$$- \left[\frac{\partial}{\partial x} \left(\frac{\partial \tilde{\phi}}{\partial x} \frac{\partial \tilde{\phi}}{\partial y} \right) - \left(\frac{\partial \tilde{\phi}}{\partial x} \frac{\partial^2 \tilde{\phi}}{\partial x \partial y} \right) \right] \hat{y} + \frac{\partial}{\partial y} \frac{1}{2} \left(\frac{\partial \tilde{\phi}}{\partial y} \right)^2 \hat{y} \Big\rangle$$

$$= - \left\langle \frac{\partial}{\partial x} \left(\frac{\tilde{\phi}_x^2}{2} \right) + \frac{\partial}{\partial y} \left(\tilde{\phi}_y \tilde{\phi}_x \right) \right\rangle \hat{x}$$

$$+ \frac{1}{2} \left\langle \frac{\partial}{\partial x} \frac{\tilde{\phi}_y^2}{2} \right\rangle \hat{x}$$

$$- \left\langle \frac{\partial}{\partial x} \left(\tilde{\phi}_x \tilde{\phi}_y \right) + \frac{\partial}{\partial y} \frac{1}{2} \left(\tilde{\phi}_y \right)^2 \right\rangle \hat{y}$$

$$+ \frac{1}{2} \left\langle \frac{\partial}{\partial y} \left(\tilde{\phi}_x \right)^2 \right\rangle \hat{y}$$

$$= - \left(\partial_x , \partial_y \right) \begin{bmatrix} \tilde{\phi}_x \tilde{\phi}_x , & \tilde{\phi}_x \tilde{\phi}_y \\ \tilde{\phi}_y \tilde{\phi}_x , & \tilde{\phi}_y \tilde{\phi}_y \end{bmatrix}$$

$$+ \frac{1}{2} \begin{bmatrix} \partial_x \left(\tilde{\phi}_x \tilde{\phi}_x + \tilde{\phi}_y \tilde{\phi}_y \right) & 0 \\ 0 , & \partial_y \left(\tilde{\phi}_x \tilde{\phi}_x + \tilde{\phi}_y \tilde{\phi}_y \right) \end{bmatrix}$$

$$= - \underline{\nabla} \cdot \underline{\underline{T}}$$

$$\underline{\underline{T}} = \begin{pmatrix} \tilde{\phi}_x \tilde{\phi}_x , & \tilde{\phi}_x \tilde{\phi}_y \\ \tilde{\phi}_x \tilde{\phi}_y , & \tilde{\phi}_y \tilde{\phi}_y \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \tilde{\phi}_x \tilde{\phi}_x + \tilde{\phi}_y \tilde{\phi}_y , & 0 \\ 0 , & \tilde{\phi}_x \tilde{\phi}_x + \tilde{\phi}_y \tilde{\phi}_y \end{pmatrix}$$

$$\underline{f} = - \underline{\nabla} \cdot \underline{\underline{T}}$$

and since periodicity in $y \Rightarrow \langle \partial_y (\,) \rangle = 0$

$\Rightarrow$

$$\langle \underline{f} \rangle = -\frac{\partial}{\partial x}\left\langle \frac{(\tilde{\phi}_x)^2}{2} \right\rangle \hat{x} - \frac{\partial}{\partial x} \langle \tilde{\phi}_x \tilde{\phi}_y \rangle \hat{y}$$

$$+ \frac{1}{2} \frac{\partial}{\partial x} \left\langle \frac{(\tilde{\phi}_y)^2}{2} \right\rangle$$

$$= -\frac{1}{2} \partial_x \left\langle \left((\tilde{\phi}_x)^2 - (\tilde{\phi}_y)^2 \right) \right\rangle \hat{x} - \partial_x \langle \tilde{\phi}_x \tilde{\phi}_y \rangle \hat{y}$$

for torque density: $\underline{T} = \tilde{x} \times \langle \underline{f} \rangle$

$$\underline{\underline{T}} = T_z \equiv -\partial_x \langle \tilde{\phi}_x \tilde{\phi}_y \rangle$$

Then,

$$\underset{\substack{\uparrow \\ \text{angular momentum density}}}{\frac{d}{dt} \mathcal{L}_\phi} = \underset{\substack{\uparrow \\ \text{torque density}}}{\underline{\underline{T}}} = -\partial_x \langle \tilde{\phi}_x \tilde{\phi}_y \rangle$$

Equivalently:

$$\frac{\partial}{\partial t} r\langle V_\theta\rangle = -\frac{\partial}{\partial r}\left\langle \tilde{v}_r (r\tilde{v}_\theta)\right\rangle$$

$$= -\frac{\partial}{\partial r}\Pi_{r,\phi}$$

$$\Pi_{r,\phi} = r\langle \tilde{v}_r \tilde{v}_\phi\rangle$$

→ formulations are clearly equivalent

→ both $\sim k_r k_\phi$