Blackbody radiation and Plank's law

"blackbody" problem:
calculating the intensity of radiation at a given wavelength emitted by a body at a specific temperature

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Max Planck, 1900
quantization of energy of radiation-emitting oscillators:
only certain energies for the radiation-emitting oscillators
in the cavity wall are allowed

Albert Einstein, 1905
extended quantization of energy
of radiation-emitting oscillators
to quantization of light → photons

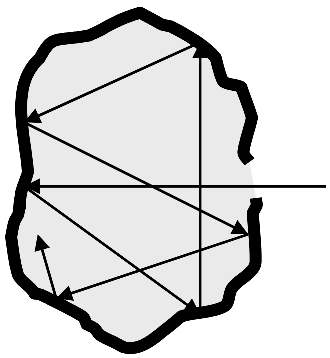
↓

explained the photoelectric effect

Niels Bohr, 1913
quantum model of the atom

Blackbody radiation and Plank's law

blackbody is an object that absorbs all electromagnetic radiation falling on it
an consequently appears black



the opening to the cavity is a good approximation of a blackbody:
after many reflections all of the incident energy is absorbed

radiation is in thermal equilibrium with the cavity (e.g. oven cavity)
because radiation has exchanged energy with the walls many times

similar to the thermal equilibrium
of a fluid with a container

spectral energy density $u(f,T)$
is energy per unit volume per unit frequency of the radiation within the blackbody cavity

$u(f,T)$ depends only on temperature and light frequency
and not on the physical and chemical makeup of the blackbody

↕

all the objects in the oven, regardless of their chemical nature, size, or shape,
emit light of the same color

↕

blackbody is a perfect absorber and ideal radiator

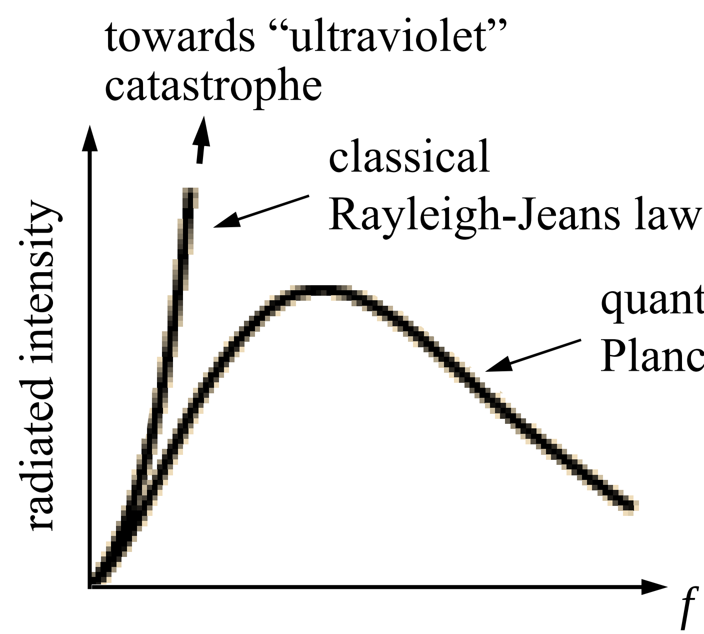
Max Planck, 1900

spectral energy density of blackbody radiation

$$u(f,T)=\frac{8\pi hf^3}{c^3}\frac{1}{e^{hf/k_BT}-1}$$

$h=6.626\times10^{-34}\ \mathrm{J\cdot s}$ is Planck's constant

$k_B=1.380\times10^{-23}$ J/K is Boltzmann's constant



limiting behaviors

at high frequencies $hf/k_BT>>1$

$$\frac{1}{e^{hf/k_BT}-1}\approx e^{-hf/k_BT}$$

$$u(f,T)=\frac{8\pi hf^3}{c^3}\frac{1}{e^{hf/k_BT}-1}\approx\frac{8\pi hf^3}{c^3}e^{-hf/k_BT}$$

Wien's exponential law, 1893

at low frequencies $hf/k_BT<<1$

$$\frac{1}{e^{hf/k_BT}-1}=\frac{1}{1+hf/k_BT+\ldots-1}\approx\frac{k_BT}{hf}$$

$$u(f,T)=\frac{8\pi hf^3}{c^3}\frac{1}{e^{hf/k_BT}-1}\approx\frac{8\pi hf^3}{c^3}\frac{k_BT}{hf}=\frac{8\pi f^2}{c^3}k_BT$$

classical Rayleigh-Jeans law

Rayleigh-Jeans law is the classical limit obtained when $h\rightarrow0$

Max Planck:
blackbody radiation is produced by vibrating submicroscopic electric charges, which he called <u>resonators</u>

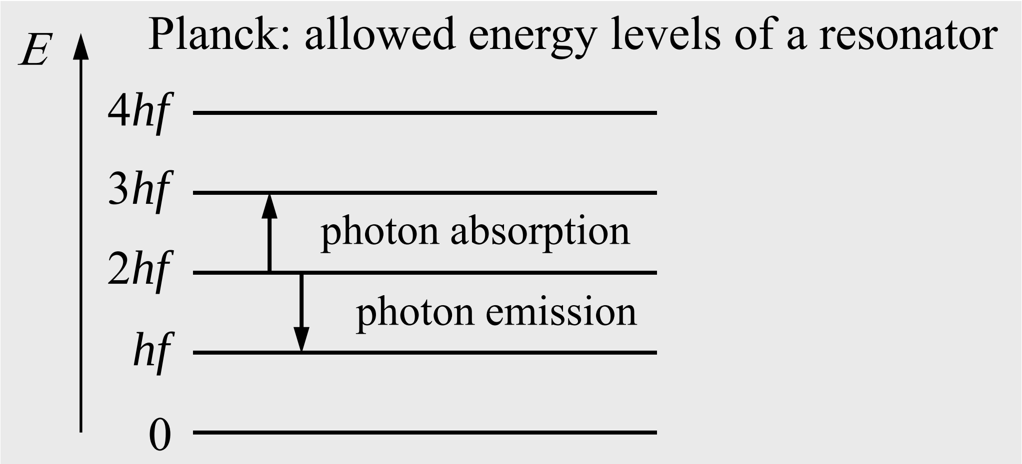
the walls of a cavity are composed of resonators vibrating at different frequency

Classical Maxwell theory:
An oscillator of frequency f could have any value of energy and could change its amplitude continuously by radiating any fraction of its energy

Planck: the total energy of a resonator with frequency f could only be an integer multiple of hf.
(During emission or absorption of light) resonator can change its energy only by the <u>quantum of energy</u> $\Delta E = hf$

$$E = nhf \quad \text{where } n = 1, 2, 3, \ldots$$

$$\Delta E = hf$$



all systems vibrating with frequency *f* are quantized
and lose or gain energy in discrete packets or quanta $\Delta E = hf$

Consider a pendulum
m=0.1 kg, l=1 m, displaced by θ=10^0

$$E = mgl(1-\cos\theta) = (0.1\text{ kg})(9.8\text{ m/s}^2)(1\text{ m})(1-\cos 10^0) = 1.5\times10^{-2}\text{ J}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{1}{2\pi}\sqrt{\frac{9.8\text{ m/s}^2}{1\text{ m}}} = 0.5\text{ Hz}$$

$$\Delta E = hf = (6.63\times10^{-34}\text{ J}\cdot\text{s})(0.5\text{ s}^{-1}) = 3.3\times10^{-34}\text{ J}$$ ← quantum of energy

$$\frac{\Delta E}{E} = \frac{3.3\times10^{-34}\text{ J}}{1.5\times10^{-2}\text{ J}} = 2.2\times10^{-32}$$

← classical physics:
quantization is unobservable and
energy loss or gain looks continuum

ΔE is large ← *f* is large
E is small ← *m* is small

← quantum physics

energy change of atomic oscillator sending out green light

$$\Delta E = hf = \frac{hc}{\lambda} = \frac{(6.63\times10^{-34}\text{ J}\cdot\text{s})(3\times10^{8}\text{ m/s})}{540\times10^{-9}\text{ m}} = 3.68\times10^{-19}\text{ J} = 2.3\text{ eV}$$

a more appropriate unit of energy for describing atomic processes $1\text{ eV} = 1.602\times10^{-19}\text{ J}$

Max Planck:

average energy emitted per oscillator: $\bar{E}$

number of oscillators with frequency between *f* and *f*+*df*: $N(f)df$

$$u(f,T)df = \bar{E}N(f)df$$

$$N(f)df = \frac{8\pi f^2}{c^3}df$$

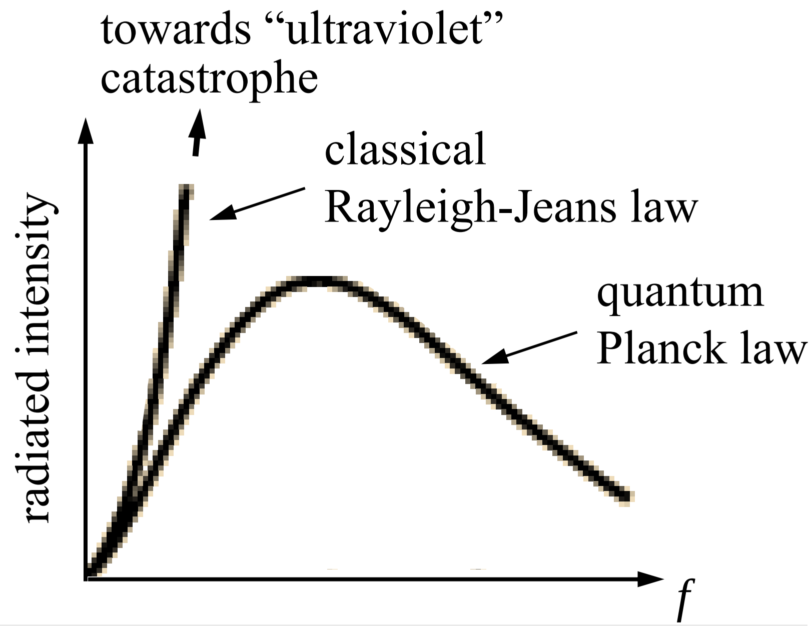
$$\bar{E} = hf\,\frac{1}{e^{hf/k_BT}-1}$$

Plank distribution function: $\frac{1}{e^{hf/k_BT}-1}$

$$u(f,T)df = \frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/k_BT}-1}df$$

$$u(\lambda,T)d\lambda = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda k_BT}-1}d\lambda$$

$$f = \frac{c}{\lambda}$$



in classical Rayleigh-Jeans theory $\bar{E} = k_BT$

$$u(f,T)df = \frac{8\pi f^2}{c^3}k_BTdf$$

for $f\rightarrow\infty$ classical theory predicts unlimited energy emission in the ultraviolet region "ultraviolet catastrophe"

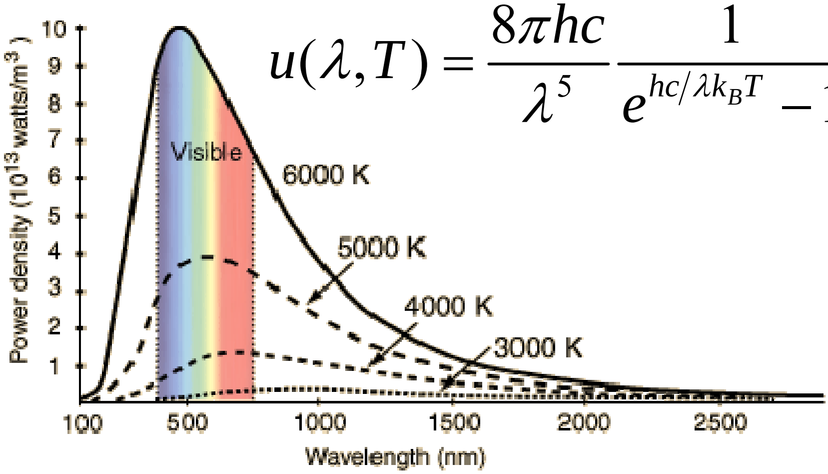
<u>in quantum theory "ultraviolet catastrophe" is avoided:</u>

$\bar{E}$ tents to zero at high *f* because the first allowed energy level $E=hf$ is so large for large *f* compared to the average thermal energy available k_BT that occupation of the first excited state is negligibly small

spectral energy density $u(f,T)$
is energy per unit volume per unit frequency of the radiation within the blackbody cavity

$$J(f,T) = \frac{c}{4} u(f,T)$$

power density $e_f = J(f,T)$
is power emitted per unit area per unit frequency



$$u(\lambda,T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}$$

Wien's displacement law, 1893:
the wavelength marking the maximum power emission of a blackbody, λ_{max}, shifts towards shorter wavelengths with increasing temperature

$$\lambda_{\max} \sim T^{-1}$$

$$\lambda_{\max} T = 2.898 \times 10^{-3} \text{ m} \cdot \text{K}$$

Stefan-Boltzmann law, 1879:
The total power per unit area emitted at all frequencies by a blackbody, e_{total},
is proportional to the forth power of its temperature

$$e_{total} = \int_0^\infty e_f df = \sigma T^4$$

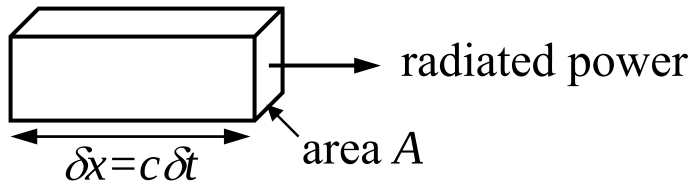
$$\sigma = 5.67 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4} \quad \leftarrow \text{The Stefan-Boltzmann constant}$$

$$e_{total} = \frac{c}{4} \int_0^\infty u(\lambda,T) d\lambda = \int_0^\infty \frac{2\pi hc^2}{\lambda^5 \left(e^{hc/\lambda k_B T} - 1\right)} d\lambda \underset{\uparrow}{=} \frac{2\pi k_B{}^4 T^4}{c^2 h^3} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1} dx}_{=\pi^4/15} = \frac{2\pi^5 k_B{}^4}{15 c^2 h^3} T^4 = \sigma T^4$$

$x = hc/\lambda k_B T$

derivation of $J(f,T)=\frac{c}{4}u(f,T)$

for perpendicular radiated energy

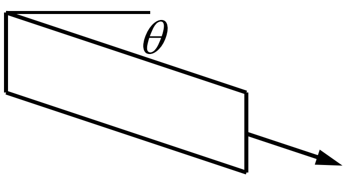


because half the power will be going in the –x direction

$$\text{radiated power} = \frac{1}{2}\left[\text{energy density}\right]\times\frac{\text{volume}}{\text{time}} =$$

$$=\frac{1}{2}u(f,T)\frac{A\cdot\delta x}{\delta t}=\frac{1}{2}u(f,T)\frac{A\cdot\delta x}{\delta x/c}=\frac{c}{2}u(f,T)A$$

for any angle



$$\text{perpendicular radiated power} = \frac{1}{2}\left[\text{energy density}\right]\times\frac{\text{volume}}{\text{time}}\times\cos\theta =$$

$$=\frac{1}{2}u(f,T)\frac{A\cdot\delta x}{\delta t}\cos\theta=\frac{1}{2}u(f,T)\frac{A\cdot\delta x}{\delta x\cdot\cos^{-1}\theta/c}\cos\theta=\frac{c}{2}u(f,T)A\cos^2\theta$$

averaging over θ and dividing by A to get power density

$$J(f,T)=\frac{\langle\text{radiated power}\rangle}{\text{A}}=\frac{\frac{c}{2}u(f,T)A\langle\cos^2\theta\rangle}{A}=\frac{c}{4}u(f,T)$$

$$\langle\cos^2\theta\rangle=1/2$$

Estimate the surface temperature of the Sun and find λ_{max} for the Sun emission.
The Sun radius R_S=7.0x10^8 m
The Earth-to-Sun distance R=1.5x10^{11} m
The total power from the Sun at the Earth e_{total}=1400 W/m^2

$$e_{total}(R_S) = \sigma T^4 \qquad \leftarrow \text{Stefan-Boltzmann law}$$

$$e_{total}(R_S)\cdot 4\pi R_S^{\,2} = e_{total}(R)\cdot 4\pi R^2 \qquad \leftarrow \text{conservation of energy}$$

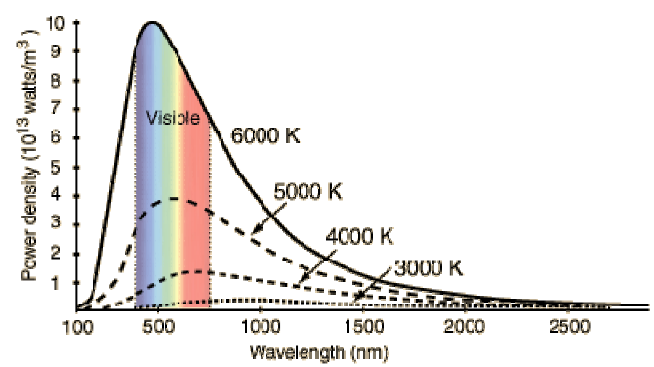
$$T = \left(\frac{1}{\sigma}\cdot e_{total}(R_S)\right)^{1/4} = \left(\frac{1}{\sigma}\cdot\frac{e_{total}(R)\cdot R^2}{R_S^{\,2}}\right)^{1/4}$$

$$T = \left(\frac{1}{5.67\times10^{-8}\ \text{W/m}^2\cdot\text{K}^4}\cdot\frac{\left(1400\ \ \text{W/m}^2\right)\cdot\left(1.5\times10^{11}\ \text{m}\right)^2}{\left(7.0\times10^8\ \text{m}\right)^2}\right)^{1/4} = \underline{5800\ \text{K}}$$

$$\lambda_{\max} = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{T} = \frac{2.898\times10^{-3}\ \text{m}\cdot\text{K}}{5800\ \text{K}} = 500\times10^{-9}\ \text{m} = \underline{500\ \text{nm}}$$

← eye's sensitivity peak

↑ Wien's displacement law



Planck, 1900: oscillators in the walls of the blackbody are quantized

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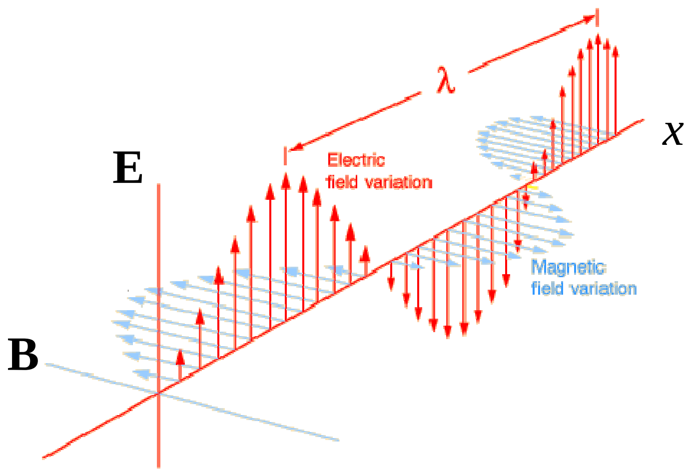
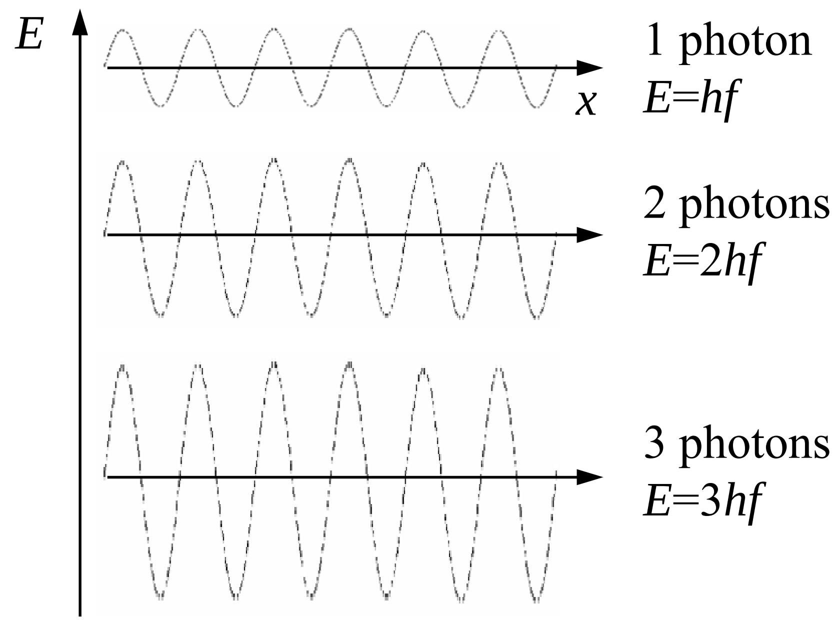
Einstein, 1905: light itself is composed of quanta of energy

photon – a quantum of electromagnetic radiation (a quantum of light)





$$E_{\text{photon}} = hf$$

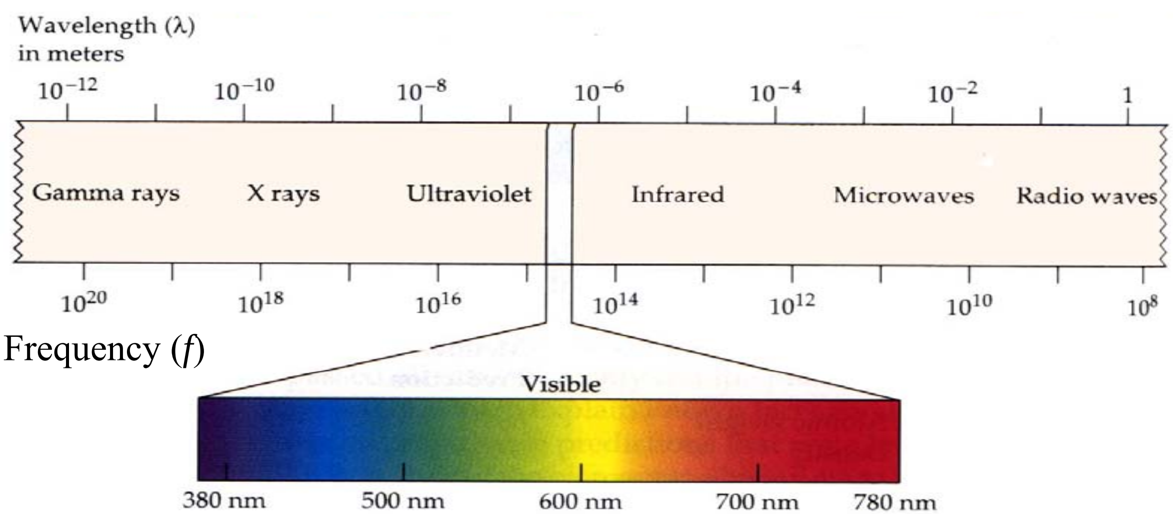
↓

the photoelectric effect

Einstein, 1906:
in addition to carrying **energy** $E=hf$
a photon carries a **momentum** $p=E/c=hf/c$ directed along its line of motion

Peter Debye, Arthur Holly Compton, 1923:
scattering of x-rays photons from electrons could be explained by treating photons as particles with energy hf and momentum hf/c and
by conserving energy and momentum of the photon-electron pair in a collision

particle properties of light



$$\text{photon energy } hf = \frac{hc}{\lambda} \approx \frac{hc}{10^{-10}\,\text{m}} \approx 12\ \text{keV}$$

x-rays – are electromagnetic waves with short wavelengths

x-rays were discovered by **Wilhelm Roentgen in 1895:**

X-rays are generated when high-speed electrons strike a metal target and give up some of their energy when they interact with the orbital electrons of an atom