Problem set #6

8.5 first, solve for radius of circular orbit:

$$U_{eff}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2}$$

$$\frac{\partial U_{eff}}{\partial r}(r) = \frac{k}{r^2} - \frac{\ell^2}{\mu r^3} = 0 \qquad \frac{k}{r^2} = \frac{\ell^2}{\mu r^3} \rightarrow r_0 = \frac{\ell^2}{\mu k}$$

$$2\pi = \tau\dot{\theta} \quad , \dot{\theta} = \frac{\ell}{\mu r_0^2} \quad \rightarrow \quad \tau = \frac{2\pi \mu r_0^2}{\ell} = \frac{2\pi \mu r_0^2}{\sqrt{r_0 \mu k}}$$

$$\therefore \; r_0^3 = \frac{\tau^2 k}{\mu 4\pi^2}$$

now, use conservation of energy

$$-\frac{k}{r_0} = \frac{1}{2}\mu\dot{r}^2 - \frac{k}{r} \quad \rightarrow \quad \frac{k}{r} - \frac{k}{r_0} = \frac{1}{2}\mu\dot{r}^2$$

$$\frac{2}{\mu}k\left(\frac{r_0 - r}{r r_0}\right) = \dot{r}^2 \quad \rightarrow \quad \frac{dr}{dt} = \sqrt{\frac{2k}{\mu}}\left(\frac{r_0 - r}{r r_0}\right)^{1/2}$$

$$\int_0^T dt = \int_0^{r_0} dr \sqrt{\frac{\mu}{2k}}\left(\frac{r r_0}{r_0 - r}\right)^{1/2}$$

$$T = \frac{\pi}{2}\sqrt{\frac{\mu}{2k}}\, r_0^{3/2} = \frac{\tau}{4\sqrt{2}}$$

8.14 $r(\theta) = k\theta^2$, k const.

from the eqn for the geometric shape of orbit we have

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F(r)$$

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{1}{k}\frac{d}{d\theta}\left(\theta^{-2}\right) = -\frac{2}{k}\theta^{-3}$$

$$\frac{d^2}{d\theta}\left(\frac{1}{r}\right) = \frac{d}{d\theta}\left(-\frac{2}{k}\theta^{-3}\right) = \frac{6}{k}\theta^{-4} = \frac{6k}{r^2}$$

$$\therefore \quad 6k\theta^{-4} + k\theta^{-2} = -\frac{\mu k^2\theta^4}{\ell^2}F(r)$$

$$\Rightarrow \quad \frac{6k}{r^2} + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)$$

$$\frac{6k\ell^2}{\mu r^4} + \frac{\ell^2}{\mu r^3} = -F(r)$$

$$\boxed{\frac{\ell^2}{\mu}\left(\frac{6k}{r^4} + \frac{1}{r^3}\right) = -F(r)}$$

8.15 $r(\theta) = b\coth\left(\frac{\theta}{\sqrt{2}}\right)$, $F = -\frac{k}{r^5}$, $\ell = \frac{\sqrt{k}}{b}$

To show that $r(\theta)$ is the trajectory, show that it satisfies the geometric eqn of the orbit.

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu}{\ell^2 u^2} F\left(\frac{1}{u}\right) \qquad \text{where } u = \frac{1}{r}$$

$$\frac{d^2u}{d\theta^2} + u = -b^2 u^3 \quad , \text{ for } \mu = 1$$

solve this for $u(\theta) = \dfrac{1}{b\coth\left(\frac{\theta}{\sqrt{2}}\right)}$

$\Rightarrow$ see attached mathematica

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r[θ_] = Coth[θ/√2]
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$\text{Coth}\left[\frac{\theta}{\sqrt{2}}\right]$

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ParametricPlot[{r[θ] Cos[θ], r[θ] Sin[θ]}, {θ, 0, 100 Pi}]
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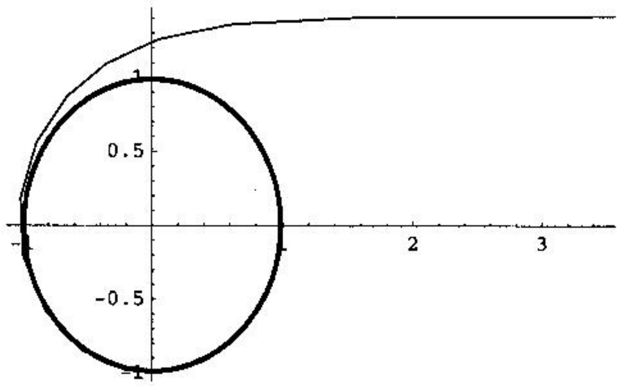

- Graphics -

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r[t_] = b Coth[t/√2]
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$b\,\text{Coth}\left[\frac{t}{\sqrt{2}}\right]$

$$\text{FullSimplify}\left[D\left[\frac{1}{r[t]}, \{t, 2\}\right] + \frac{1}{r[t]} - b^2 \frac{1}{r[t]^3}\right]$$

0

In[1]:= $u[t\_] = \frac{1}{b\,\text{Coth}[t/\sqrt{2}]}$

Out[1]= $\frac{\text{Tanh}\left[\frac{t}{\sqrt{2}}\right]}{b}$

In[2]:= $\text{FullSimplify}[D[u[t], \{t, 2\}] + u[t] - b^2\, u[t]^3]$

Out[2]= 0

8.214

a) $R_e = 6.38 \times 10^3$ km

$r_p = R_e + r_{min} = 6680$ km

$r_a = R_e + r_{max} = 9880$ km

$$a = \frac{r_p + r_a}{2} = 8280 \text{ km}$$

$$\epsilon = \frac{r_a}{a} - 1 = .1932$$

$$\alpha = r_p(1+\epsilon) = 7970 \text{ km}$$

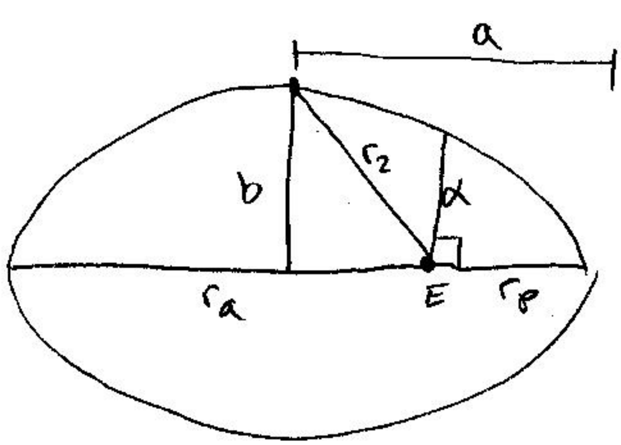
$$\epsilon \cos\theta = \alpha \cdot \frac{1}{r} - 1 \rightarrow @ \ \theta = \frac{\pi}{2}, \ r = \alpha$$

$$r_1 \text{ above earth} = \alpha - r_e = 1590 \text{ km}$$

b) $$r_2 \text{ above earth} = \sqrt{b^2 - (a - r_p)^2} - r_e$$

where $b = \sqrt{\alpha a}$

$$r_2 = \sqrt{\alpha a + (a + r_p)^2} - r_e = 8280 - 6380 = 1900 \text{ km}$$



8.39
$$\Delta V_1 = \sqrt{\frac{2k}{mr_1}\left(\frac{r_2}{r_1+r_2}\right)} + \sqrt{\frac{k}{mr_1}}$$

now solve numerically for smallest $|\Delta V_1|$.

for $\frac{k}{m} = GM_\odot = (6.67\times10^{-11}\, m^3/s^2\, kg)(2\times10^{30}\, kg)$

$r_1 =$ average distance between Earth & Sun

$= 150\times10^9$ m

$r_2 =$ average $\begin{pmatrix}\text{Venus}\\ \text{Mars}\end{pmatrix}$ - sun distance

$= \begin{pmatrix}108\\ 228\end{pmatrix}\times 10^9$ m

$\therefore\ \Delta V_{venus} = -2.53$ km/sec

$\Delta V_{mars} = 2.92$ km/sec

where the sign denotes the boost direction.

we see that a venus rendezvous requires less energy