PHYSICS 110A : CLASSICAL MECHANICS
PROBLEM SET #5

[1] A bead of mass m slides frictionlessly along a wire curve $z = x^2/2b$, where $b > 0$. The wire rotates with angular frequency ω about the $\hat{\boldsymbol{z}}$ axis.

(a) Find the Lagrangian of this system.

(b) Find the Hamiltonian.

(c) Find the effective potential $U_{\text{eff}}(x)$.

(d) Show that the motion is unbounded for $\omega^2 > \omega_{\text{c}}^2$ and find the critical value ω_{c}.

(e) Sketch the phase curves for this system for the cases $\omega^2 < \omega_{\text{c}}^2$ and $\omega^2 > \omega_{\text{c}}^2$.

(f) Find an expression for the period of the motion when $\omega^2 < \omega_{\text{c}}^2$.

(g) Find the force of constraint which keeps the bead on the wire.

Solution :

We will solve this problem for a general shape $z(x)$. Since the curve is rotating, we will use the radial coordinate ρ instead of x, keeping in mind that the wire is a one-dimensional object and not a two-dimensional surface. The coordinate ρ then indicates the direction along the wire but perpendicular to the $\hat{\boldsymbol{z}}$ axis. Note that $\rho \in \mathrm{R}$ may be positive or negative.

(a) The Lagrangian is

$$L(\rho, z, \dot{\rho}, \dot{z}) = \tfrac{1}{2} m\dot{\rho}^2 + \tfrac{1}{2} m\dot{z}^2 + \tfrac{1}{2} m\omega^2\rho^2 - mgz \ . \tag{1}$$

This is supplemented by the constraint

$$G(\rho, z) = z - z(\rho) = 0 \ . \tag{2}$$

Of course, we could eliminate z as an independent degree of freedom from the outset, and write

$$L(\rho, \dot{\rho}) = \tfrac{1}{2} m\Big[\big(1 + [z'(\rho)]^2\big)\dot{\rho}^2 + \omega^2\rho^2\Big] - mgz(\rho) \ . \tag{3}$$

(b) The Hamiltonian is

$$\begin{aligned} H &= p_\sigma \dot{q}_\sigma - L \\ &= \tfrac{1}{2} m\dot{\rho}^2 + \tfrac{1}{2} m\dot{z}^2 - \tfrac{1}{2} m\omega^2\rho^2 + mgz \\ &= \tfrac{1}{2} m\big(1 + [z'(\rho)]^2\big)\dot{\rho}^2 + U_{\text{eff}}(\rho) \ . \end{aligned} \tag{4}$$

(c) The effective potential is

$$\begin{aligned} U_{\text{eff}}(\rho) &= mgz(\rho) - \tfrac{1}{2}m\omega^2\rho^2 \\ &= \tfrac{1}{2}m\left(\omega_{\text{c}}^2 - \omega^2\right)\rho^2 \ , \end{aligned} \tag{5}$$

where $\omega_{\text{c}} \equiv \sqrt{g/b}$. Note that we do not have $m\ddot{\rho} = -U'_{\text{eff}}(\rho)$. This is because

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\big(1 + [z'(\rho)]^2\big)\dot{\rho} \ , \tag{6}$$

and thus

$$\dot{p}_\rho = \frac{\partial L}{\partial \rho} \quad \Rightarrow \quad \Big(1 + \big[z'(\rho)\big]^2\Big)\,\ddot{\rho} = \omega^2\rho - gz'(\rho) - z'(\rho)\,z''(\rho)\,\dot{\rho}^2 \ . \tag{7}$$

(d) Since L has no explicit time dependence, H is a constant of the moton:

$$\begin{aligned} H &= \tfrac{1}{2}m\big(1 + [z'(\rho)]^2\big)\dot{\rho}^2 + U_{\text{eff}}(\rho) \\ &= \tfrac{1}{2}m\left(1 + \frac{\rho^2}{b^2}\right)\dot{\rho}^2 + \tfrac{1}{2}m(\omega_{\text{c}}^2 - \omega^2)\,\rho^2 \ . \end{aligned} \tag{8}$$

Note that if $\omega^2 > \omega_{\text{c}}^2$ that the level sets of $H(\rho,\dot{\rho})$ are unbounded. Hence the motion of the system, which takes place along these level sets, is also unbounded.

(e) Let us define the dimensionless coordinate $u \equiv \rho/b$ and dimensionless time variable $s \equiv |\omega_{\text{c}}^2 - \omega^2|^{1/2}\,t$. Then conservation of H means that

$$C = (1 + u^2)\,v^2 - \sigma u^2 \tag{9}$$

is constant, where $v = \frac{du}{ds}$ is the dimensionless velocity, and where $\sigma \equiv \text{sgn}\big(\omega^2 - \omega_{\text{c}}^2\big)$. Setting $\frac{dC}{ds} = 0$, we obtain

$$\frac{du}{ds} = v \qquad , \qquad \frac{dv}{ds} = \frac{(\sigma - v^2)\,u}{1 + u^2} \ . \tag{10}$$

This phase flow has a single fixed point, at $(u, v) = (0, 0)$, which is either a center ($\omega^2 < \omega_{\text{c}}^2$) or a saddle point ($\omega^2 > \omega_{\text{c}}^2$).

A sketch of the phase flow for $\omega^2 < \omega_{\text{c}}^2$ is shown in Fig. 1; the flow for $\omega^2 > \omega_{\text{c}}^2$ is shown in Fig. 2. The `Mathematica` plot in Fig. 1 was obtained from the following commands:

```
<<Graphics`PlotField`
G1 = ContourPlot[ (1+x^2) y^2 + x^2, {x,-4,4}, {y,-4,4}, PlotPoints -> 50,
Contours -> {0.1, 1, 4, 10, 20, 50, 100}, ContourShading -> False];
G2 = PlotVectorField[ {y, -(1+y^2) x / (1+x^2)}, {x,-4,4}, {y,-4,4},
PlotPoints -> 30, ColorFunction -> Hue, ScaleFactor -> 0.55];
Show[ {G1, G2} ]
```

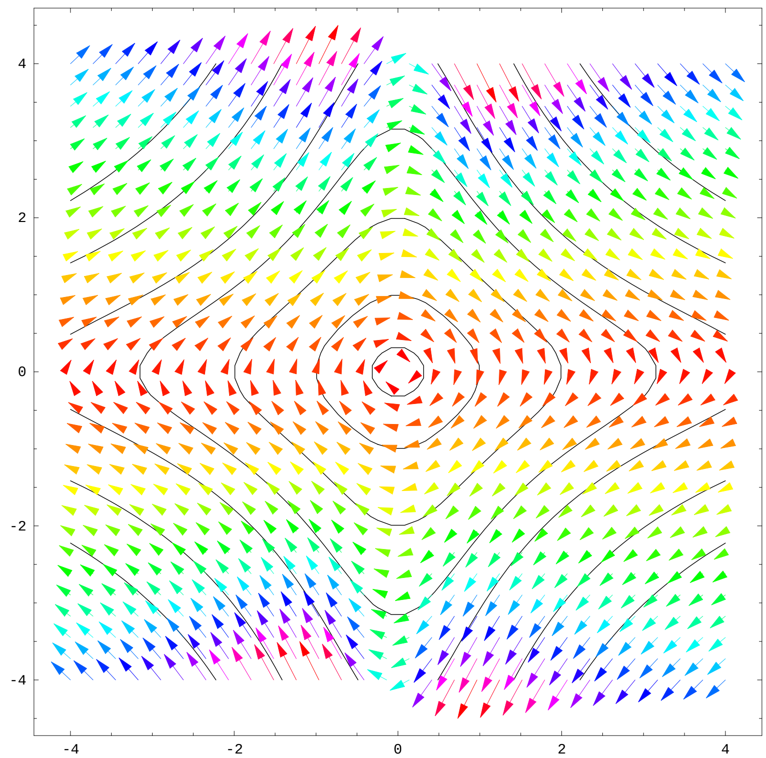


Figure 1: Level sets of the function $C(u,v) = (1+u^2)\,v^2 + u^2$ superimposed on the phase flow $\dot{u} = v$, $\dot{v} = -u\,(1+v^2)/(1+u^2)$. Note that the phase curves are bounded.

It is worthwhile noting that other shapes $z(\rho)$ may have fixed points for $\rho \neq 0$. For example, consider the shape

$$z(\rho) = \frac{\rho^4}{4\,b^3} \ . \tag{11}$$

If we define $u = \rho/b$ and $\omega_{\rm c}^2 = g/b$ as before, but this time write $s = \omega_{\rm c}\,t$, and define the new dimensionless parameter $\varepsilon \equiv \omega^2/\omega_{\rm c}^2$, we have that

$$C(u,v) = (1+u^6)\,v^2 + \tfrac{1}{4}u^2 - \tfrac{1}{2}\varepsilon u^2 \tag{12}$$

is constant, and the dynamics is given by

$$\frac{du}{ds} = v \qquad , \qquad \frac{dv}{ds} = \frac{(\varepsilon - u^2 - 6\,u^4\,v^2)\,u}{2\,(1+u^6)} \ . \tag{13}$$

This flow, shown in Fig. 3, exhibits a saddle point at $(u,v) = (0,0)$ and two centers at $(u,v) = (\pm\sqrt{\varepsilon}, 0)$. The separatrix, which flows through $(0,0)$, has $C = 0$. All the phase curves are bounded.

(e) The equation of motion can be taken as $\dot{H} = 0$, which yields

$$\Big(1 + \big[z'(\rho)\big]^2\Big)\,\ddot{\rho} + z'(\rho)\,z''(\rho)\,\dot{\rho}^2 = \omega^2\rho - g\,z'(\rho) \ . \tag{14}$$

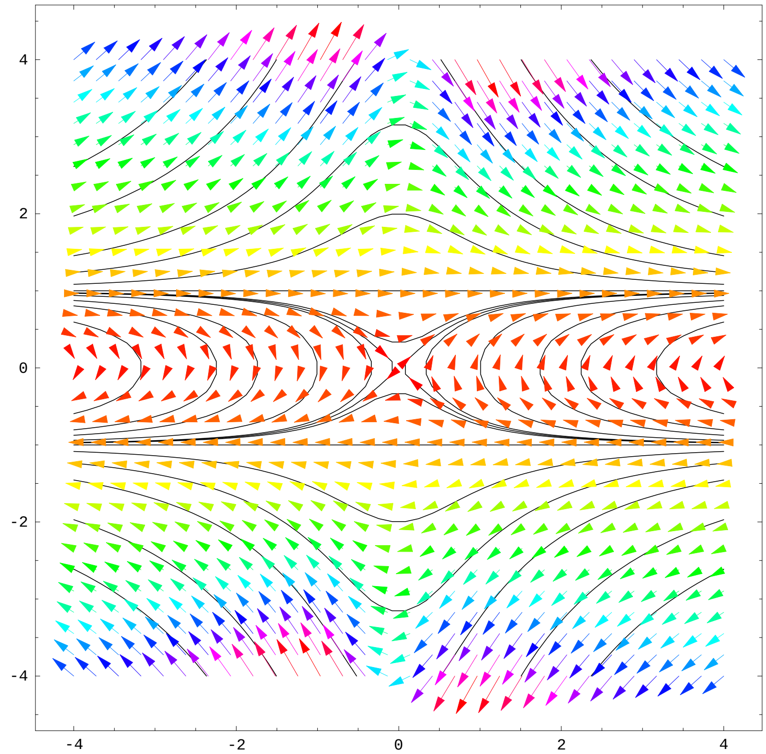


Figure 2: Level sets of the function $C(u,v) = (1+u^2)\,v^2 - u^2$ superimposed on the phase flow $\dot{u} = v$, $\dot{v} = u\,(1-v^2)/(1+u^2)$. Note that the phase curves are unbounded.

We can expand about an equilibrium solution $gz'(\rho^*) = \omega^2 \rho^*$, writing $\rho = \rho^* + \delta\rho$, in which case

$$\delta\ddot{\rho} = -\Omega^2\,\delta\rho \qquad , \qquad \Omega^2 = \frac{gz''(\rho^*) - \omega^2}{1 + \big[z'(\rho^*)\big]^2} \ . \tag{15}$$

Thus, the equilibrium at ρ^* is stable if $\omega^2 < gz''(\rho^*)$ and unstable if $\omega^2 > gz''(\rho^*)$.

We can go even farther in this analysis, using the conservation of H, which allows us to write the motion as a first order ODE,

$$dt = \pm \frac{\sqrt{1 + \big[z'(\rho)\big]^2}}{\sqrt{\frac{2}{m}\big[H - U_{\text{eff}}(\rho)\big]}}\, d\rho \ . \tag{16}$$

Identifying the turning points as solutions to

$$H = U_{\text{eff}}(\rho_\pm) \ , \tag{17}$$

we have the period for motion $T(H)$ is

$$T(H) = \sqrt{\frac{m}{2}} \int\limits_{\rho_-(H)}^{\rho_+(H)} d\rho\, \sqrt{\frac{1 + \big[z'(\rho)\big]^2}{H - U_{\text{eff}}(\rho)}} \ . \tag{18}$$

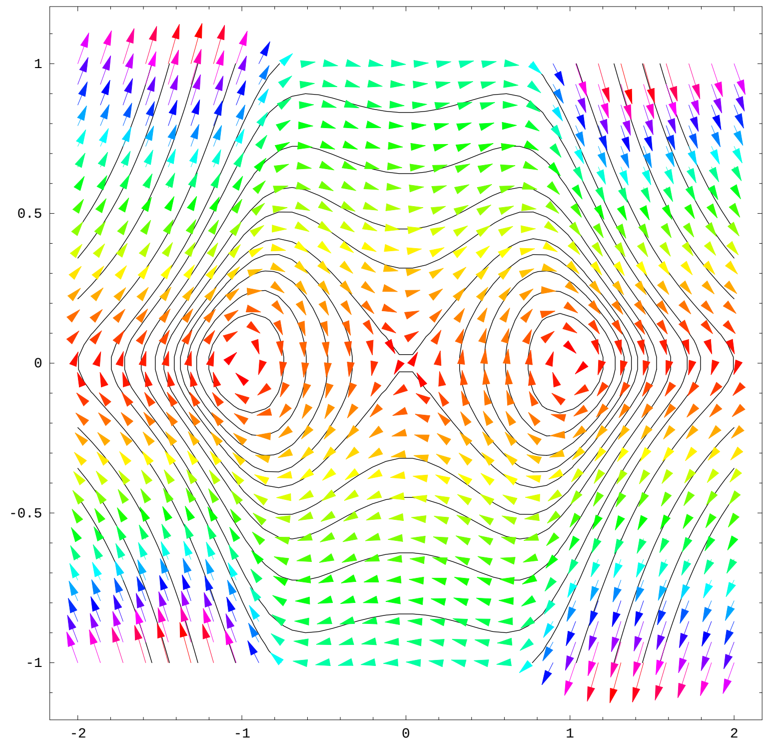


Figure 3: Level sets of the function $C(u,v) = (1+u^6)\,v^2 + \frac{1}{4}u^4 - \frac{1}{2}\varepsilon u^2$ superimposed on the phase flow $\dot{u} = v$, $\dot{v} = \frac{1}{2}u\,(\varepsilon - u^2 - 6\,u^4\,v^2)/(1+u^6)$, for $\varepsilon = 1$. There are two centers, at $(\pm 1, 0)$, and a saddle at $(0,0)$. All phase curves are bounded.

For the case $z(\rho) = \rho^2/2b$, we have

$$T(H) = \frac{4}{\sqrt{\omega_c^2 - \omega^2}} \int\limits_0^{\pi/2} \! d\theta \, \sqrt{1 + \frac{2H \sin^2\!\theta}{mb^2(\omega_c^2 - \omega^2)}} \ . \tag{19}$$

(g) If we write $G(\rho, z) = z - z(\rho) = 0$ as a constraint, the equations of motion are

$$m\ddot{\rho} = m\omega^2\rho - \lambda z'(\rho) \tag{20}$$

$$m\ddot{z} = -mg + \lambda \ . \tag{21}$$

We now eliminate $z = z(\rho)$, in which case

$$\dot{z} = z'(\rho)\,\dot{\rho} \qquad , \qquad \ddot{z} = z'(\rho)\,\ddot{\rho} + z''(\rho)\,\dot{\rho}^2 \ . \tag{22}$$

We may now write

$$\lambda = mg + mz'(\rho)\,\ddot{\rho} + mz''(\rho)\,\dot{\rho}^2 \tag{23}$$

and, substituting this into the first of the equations of motion and collecting terms, we find

$$\Big(1 + \big[z'(\rho)\big]^2\Big)\,\ddot{\rho} = \omega^2\rho - gz'(\rho) - z'(\rho)\,z'(\rho)\,\dot{\rho}^2 \ . \tag{24}$$

As we have seen above, this result also follows from $\dot{H} = 0$. We may now solve for λ in terms of ρ and $\dot{\rho}$:

$$\lambda = \frac{m}{1 + \big[z'(\rho)\big]^2} \left(g + z''(\rho)\,\dot{\rho}^2 + \omega^2 \rho\, z'(\rho)\right) . \tag{25}$$

The force of constraint supplied by the wire is

$$\boldsymbol{Q} = Q\,\hat{\mathbf{n}}_\perp = (Q_\rho\,\hat{\boldsymbol{\rho}} + Q_z\,\hat{\boldsymbol{z}}) \ , \tag{26}$$

where

$$\hat{\mathbf{n}} = \frac{-z'(\rho)\,\hat{\boldsymbol{\rho}} + \hat{\mathbf{z}}}{\sqrt{1 + \big[z'(\rho)\big]^2}} \tag{27}$$

is the unit vector locally orthogonal to the tangent to the curve. Thus,

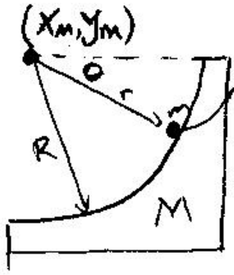
$$\begin{aligned} Q &= \lambda \cdot \sqrt{1 + \big[z'(\rho)\big]^2} \\ &= \frac{m\left(g + z''(\rho)\,\dot{\rho}^2 + \omega^2 \rho\, z'(\rho)\right)}{\sqrt{1 + \big[z'(\rho)\big]^2}} \ . \end{aligned} \tag{28}$$

We may further eliminate $\dot{\rho}$ in favor of ρ by invoking conservation of H, which says

$$\dot{\rho}^2 = \frac{\frac{2H}{m} - 2gz(\rho) + \omega^2 \rho^2}{1 + \big[z'(\rho)\big]^2} \ . \tag{29}$$

Problem set #5

7.34



$x_M = X \;,\; y_M = 0$

$x_m = r\cos\theta + X \;,\; y_m = -r\sin\theta$

$\dot{x}_m = \dot{r}\cos\theta - r\dot{\theta}\sin\theta + \dot{X}$

$\dot{y}_m = -\dot{r}\sin\theta - r\dot{\theta}\cos\theta$

$\dot{x}_m^2 = \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + \dot{X}^2 - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + 2\dot{r}\dot{X}\cos\theta$
$\quad - 2r\dot{X}\dot{\theta}\sin\theta$

$\dot{y}_m^2 = \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

a) $L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2) - mgy_m$

$= \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{r}\dot{X}(\cos\theta - \dot{\theta}\sin\theta)\right) + mgr\sin\theta$

In order to find the reaction of the wedge on the mass m, we cannot assume r is constant.

$\therefore$ our constraint $r - R = 0 = G_1(X, r, \theta)$

To get the eqns. of motion, we can set $r = R$ with $\dot{r} = \ddot{r} = 0$.

X: $\quad \ddot{X} = aR(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

θ: $\quad \ddot{\theta} = \dfrac{\ddot{X}\sin\theta + g\cos\theta}{R}$

for $a = \dfrac{m}{M+m}$

b) $$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \lambda \frac{\partial G}{\partial r}$$

$$\Rightarrow m\ddot{x}\cos\theta - mR\dot{\theta}^2 - mg\sin\theta = \lambda \quad , \quad \text{for } r = R, \; \dot{r} = \ddot{r} = 0$$

plug in $\ddot{x}$ in terms of $\theta, \dot{\theta}$:

$$\lambda = \left[\frac{a-1}{1-a\sin^2\theta}\right]\left(R\dot{\theta}^2 + g\sin\theta\right)$$

now, find an expression for $\dot{\theta}$ using conservation of energy:

$$\frac{M+m}{2}\dot{x}^2 + \frac{m}{2}\left(R^2\dot{\theta}^2 - 2\dot{x}R\dot{\theta}\sin\theta\right) - mgR\sin\theta = -mgR\sin\theta_0$$

where θ_0 is the initial position of m.

we also have $\dot{x} = aR\dot{\theta}\sin\theta$ from of eqn of motion

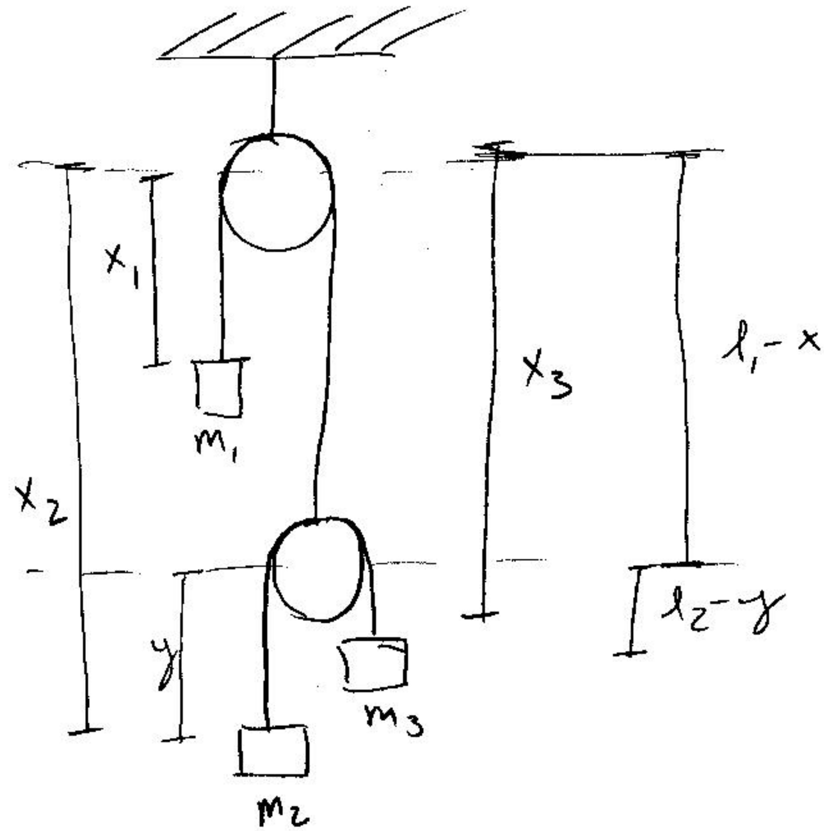
plug this into the energy expression:

$$\dot{\theta}^2 = \frac{2g(\sin\theta - \sin\theta_0)}{R(1-a\sin^2\theta)}$$

$\therefore$ we have the mass reaction:

$$\lambda = -\frac{mMg(3\sin\theta - a\sin^3\theta - 2\sin\theta_0)}{(m+M)(1-a\sin^2\theta)^2}$$

7.37



$$L = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2\right) + g(mx_1 + mx_2 + mx_3)$$

Constraints are $x_1 + y = l_1$; $x_2 - y + x_3 - y = l_2$

eqns of motion are

① $m_1 g - m_1 \frac{d^2 x}{dt^2} + 2\lambda = 0$

② $m_2 g - m_2 \frac{d^2 x_2}{dt^2} + \lambda = 0$

③ $m_3 g - m_3 \frac{d^2 x_3}{dt^2} + \lambda = 0$

The constraints can be combined to give

$$2x_1 + x_2 + x_3 - (2l_1 + l_2) = 0 \Rightarrow 2\ddot{x}_1 + \ddot{x}_2 + \ddot{x}_3 = 0 \quad (4)$$

combine eqns ① → ④ to get

$$\lambda = \frac{-4g}{\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}$$

and

$$T_1 = m_1 g - m_1 \ddot{x}_1 = -2\lambda = \frac{8g}{\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}$$