Physics 2D Lecture Slides
Week of May 18, 2009

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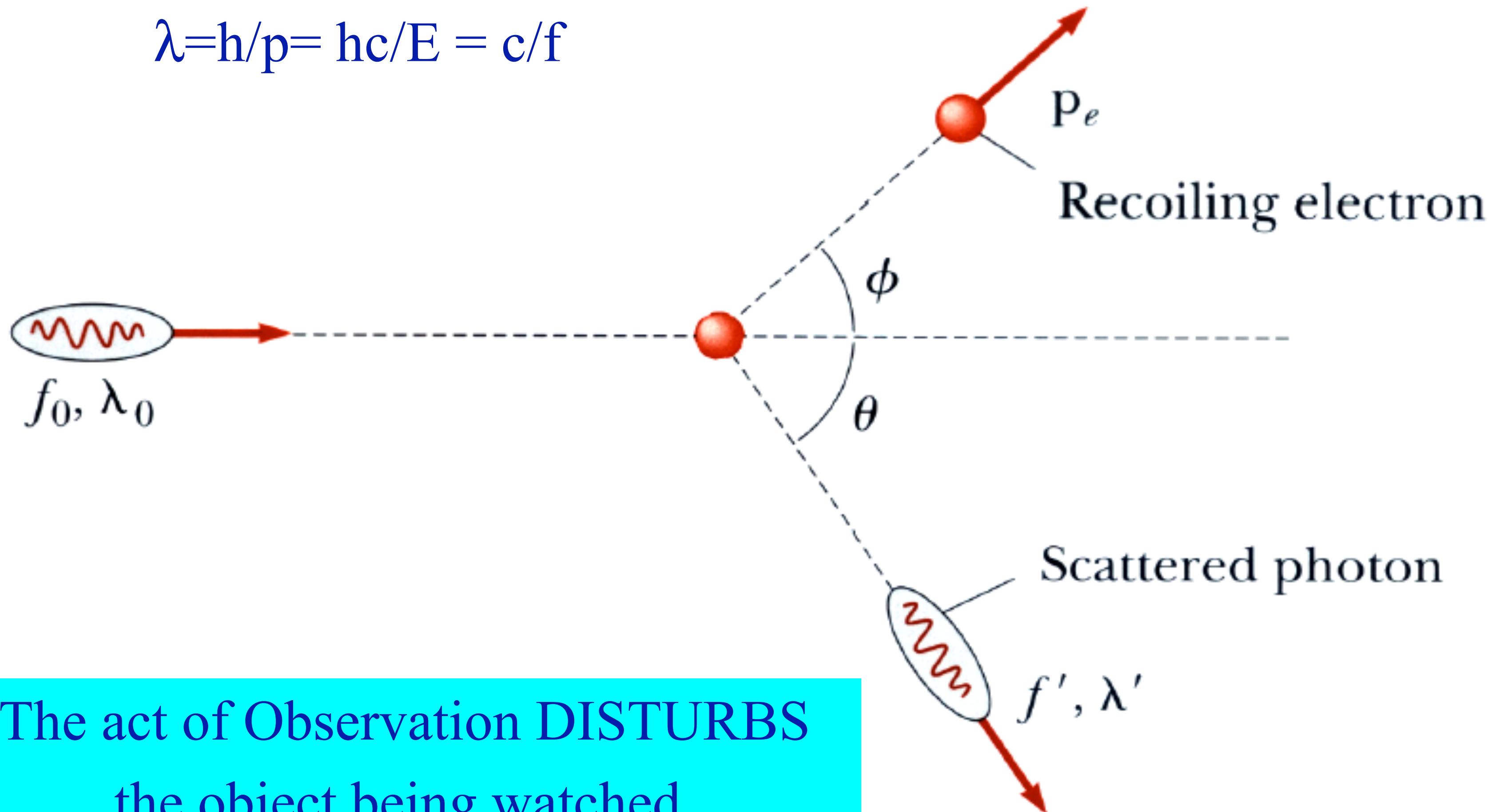
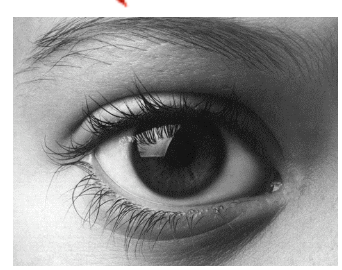
Compton Scattering: Shining light to observe electron

$\lambda = h/p = hc/E = c/f$

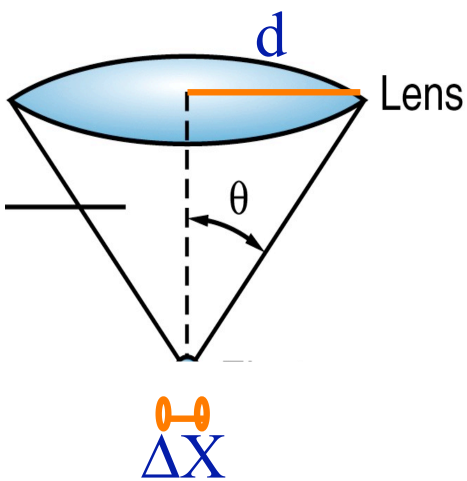


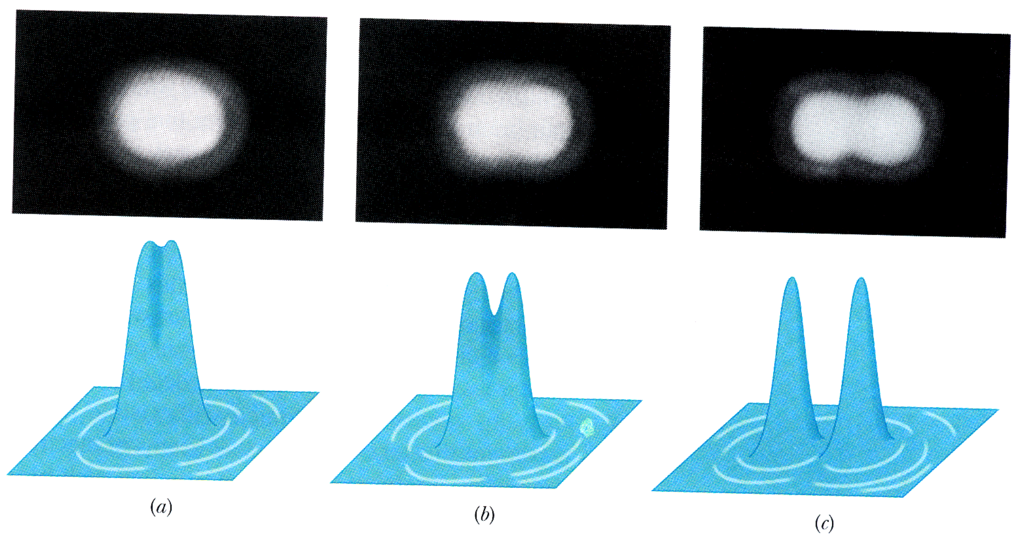
The act of Observation DISTURBS
the object being watched,
here the electron moves away from
where it was originally



Resolving Power of Light Thru a Lens

Image of 2 separate point sources formed by a converging lens of diameter d, ability to resolve them depends on λ & d because of the Inherent diffraction in image formation



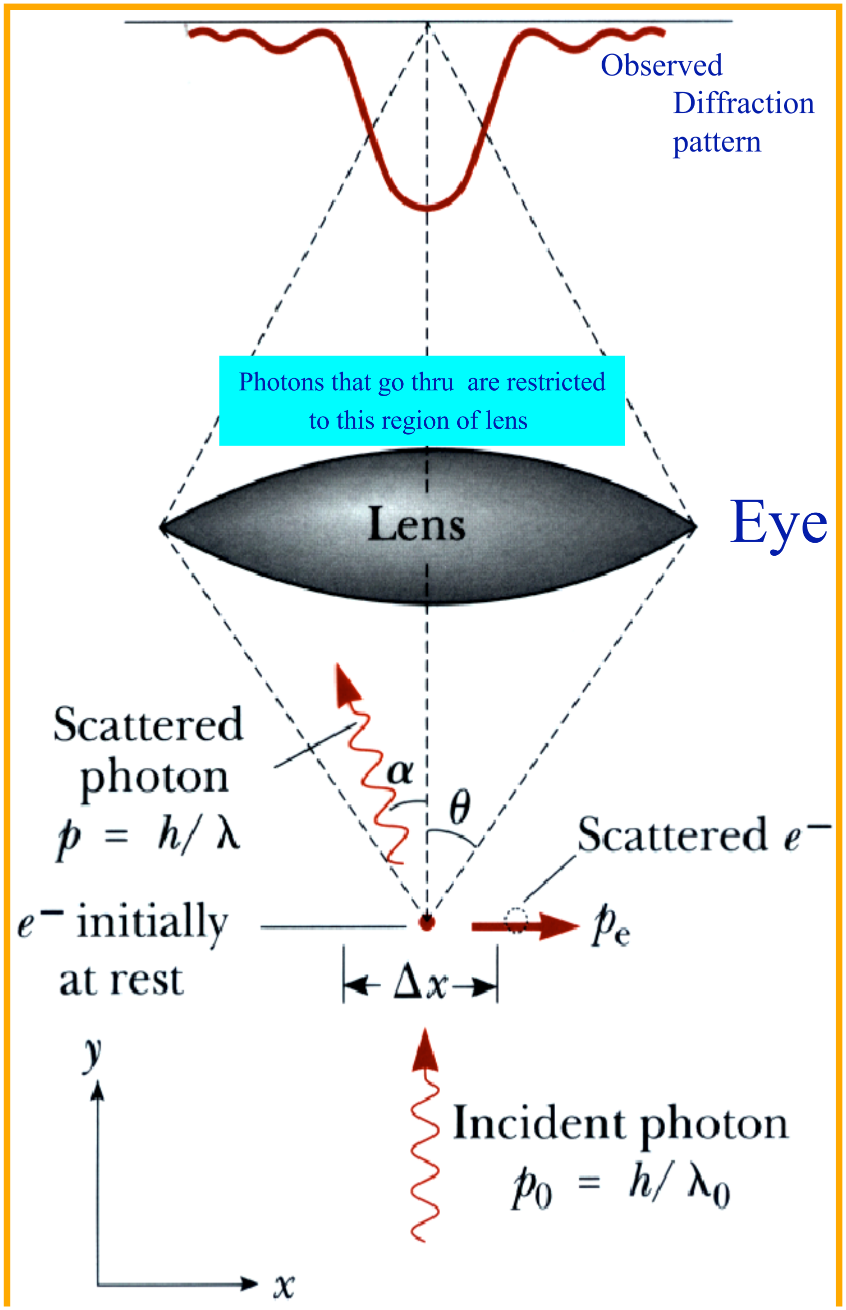


Not resolved   barely resolved   resolved

$$\text{Resolving power } \Delta x \simeq \frac{\lambda}{2\sin\theta}$$

ϑ Depends on d

Putting it all together: act of Observing an electron



- Incident light (p,λ) scatters off electron
- To be collected by lens → γ must scatter thru angle α
    - $-\vartheta \le \alpha \le \vartheta$
- Due to Compton scatter, electron picks up momentum
    - P_X , P_Y

$$-\frac{h}{\lambda}\sin\theta \le P_x \le \frac{h}{\lambda}\sin\theta$$

electron momentum uncertainty is

$$\Delta p \cong \frac{\sim 2h}{\lambda}\sin\theta$$

- After passing thru lens, photon diffracts, lands somewhere on screen, image (of electron) is fuzzy
- How fuzzy ? Optics says shortest distance between two resolvable points is :

$$\Delta x = \frac{\lambda}{2\sin\theta}$$

- Larger the lens radius, larger the $\vartheta \Rightarrow$ better resolution

$$\Rightarrow \Delta p.\Delta x \simeq \left(\frac{2h\sin\theta}{\lambda}\right)\left(\frac{\lambda}{2\sin\theta}\right) = h$$

$$\Rightarrow \Delta p.\Delta x \ge \hbar/2$$

Pseudo-Philosophical Aftermath of Uncertainty Principle

- Newtonian Physics & Deterministic physics topples over
  - Newton’s laws told you all you needed to know about trajectory of a particle
    - Apply a force, watch the particle go !
      - Know every thing ! X, v, p , F, a
      - Can predict exact trajectory of particle if you had perfect device

- No so in the subatomic world !
  - Of small momenta, forces, energies
  - Can’t predict anything exactly
    - Can only predict probabilities
      - There is so much chance that the particle landed here or there
      - Cant be sure !.... Be cognizant of the errors of your observations

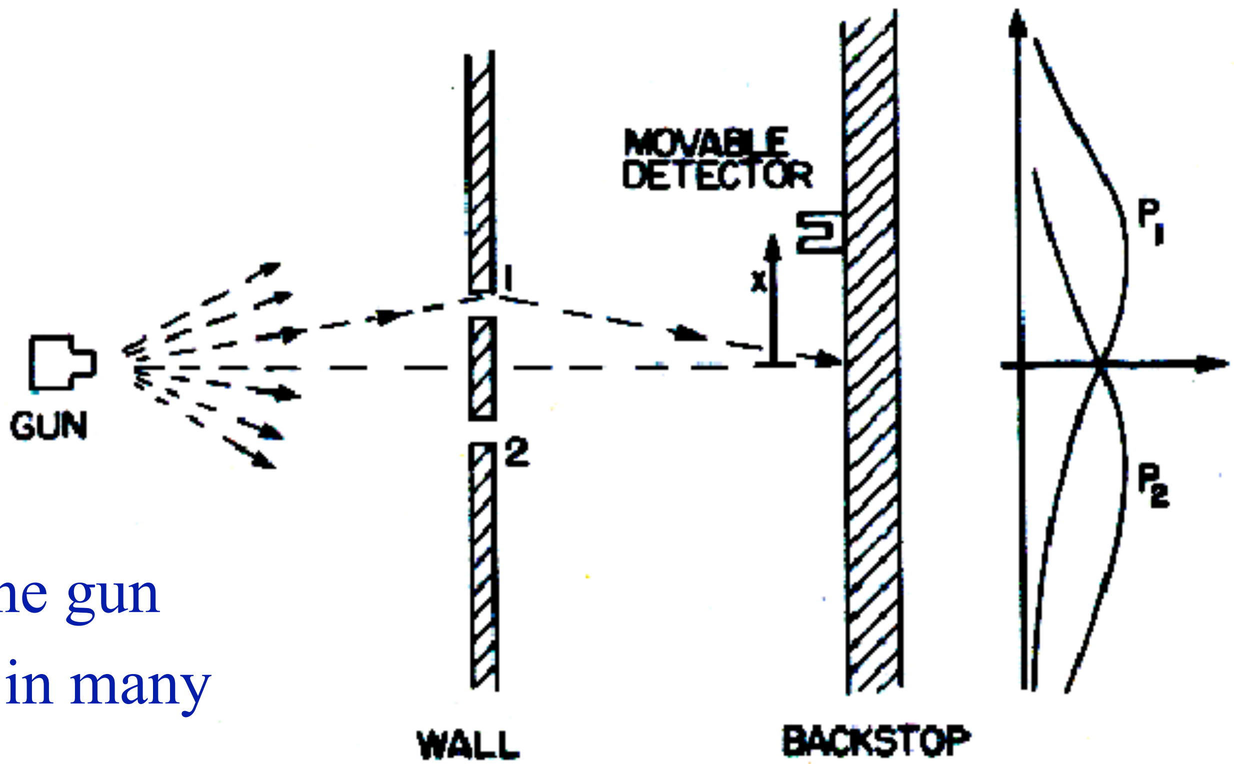
All Measurements Have Associated Errors

- If your measuring apparatus has an intrinsic inaccuracy (error) of amount Δp
- Then results of measurement of momentum p of an object at rest can easily yield a range of values accommodated by the measurement imprecision :
  - $-\Delta p \leq p \leq \Delta p$

- Similarly for all measurable quantities like x, t, Energy !

An Experiment with Indestructible Bullets

Probability P_{12} when
Both holes open



Erratic
Machine gun
sprays in many
directions

Made of
Armor
plate

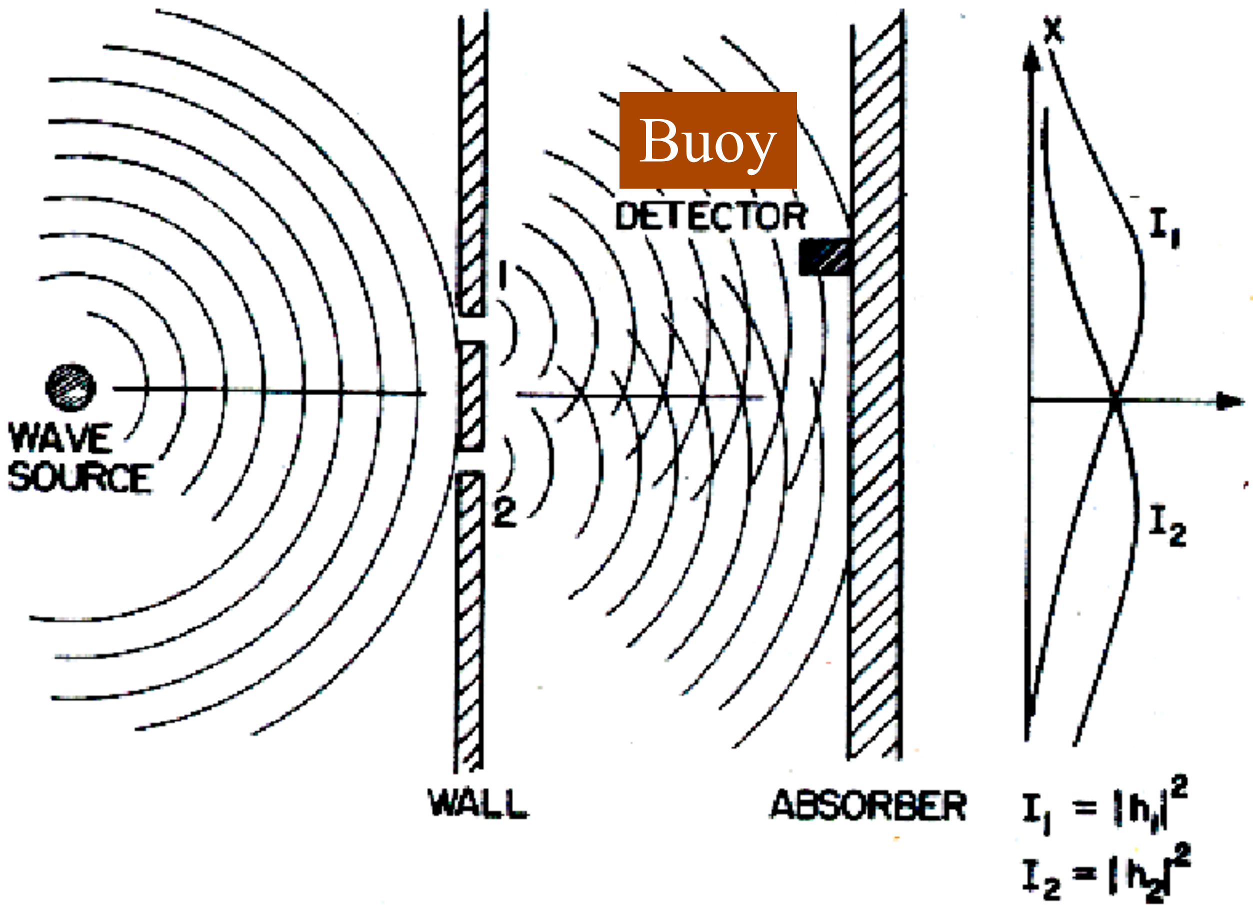
$P_{12} = P_1 + P_2$

An Experiment With Water Waves

Measure Intensity of Waves

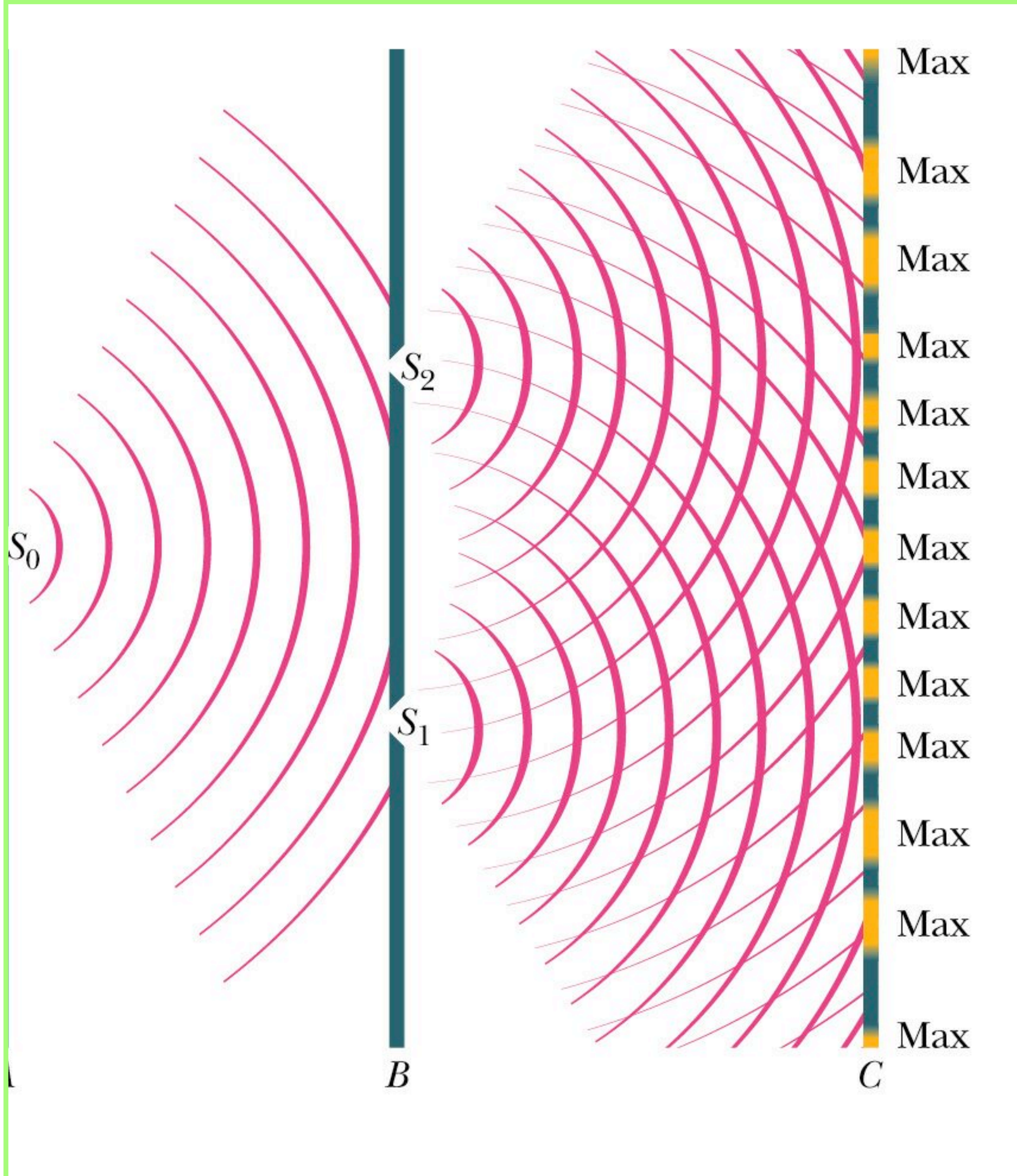
(by measuring amplitude of displacement)

Intensity I_{12} when Both holes open

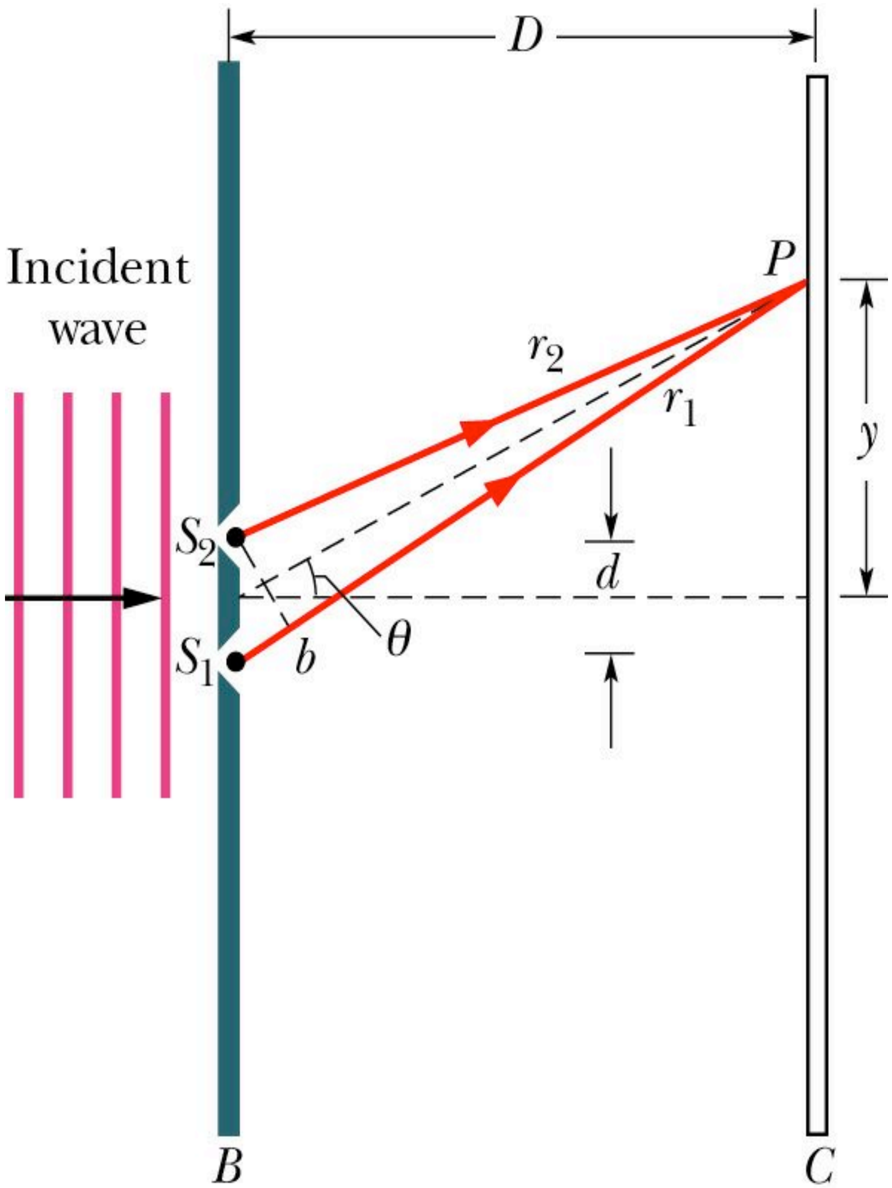


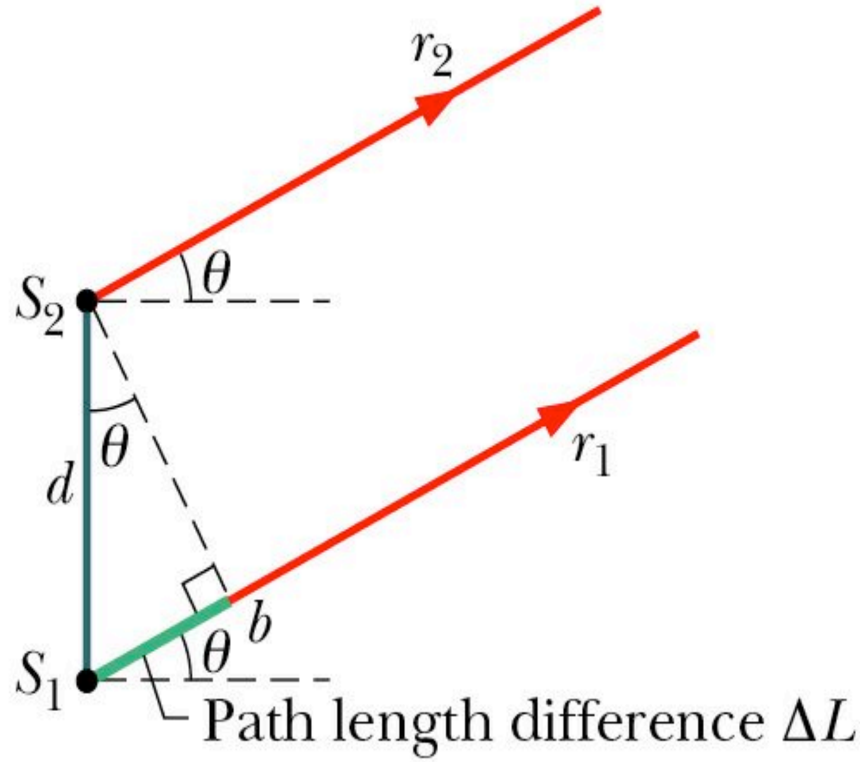
$$I_{12} = |h_1 + h_2|^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta$$

Interference and Diffraction: Ch 36 & 37, RHW



Interference Phenomenon in Waves



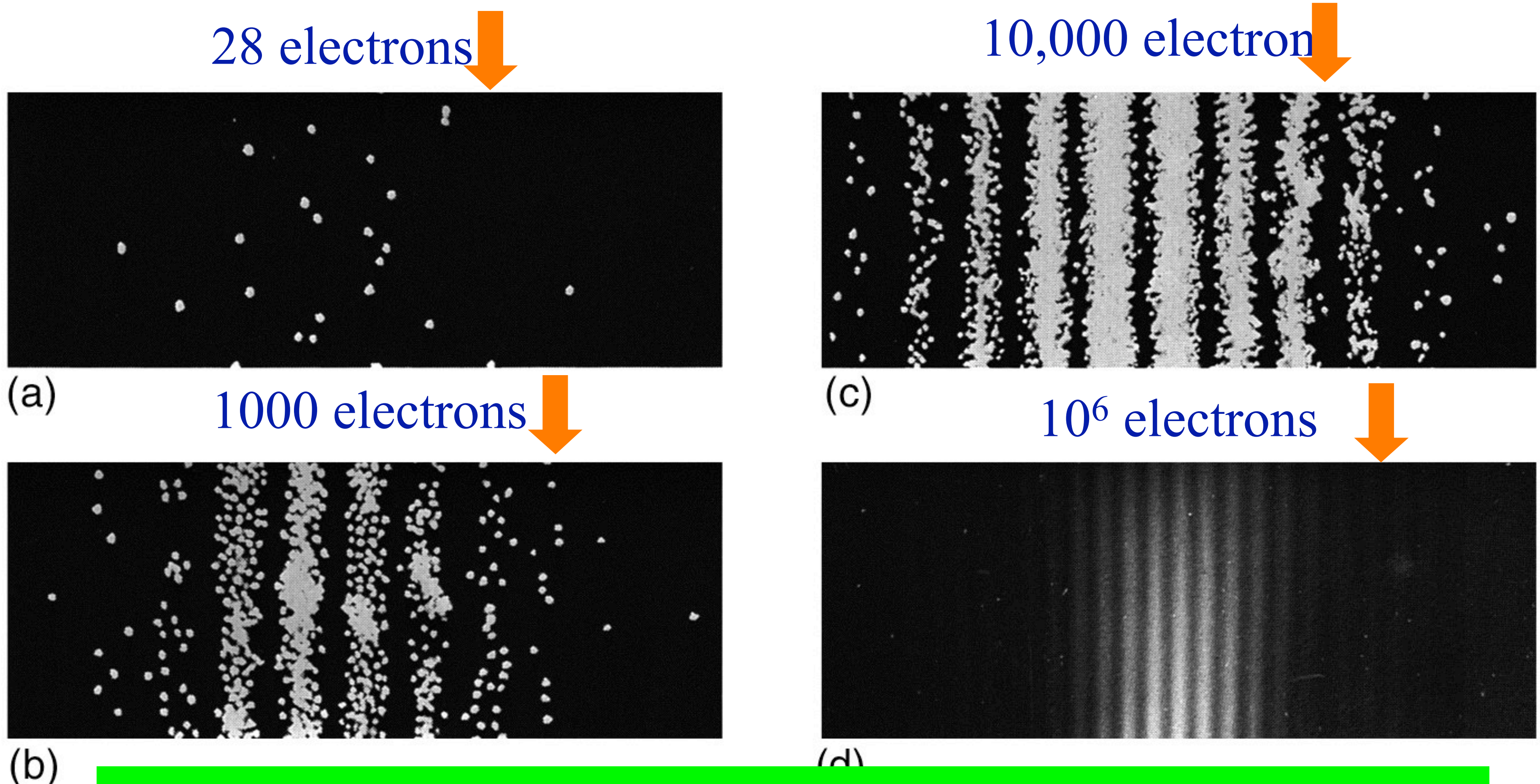


$$n\lambda = d\sin\theta$$

Interference Pattern of Electrons When Both slits open

Growth of 2-slit Interference pattern thru different exposure periods

Photographic plate (screen) struck by:



White dots simulate presence of electron

No white dots at the place of destructive Interference (minima)

Summary

- Probability of an event in an ideal experiment is given by the square of the absolute value of a complex number Ψ which is call probability amplitude
  - P = probability
  - Ψ= probability amplitude,
  - $P=|\Psi|^2$

- When an even can occur in several alternative ways, the probability amplitude for the event is the sum of the probability amplitudes for each way considered separately. There is interference:
  - $\Psi= \Psi_1+ \Psi_2$
  - $P=|\Psi_1+ \Psi_2|^2$

- If an experiment is performed which is capable of determining whether one or other alternative is actually taken, the probability of the event is the sum of probabilities for each alternative. The interferenence is lost: $P = P_1 + P_2$

The Lesson Learnt

- In trying to determine which slit the particle went through, we are examining particle-like behavior
- In examining the interference pattern of electron, we are using wave like behavior of electron

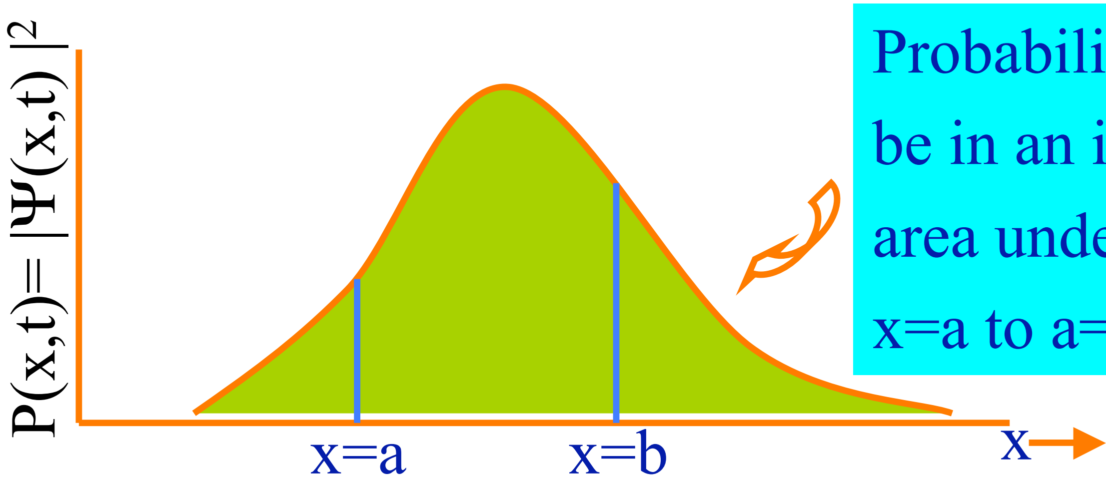
Bohr's Principle of Complementarity:

It is not possible to simultaneously determine physical observables in terms of both particles and waves

Quantum Mechanics of Subatomic Particles

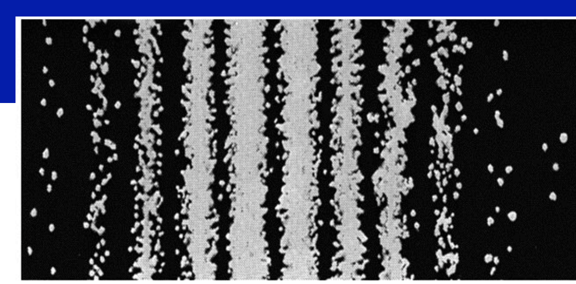
- Act of Observation destroys the system (No watching!)
- If can't watch then all conversations can only be in terms of Probability P
- Every particle under the influence of a force is described by a Complex wave function $\Psi(x,y,z,t)$
- Ψ is the ultimate DNA of particle: contains all info about the particle under the force (in a potential e.g Hydrogen)
- Probability of per unit volume of finding the particle at some point (x,y,z) and time t is given by
  - $P(x,y,z,t) = \Psi(x,y,z,t) . \Psi^{*}(x,y,z,t) =| \Psi(x,y,z,t) |^2$
- When there are more than one path to reach a final location then the probability of the event is
  - $\Psi = \Psi_1 + \Psi_2$
  - $P = | \Psi^{*} \Psi| = |\Psi_1|^2 + |\Psi_2|^2 +2 |\Psi_1 |\Psi_2| \cos\phi$

Wave Function of "Stuff" & Probability Density



Probability of a particle to be in an interval a ≤ x ≤b is area under the curve from x=a to a=b

- Although not possible to specify with certainty the location of particle, its possible to assign probability P(x)dx of finding particle between x and x+dx
- $P(x)\,dx = |\Psi(x,t)|^2\,dx$
- E.g intensity distribution in light diffraction pattern is a measure of the probability that a photon will strike a given point within the pattern



Ψ: The Wave function Of A Particle

- The particle must be some where

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \, dx = 1$$

- Any Ψ satisfying this condition is NORMALIZED
- Prob of finding particle in finite interval

$$P(a \le x \le b) = \int_a^b \psi^*(x,t) \;\; \psi(x,t) dx$$

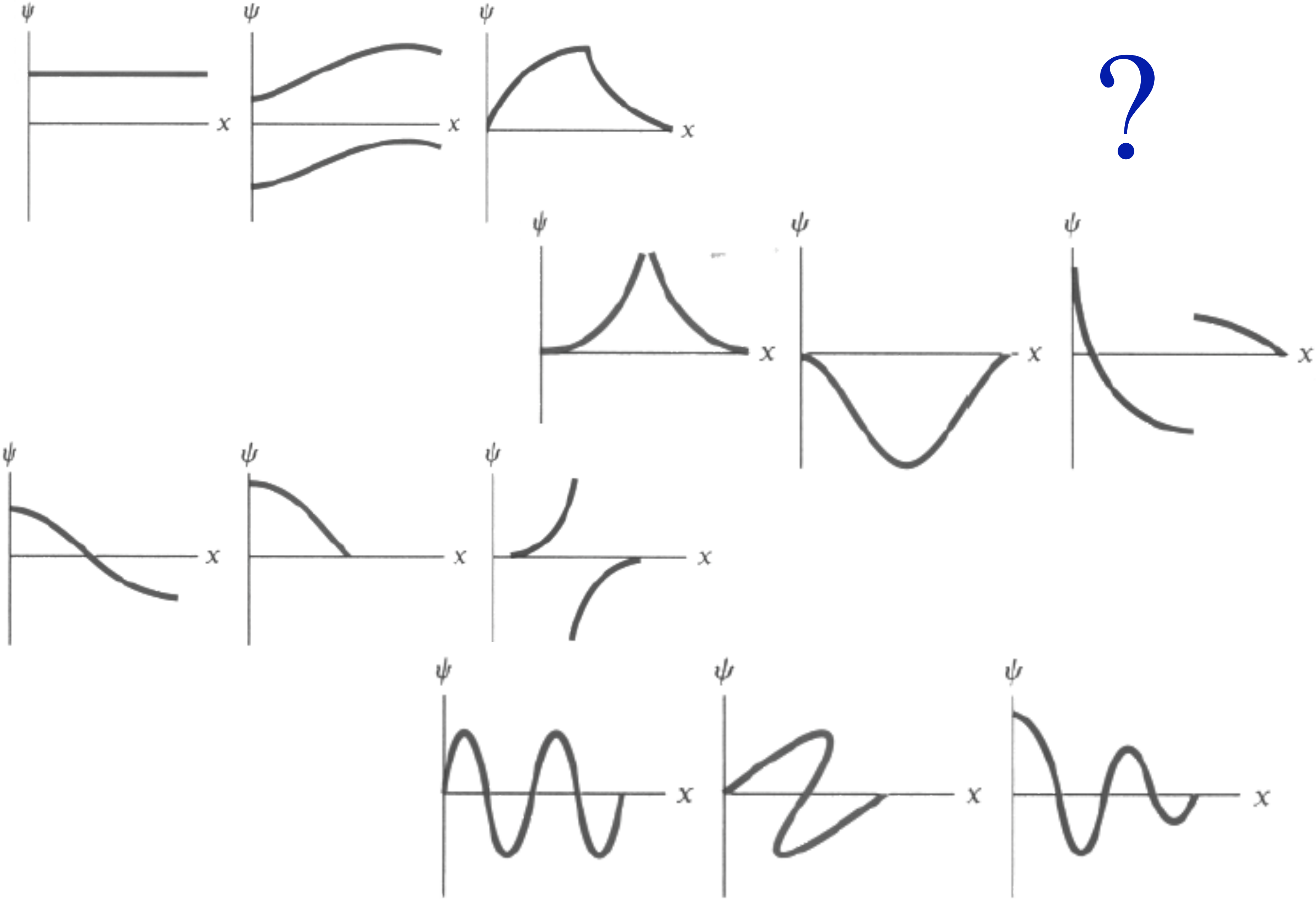
- Fundamental aim of Quantum Mechanics
  - Given the wavefunction at some instant (say t=0) find Ψ at some subsequent time t
  - Ψ(x,t=0) → Ψ(x,t) …evolution
  - Think of a probabilistic view of particle's "newtonian trajectory"
    - We are replacing Newton's 2nd law for subatomic systems

The Wave Function is a mathematical function that describes a physical object → Wave function must have some rigorous properties :

- Ψ must be finite
- Ψ must be continuous fn of x,t
- Ψ must be single-valued
- Ψ must be smooth fn → $\dfrac{d\psi}{dx}$ must be continuous

WHY ?

Bad (Mathematical) Wave Functions Of a Physical System : You Decide Why



A Simple Wave Function : Free Particle

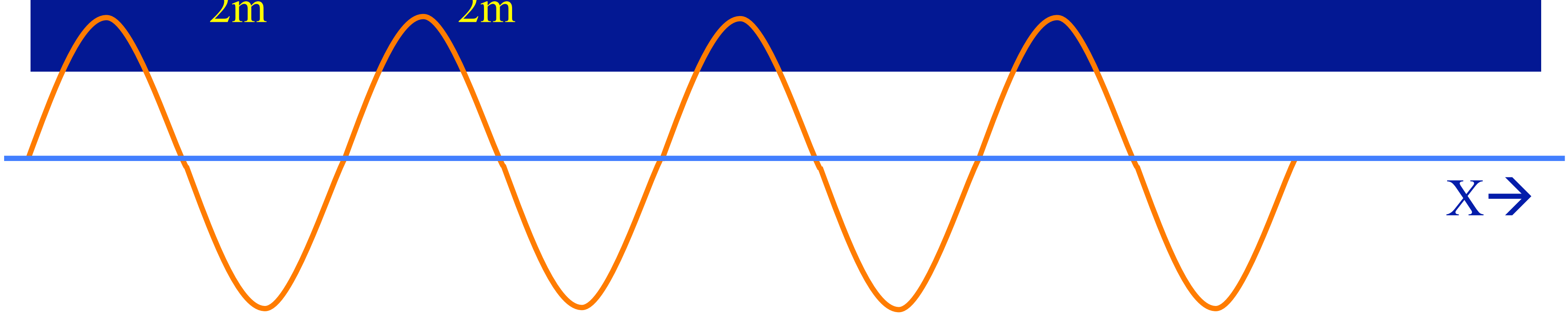
- Imagine a free particle of mass m , momentum p and $K=p^2/2m$
- Under no force , no attractive or repulsive potential to influence it
- Particle is where it wants : can be any where $[-\infty \le x \le +\infty]$
  - Has No relationship, no mortgage , no quiz, no final exam….its essentially a **bum** !
  - how to describe a **quantum mechanical bum** ?
    - $\Psi(x,t) = Ae^{i(kx-\omega t)} = A(\cos(kx-\omega t) + i\sin(kx-\omega t))$

$$k = \frac{p}{\hbar}; \quad \omega = \frac{E}{\hbar}$$

For non-relativistic particles

$$E = \frac{p^2}{2m} \Rightarrow \omega(k) = \frac{\hbar k^2}{2m}$$

Has definite momentum and energy but location unknown !



Wave Function of Different Kind of Free Particle : Wave Packet

Sum of Plane Waves:

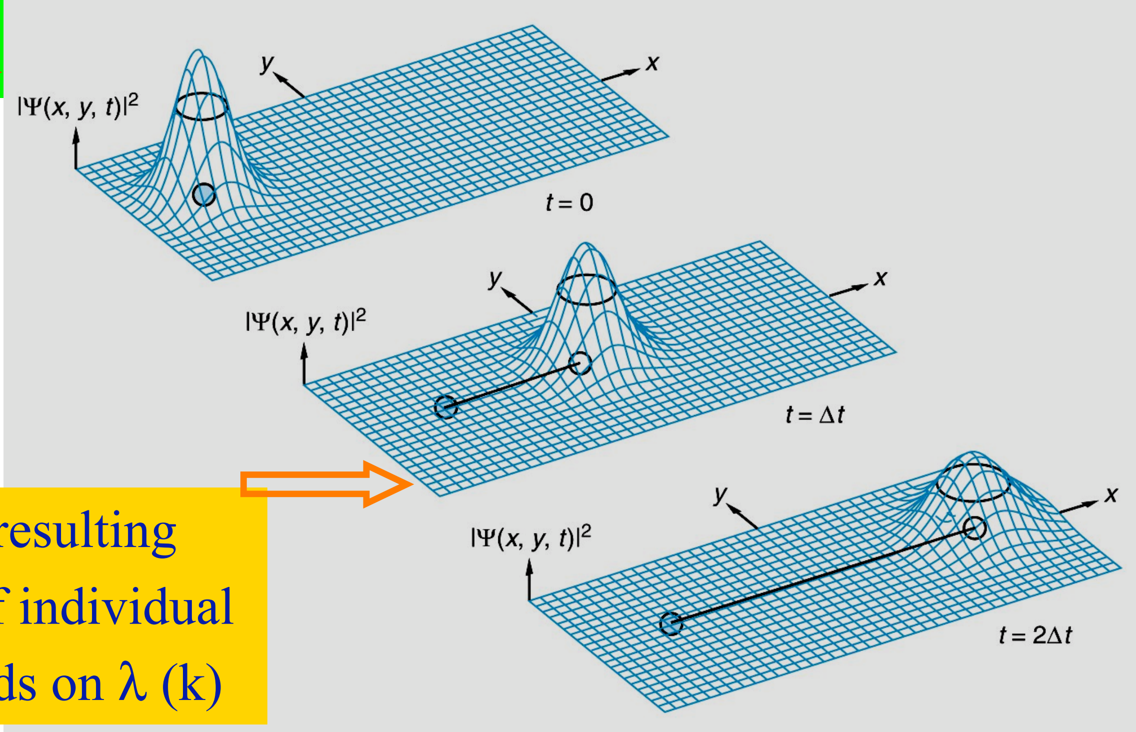
$$\Psi(x,0)=\int_{-\infty}^{+\infty} a(k)e^{ikx}\,dk$$

$$\Psi(x,t)=\int_{-\infty}^{+\infty} a(k)e^{i(kx-\omega t)}\,dk$$

Wave Packet initially localized in ΔX, Δt undergoes dispersion

Combine many free waves to create a Localized wave packet (group)

The more you know now,
The less you will know later
Why ?



Spreading is due to DISPERSION resulting from the fact that phase velocity of individual waves making up the packet depends on λ (k)

Normalization Condition: Particle Must be Somewhere

$Example: \ \psi(x,0) = Ce^{-\left|\frac{x}{x_0}\right|}$, C & x_0 are constants

This is a symmetric wavefunction with diminishing amplitude

The Amplitude is maximum at x =0 $\Rightarrow$ Probability is max too

Normalization Condition: How to figure out C ?

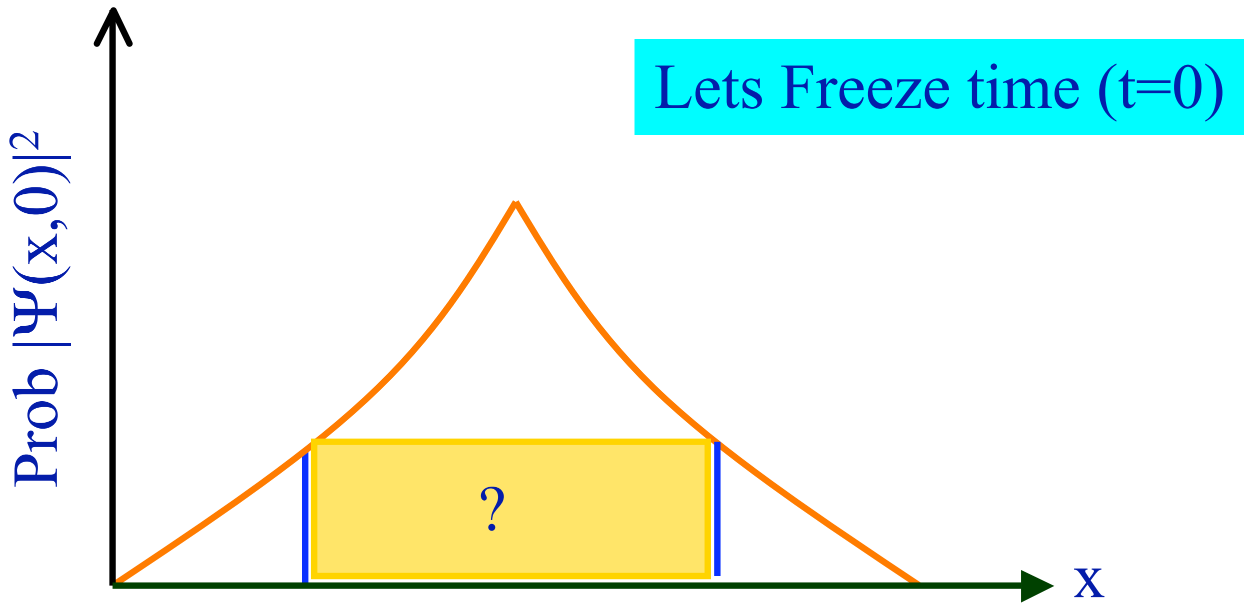
A real particle must be somewhere: Probability of finding

particle is finite $P(-\infty \le x \le +\infty) = \int_{-\infty}^{+\infty} |\psi(x,0)|^2 dx = \int_{-\infty}^{+\infty} C^2 e^{-2\left|\frac{x}{x_0}\right|} dx = 1$

$$\Rightarrow 1 = 2C^2 \int_0^{\infty} e^{-2\left|\frac{x}{x_0}\right|} dx = 2C^2 \left[\frac{x_0}{2}\right] = C^2 x_0$$

$$\Rightarrow \boxed{\psi(x,0) = \frac{1}{\sqrt{x_0}} e^{-\left|\frac{x}{x_0}\right|}}$$

Where is the particle within a certain location x ± Δx



$$P(-x_0 \le x \le +x_0) = \int_{-x_0}^{+x_0} \left|\psi(x,0)\right|^2 dx = \int_{-x_0}^{+x_0} C^2 e^{-2\left|\frac{x}{x_0}\right|} dx$$

$$= 2C^2\left[\frac{x_0}{2}\right]\left[1-e^{-2}\right] = \left[1-e^{-2}\right] = 0.865 \Rightarrow 87\%$$

Where Do Wave Functions Come From ?

- Are solutions of the time dependent Schrödinger Differential Equation (inspired by Wave Equation seen in 2C)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+U(x)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

- Given a potential U(x) → particle under certain force
  - $F(x) = -\dfrac{\partial U(x)}{\partial x}$

Schrodinger had an interesting life

Schrodinger Wave Equation

Wavefunction ψ which is a sol. of the Sch. Equation embodies all modern physics experienced/learnt so far:

$$\boxed{E=hf, \quad p=\frac{h}{\lambda}, \quad \Delta x.\Delta p \sim \hbar \,,\ \Delta E.\Delta t \sim \hbar, \ \text{quantization etc}}$$

Schrodinger Equation is a Dynamical Equation much like Newton's Equation $\vec{F} = m\,\vec{a}$

$$\psi(x,0) \quad \rightarrow \overrightarrow{\text{Force}}\,(\text{potential}) \rightarrow \psi(x,t)$$

Evolves the System as a function of space-time

The Schrodinger Eq. propagates the system forward & backward in time:

$$\psi(x,\delta t) = \psi(x,0) \pm \left[\frac{d\psi}{dt}\right]_{t=0} \delta t$$

Where does it come from ?? ..."First Principles"..no real derivation exists

Time Independent Sch. Equation

$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2}+U(x)\Psi(x,t)=i\hbar\frac{\partial\Psi(x,t)}{\partial t}}$$

Sometimes (depending on the character of the Potential U(x,t))

The Wave function is factorizable: can be broken up

$$\Psi\left(\mathrm{x,t}\right)=\psi(x)\ \phi(t)$$

Example: Plane Wave $\Psi(\mathrm{x,t})=\mathrm{e}^{\mathrm{i(kx-}\omega\mathrm{t)}}=\mathrm{e}^{\mathrm{i(kx)}}\mathrm{e}^{\mathrm{-i(}\omega\mathrm{t)}}$

In such cases, use seperation of variables to get :

$$\frac{-\hbar^2}{2\mathrm{m}}\phi(t).\frac{\partial^2\psi(x)}{\partial^2 x}+U(x)\psi(x)\phi(t)=i\hbar\psi(x)\frac{\partial\phi(t)}{\partial t}$$

Divide Throughout by $\Psi(\mathrm{x,t})=\psi(\mathrm{x})\phi(\mathrm{t})$

$$\Rightarrow\ \boxed{\frac{-\hbar^2}{2\mathrm{m}}\frac{1}{\psi(x)}.\frac{\partial^2\psi(x)}{\partial^2 x}+U(x)=i\hbar\frac{1}{\phi(t)}\frac{\partial\phi(t)}{\partial t}}$$

LHS is a function of x; RHS is fn of t

x and t are independent variables, hence :

$\Rightarrow$ RHS = LHS = Constant = E

Factorization Condition For Wave Function Leads to:

$$\boxed{\frac{-\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial^2 x}+U(x)\psi(x)=E\ \psi(x)}$$

$$\boxed{i\hbar\frac{\partial\phi(t)}{\partial t}=E\phi(t)}$$

What is the Constant E ? How to Interpret it ?

Back to a Free particle :

Ψ(x,t)= $Ae^{ikx}e^{-i\omega t}$, ψ(x)= Ae^{ikx}

U(x,t) = 0

Plug it into the Time Independent Schrodinger Equation (TISE) $\Rightarrow$

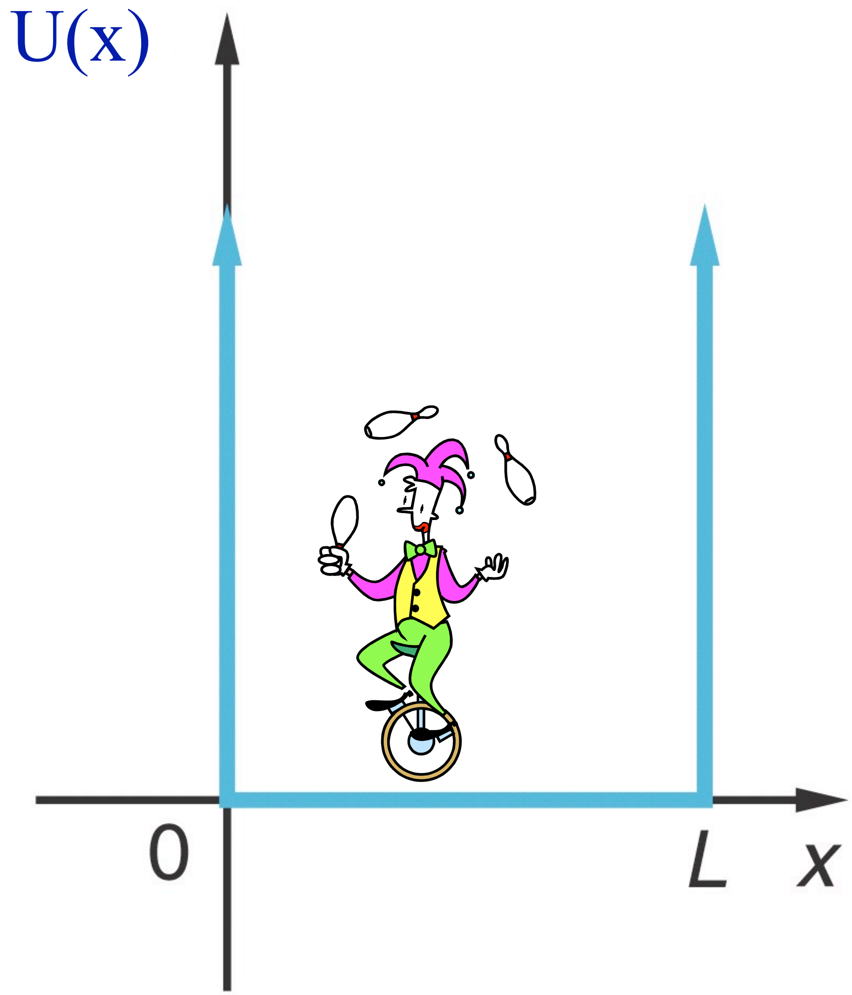
$$\frac{-\hbar^2}{2m}\frac{d^2(Ae^{(ikx)})}{dx^2}+0=E\ Ae^{(ikx)}\quad\Rightarrow E=\frac{\hbar^2k^2}{2m}=\frac{p^2}{2m}=\text{(NR Energy)}$$

Stationary states of the free particle: $\Psi(x,t)=\psi(x)e^{-i\omega t}$

$$\Rightarrow\left|\Psi(x,t)\right|^2=\left|\psi(x)\right|^2$$

Probability is static in time t, character of wave function depends on $\psi(x)$

A More Interesting Potential : Particle In a Box



Write the Form of Potential: Infinite Wall

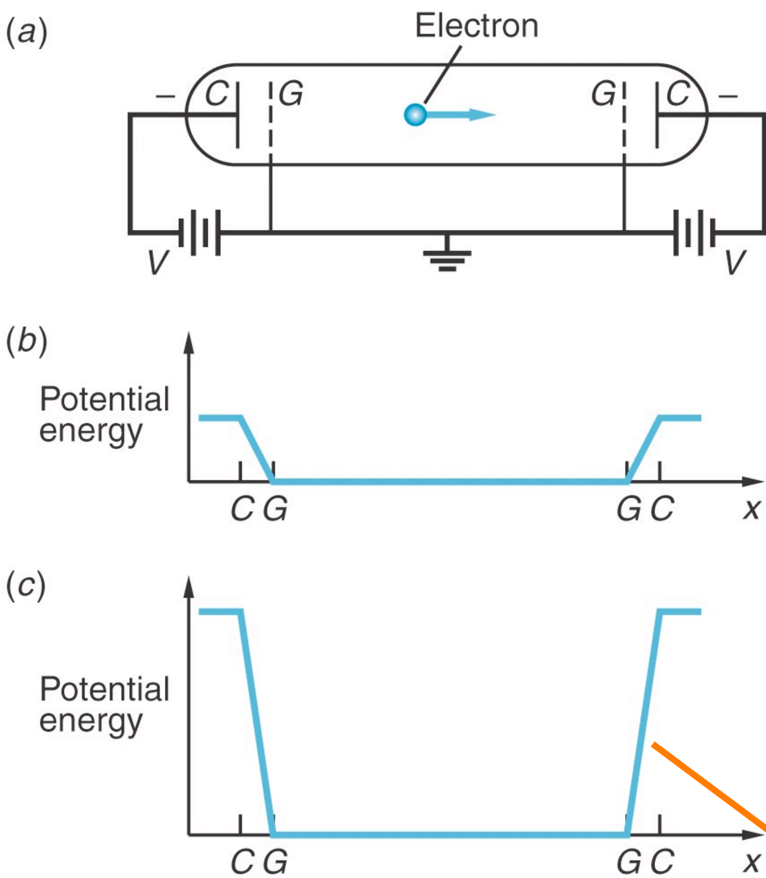
$U(x,t) = \infty; \;\; x \le 0, \;\; x \ge L$

$U(x,t) = 0 \;\; ; \;\; 0 < X < L$

- Classical Picture:
  - Particle dances back and forth
  - Constant speed, const KE
  - Average <P> = 0
  - No restriction on energy value
    - E=K+U = K+0
  - Particle can not exist outside box
    - Can't get out because needs to borrow infinite energy to overcome potential of wall

What happens when the joker is subatomic in size ??

Example of a Particle Inside a Box With Infinite Potential



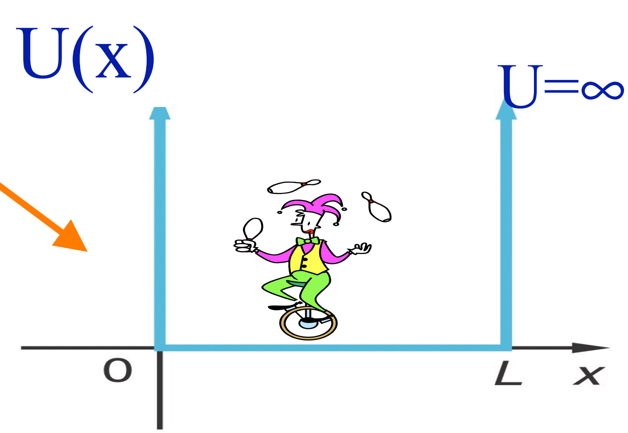
(a) Electron placed between 2 set of electrodes C & grids G experiences no force in the region between grids, which are held at Ground Potential

However in the regions between each C & G is a repelling electric field whose strength depends on the magnitude of V

(b) If V is small, then electron's potential energy vs x has low sloping "walls"

(c) If V is large, the "walls"become very high & steep becoming infinitely high for $V \rightarrow \infty$

(d) The straight infinite walls are an approximation of such a situation



$\Psi(x)$ for Particle Inside 1D Box with Infinite Potential Walls

Inside the box, no force $\Rightarrow$ U=0 or constant (same thing)

$$\Rightarrow \boxed{\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + 0\ \psi(x) = E\ \psi(x)}$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)\ ;\ k^2 = \frac{2mE}{\hbar^2}$$

$$or\ \boxed{\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0} \Leftarrow \text{ figure out what } \psi(x) \text{ solves this diff eq.}$$

In General the solution is $\psi(x) = A\ sinkx + B\ coskx$ (A,B are constants)

Need to figure out values of A, B : How to do that ?

Apply BOUNDARY Conditions on the Physical Wavefunction

We said $\psi(x)$ must be continuous everywhere

So match the wavefunction just outside box to the wavefunction value just inside the box

$\Rightarrow$ At x = 0 $\Rightarrow \psi(x=0) = 0$ & At x = L $\Rightarrow \psi(x=L) = 0$

$\therefore\ \psi(x=0) = B = 0$ (Continuity condition at x =0)

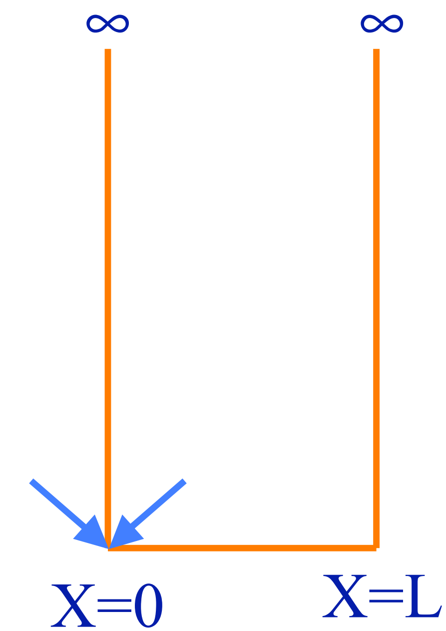
& $\psi(x=L) = 0 \Rightarrow$ A Sin kL = 0 (Continuity condition at x =L)

$$\Rightarrow kL = n\pi \Rightarrow k = \frac{n\pi}{L}, n = 1,2,3,\ldots\infty$$

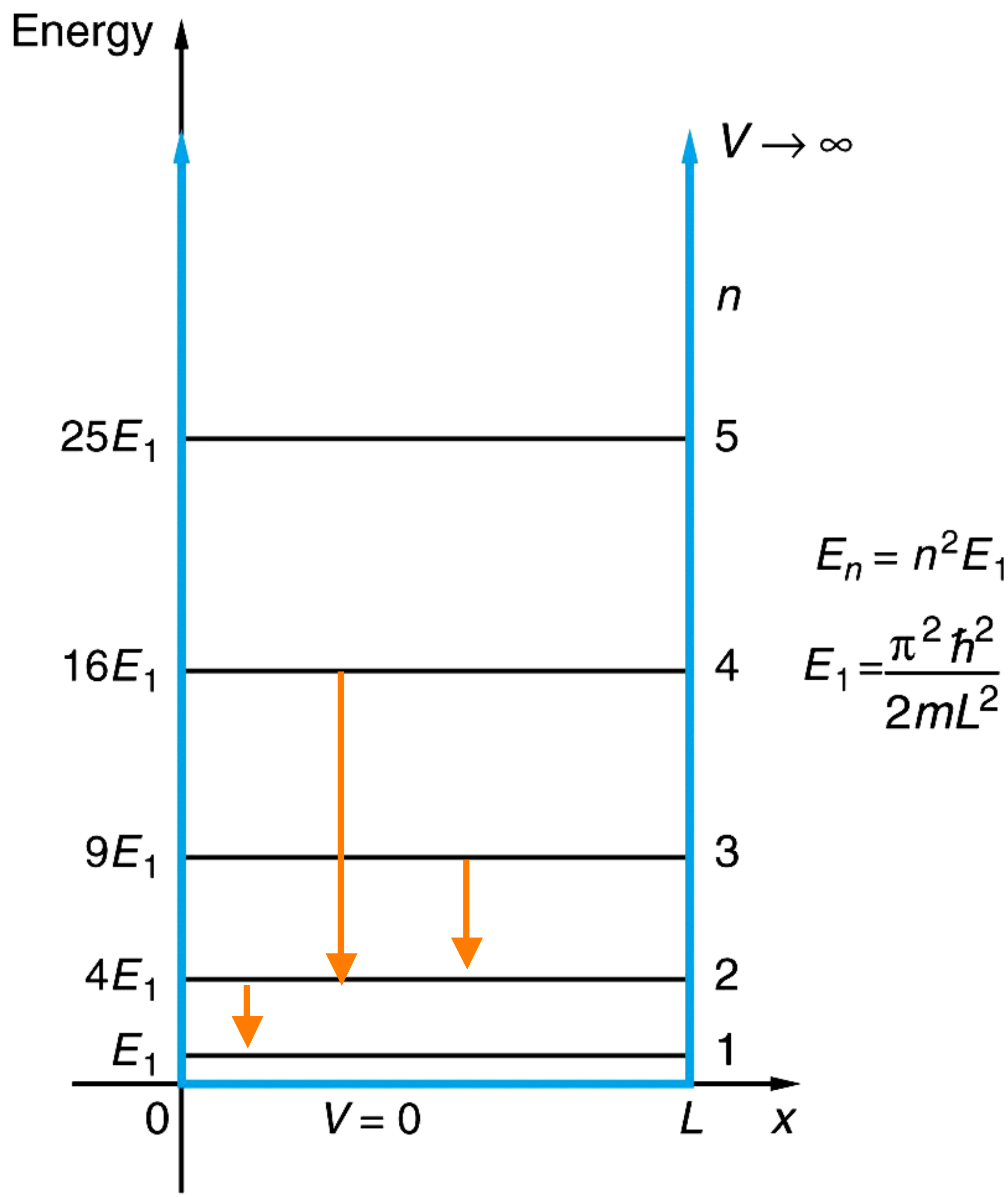
So what does this say about Energy E ? : $\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}}$ Quantized (not Continuous)!

Why can't the particle exist Outside the box ?

→ E Conservation



Quantized Energy levels of Particle in a Box



What About the Wave Function Normalization ?

The particle's Energy and Wavefunction are determined by a number n

We will call n $\rightarrow$ Quantum Number , just like in Bohr's Hydrogen atom

What about the wave functions corresponding to each of these energy states?

$$\boxed{\begin{aligned} \psi_n &= A\sin(kx) = A\sin(\frac{n\pi x}{L}) && \text{for } 0<x<L \\ &= 0 && \text{for } x \geq 0,\ x \geq L \end{aligned}}$$

Normalized Condition :

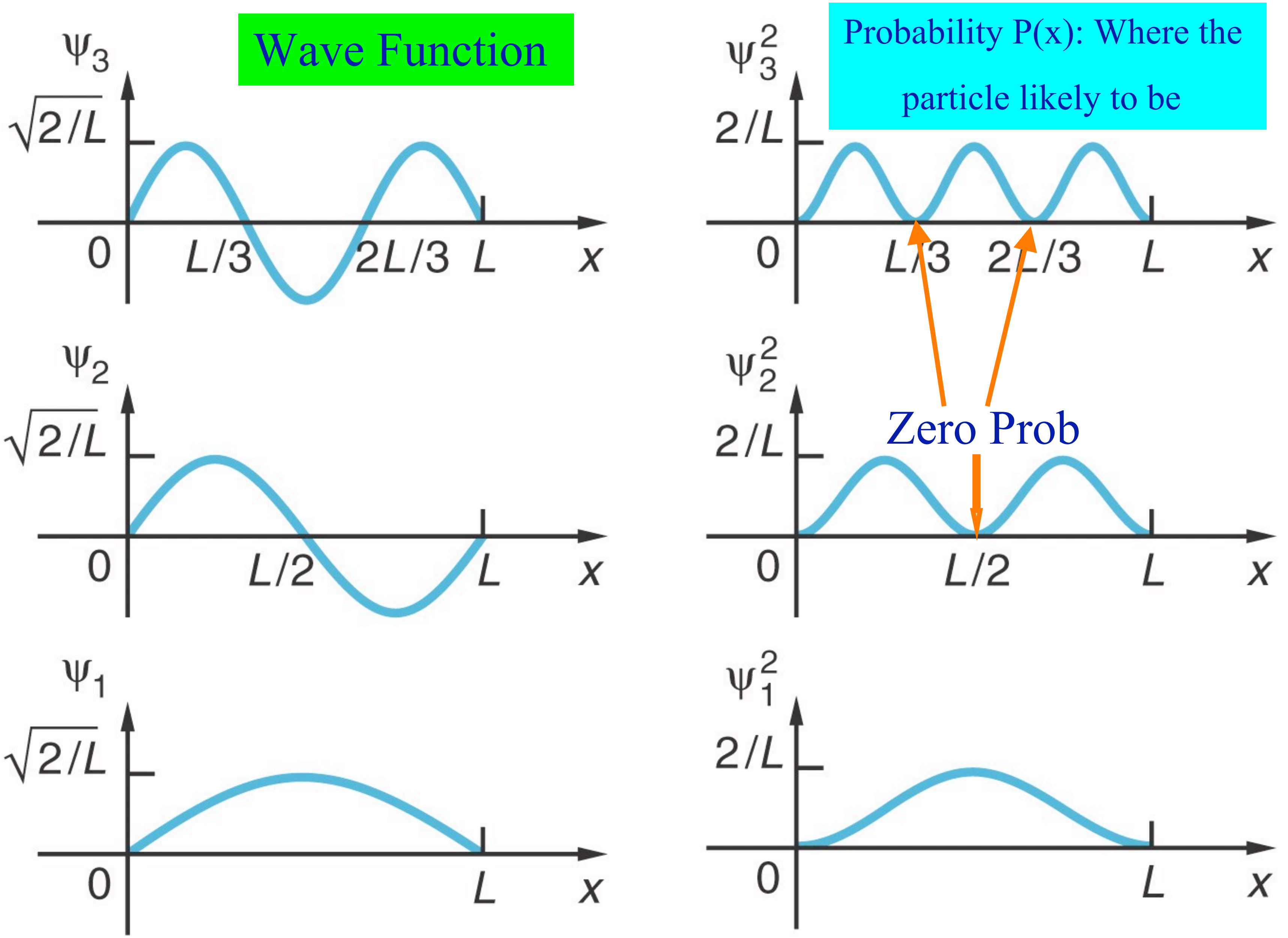
$$1 = \int_0^L \psi_n^* \psi_n dx = A^2 \int_0^L Sin^2(\frac{n\pi x}{L}) \qquad \boxed{\text{Use } 2\text{Sin}^2\theta = 1 - 2Cos2\theta}$$

$$1 = \frac{A^2}{2}\int_0^L \left(1 - \cos(\frac{2n\pi x}{L})\right) \quad \text{and since } \int \cos\theta = \sin\theta$$

$$1 = \frac{A^2}{2}L \quad \Rightarrow A = \sqrt{\frac{2}{L}}$$

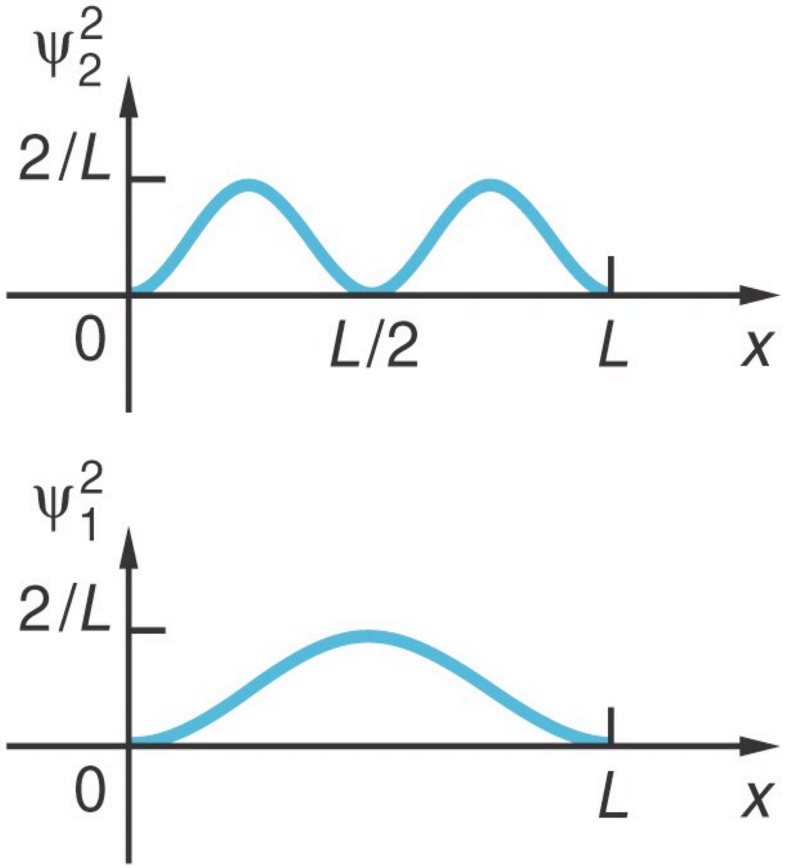
So $\psi_n = \sqrt{\frac{2}{L}}\sin(kx) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$ ...What does this look like?

Wave Functions : Shapes Depend on Quantum # n



Where in The World is Carmen San Diego?

- We can only guess the probability of finding the particle somewhere in x
  - For n=1 (ground state) particle most likely at x = L/2
  - For n=2 (first excited state) particle most likely at L/4, 3L/4
    - Prob. Vanishes at x = L/2 & L
      - How does the particle get from just before x=L/2 to just after?
        - » QUIT thinking this way, particles don't have trajectories
        - » Just probabilities of being somewhere



Classically, where is the particle most Likely to be : Equal prob of being anywhere inside the Box

NOT SO says Quantum Mechanics!