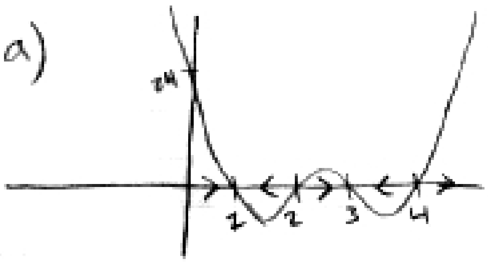
110A Problem Set #1 due 10/4

1. $\dot{u} = (u-1)(u-2)(u-3)(u-4)$

a)



b) $u^* = 1$, stable fixed point

$u^* = 2$, unstable fixed point

$u^* = 3$, stable fixed point

$u^* = 4$, unstable fixed point

c) $u_0 = 1/2$, $u(\infty) = 1$

d) $u_0 = 3/2$, $u(\infty) = 1$

e) $u_0 = 5/2$, $u(\infty) = 3$

f) $u_0 = 7/2$, $u(\infty) = 3$

g) $u_0 = 9/2$, $u(\infty) = 9/2$

2. $\ddot{x} + \Omega^2 x = 0$

a) $\ddot{x} = -\Omega^2 x \rightarrow \frac{d^2x}{dt^2} = -\Omega^2 x$, $v = \dot{x} = \frac{dx}{dt}$

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\Omega^2 x \end{pmatrix}$$

where $\dot{\phi} = \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix}$, $M = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix}$, $\phi = \begin{pmatrix} x \\ v \end{pmatrix}$

b) $\dot{\phi} = M\phi$, $\phi(t) = e^{Mt}\phi(0)$

$$e^{Mt} = 1 + Mt + \frac{1}{2!}M^2t^2 + \frac{1}{3!}M^3t^3 + \cdots$$

$$M^2 = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix} = \begin{pmatrix} -\Omega^2 & 0 \\ 0 & -\Omega^2 \end{pmatrix} = -\Omega^2 \mathbb{1}$$

$\therefore$ $M^{2k} = (-\Omega^2)^k \mathbb{1} \rightarrow$ even

$M^{2k+1} = (-\Omega^2)^k M \rightarrow$ odd

$$e^{Mt}_{even} = 1 + \frac{1}{2!}M^2t^2 + \frac{1}{4!}M^4t^4 + \cdots$$

$$e^{Mt}_{odd} = Mt + \frac{1}{3!}M^3t^3 + \frac{1}{5!}M^5t^5 + \cdots$$

$\rightarrow$ Remember: $\begin{cases} \cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \end{cases}$

$$e^{Mt}_{even} = 1 + \frac{1}{2!}M^2t^2 + \cdots = \mathbb{1}\left(1 - \frac{\Omega^2t^2}{2!} + \frac{\Omega^4t^4}{4!} - \cdots\right)$$

$$e^{Mt}_{odd} = Mt + \frac{1}{3!}M^3t^3 + \cdots = \frac{M}{\Omega}\left(\Omega t - \frac{\Omega^3t^3}{3!} + \frac{\Omega^5t^5}{5!} - \cdots\right)$$

$\therefore$ $e^{Mt}_{even} = \mathbb{1}\cos(\Omega t)$, $e^{Mt}_{odd} = \frac{M}{\Omega}\sin(\Omega t)$

2 b) continued.

$$e^{Mt}_{even} + e^{Mt}_{odd} = \mathbb{1}\cos(\Omega t) + \frac{M}{\Omega}\sin(\Omega t) = e^{Mt}$$

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 \\ 0 & \cos\Omega t \end{pmatrix}\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + M\phi(0)\frac{\sin(\Omega t)}{\Omega}$$

note: $\dot{\phi}(0) = M\phi(0)$

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos\Omega t & 0 \\ 0 & \cos\Omega t \end{pmatrix}\begin{pmatrix} x_0 \\ v_0 \end{pmatrix} + \frac{\sin(\Omega t)}{\Omega}\begin{pmatrix} v_0 \\ -\Omega^2 x_0 \end{pmatrix}$$

$$\therefore \quad x(t) = x_0\cos(\Omega t) + \frac{v_0}{\Omega}\sin(\Omega t)$$

$$v(t) = v_0\cos(\Omega t) - \Omega x_0\sin(\Omega t)$$

3. $\dot{A} = \epsilon A - gA^3$: supercrit for $g > 0$.

$\dot{A} = \epsilon A - gA^3 - kA^5$: subcrit for $g < 0$.

a) Landau eqn: $\dot{A} = \epsilon A - gA^3$

$A^* \propto \epsilon^\beta \rightarrow \epsilon A - gA^3 = 0$, $A^* = 0$

$$A^*_\pm = \pm \frac{\sqrt{4\epsilon g}}{2g} = \pm \varepsilon^{1/2}$$

$\therefore$ the landau eqn predicts $A^* \propto \varepsilon^\beta$ where $\beta = .5$, confirming the experimental results of $\beta = .5 \pm .01$.

b) $\dot{A} = \epsilon A - gA^3 - kA^5$

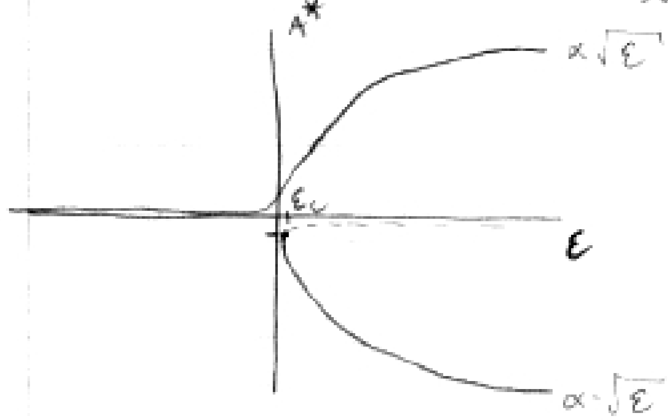
for $g = 0$, $\dot{A} = \epsilon A - kA^5$

$A^* = 0$, $A^*_\pm = \pm \sqrt[4]{\frac{\varepsilon}{k}} \propto \varepsilon^{1/4}$

c) $\dot{A} = h + \epsilon A - gA^3 - kA^5$ $k > 0$ fixed, $h > 0$, $h \ll 1$

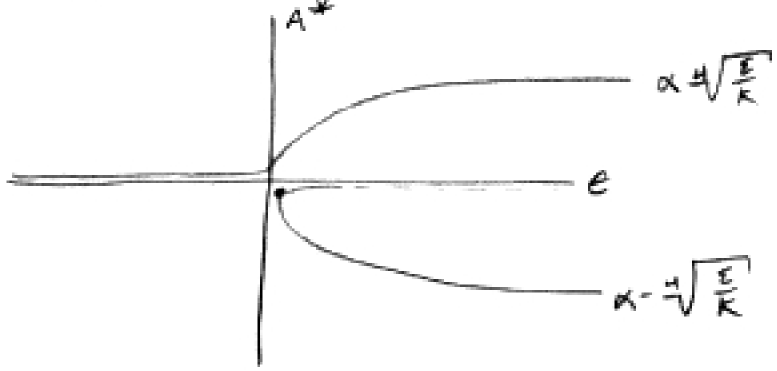
$\dot{A} = h + \epsilon A - gA^3 - kA^5 = 0$ in order to solve for roots analytically, let $h \rightarrow +0$.

i) $A^* \approx 0$, $\pm\sqrt{\varepsilon}$ we have an imperfect supercritical pitchfork.



where $\varepsilon_c \propto \frac{3}{2^{2/3}} |h|^{2/3}$

ii) $g=0, \quad A^* \approx 0, \pm\sqrt[4]{\frac{\varepsilon}{K}}$

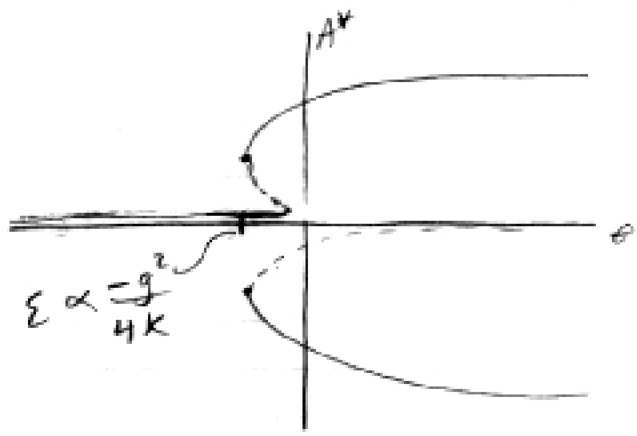


iii) $g<0, \quad A^* \approx 0, \left(+\frac{g}{2K} \pm \sqrt{\frac{g^2}{4K^2} + \frac{\varepsilon}{K}}\right)^{1/2}$

$\varepsilon < -\frac{g^2}{4K}$, one real root

$0 > \varepsilon > -\frac{g^2}{4K}$, 5 real roots

$\varepsilon > 0$, 3 real roots



4. a) $\dot{x} = x - y = 0$ $\quad (x^*, y^*) = (2, 2)$
$\dot{y} = x^2 - 4 = 0$

$$M = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$$

for $(x^*, y^*) = (2, 2)$, $T = 1$, $D = 4$

here, $D > \frac{1}{4}T^2$ $\therefore$ unstable spiral @ $(2,2)$

b) $\dot{x} = 1 + y - e^{-x} = 0$
$\dot{y} = x^3 - y = 0$ $\quad M = \begin{pmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{pmatrix}$

$(x^*, y^*) = (0, 0)$

for $(0,0)$, $T = 0$, $D = -1$

here, $D < 0$ we have a saddle @ $(0,0)$

c) $\dot{x} = \sin y = 0$ $\quad (x^*, y^*) = (\frac{\pi}{2} + n\pi, n\pi)$, $n = 0, 1, 2, \cdots$
$\dot{y} = \cos x = 0$ $\quad$ ↳ no flow in this case. b/c $\lambda_{\pm} = 0$

$$M = \begin{pmatrix} 0 & \cos y \\ -\sin x & 0 \end{pmatrix}$$

$\sin y - \cos x = 0 \rightarrow x = y = \frac{\pi}{4} + n\pi$

$$M = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$ $\quad T = 0$, $D = \frac{2}{4} = \frac{1}{2}$

since $T = 0$, $D > 0$, $\lambda_{\pm} = \pm i\sqrt{\frac{1}{2}}$
$\therefore$ centers @ $(x^*, y^*) = (\frac{\pi}{4} + n\pi, \frac{\pi}{4} + n\pi)$

5. $\dot{N}_1 = r_1 N_1\left(1 - \frac{N_1}{k_1}\right) - b_1 N_1 N_2$

$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2$

a) $t \to \alpha\tau$, $N_1 \to \beta n_1$, $N_2 \to \gamma n_2$

$$\frac{\beta\, dn_1}{\alpha\, d\tau} = r_1 \beta n_1\left(1 - \frac{\beta n_1}{k_1}\right) - b_1 \beta n_1 \gamma n_2$$

$$\frac{dn_1}{d\tau} = \alpha r_1 n_1\left(1 - \frac{\beta n_1}{k_1}\right) - b_1 \alpha\gamma n_1 n_2$$

$$\frac{\gamma\, dn_2}{\alpha\, d\tau} = r_2 \gamma n_2 - b_2 \beta n_1 \gamma n_2$$

$$\frac{dn_2}{d\tau} = \alpha r_2 n_2 - b_2 \alpha\beta n_1 n_2$$

$b_1 \alpha \gamma = 1$ $\Rightarrow$ $b_1\gamma = b_2\beta$ $\quad \alpha r_1 = 1$, $\quad \frac{r_2}{r_1} = \varepsilon$

$b_2 \alpha\beta = 1$

$\beta/k_1 = K$

$$\dot{n}_1 = n_1\left(1 - \frac{n_1}{K}\right) - n_1 n_2$$

$$\dot{n}_2 = \varepsilon n_2 - n_1 n_2$$

b) $\dot{n}_1 = 0$, $\dot{n}_2 = 0$

$n_1\left(1 - \frac{n_1}{K}\right) - n_1 n_2 = 0$ $\rightarrow$ K is the carrying capacity.

$\varepsilon n_2 - n_1 n_2 = 0$

for K large

$\dot{n}_1 \simeq n_1 - n_1 n_2 = n_1(1 - n_2) = 0$

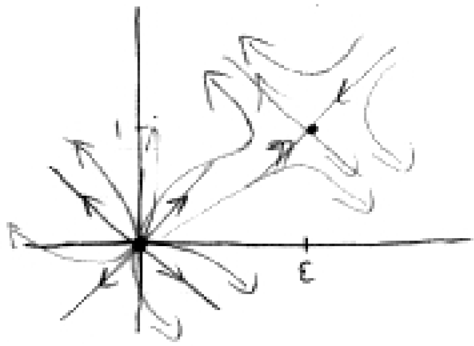
$\dot{n}_2 = \varepsilon n_2 - n_1 n_2 = n_2(\varepsilon - n_1) = 0$, $\quad M = \begin{pmatrix} 1 - n_2 & -n_1 \\ -n_2 & \varepsilon - n_1 \end{pmatrix}$

$(n_1^*, n_2^*) = (0,0), (\varepsilon, 1)$

pt $(0,0)$: $M = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ $\quad T = 1 + \varepsilon$, $D = \varepsilon$ $\quad$ for $\varepsilon > 0$,

we get $0 < D < \frac{1}{4}T^2 \rightarrow$ unstable node pt $(\varepsilon, 1)$: $M = \begin{pmatrix} 0 & -\varepsilon \\ -1 & 0 \end{pmatrix}$ $T=0$, $D=-\varepsilon$

we get a saddle point



for K small:

$$\dot{n}_1 = n_1\left(1 - \frac{n_1}{K}\right) - n_1 n_2 = 0$$
$$\dot{n}_2 = \varepsilon n_2 - n_1 n_2 = 0$$

$$(n_1^*, n_2^*) = (0,0), (K,0), \left(\varepsilon, \frac{\varepsilon}{K} - 1\right)$$

$$M = \begin{pmatrix} 1 - \frac{2n_1}{K} - n_2 & -n_1 \\ -n_2 & \varepsilon - n_1 \end{pmatrix}$$

pt $(0,0)$: $M = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$ $\to$ unstable node

pt $(K,0)$: $M = \begin{pmatrix} -1 & -K \\ 0 & \varepsilon - K \end{pmatrix}$ $T = \varepsilon - K - 1$, $D = K - \varepsilon$, $T > 0$

$\frac{1}{4}T^2 > D > 0$ ∴ unstable node

pt $\left(\varepsilon, \frac{\varepsilon}{K} - 1\right)$: $M = \begin{pmatrix} 2 - \frac{3\varepsilon}{K} & -\varepsilon \\ 1 - \frac{\varepsilon}{K} & 0 \end{pmatrix}$ $T = 2 - \frac{3\varepsilon}{K}$, $D = \varepsilon - \frac{\varepsilon^2}{K}$

$D < 0$ ∴ saddle