PHYSICS 110A : CLASSICAL MECHANICS
FALL 2007 FINAL EXAM SOLUTIONS

[1] Two masses and two springs are configured linearly and externally driven to rotate with angular velocity ω about a fixed point on a horizontal surface, as shown in fig. 1. The unstretched length of each spring is a.

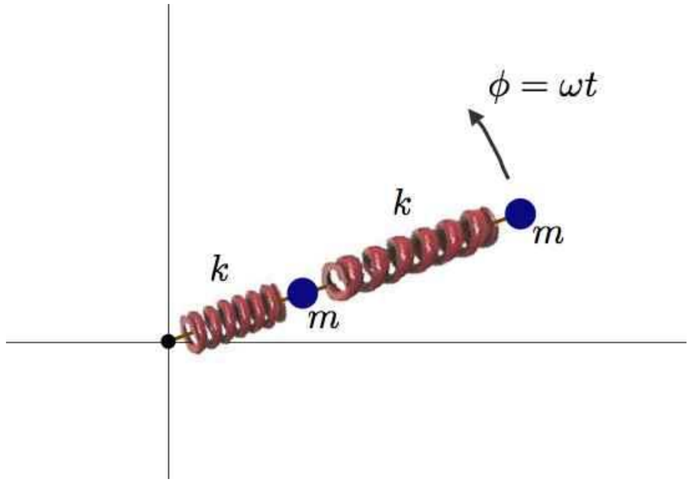

Figure 1: Two masses and two springs rotate with angular velocity ω.

(a) Choose as generalized coordinates the radial distances $r_{1,2}$ from the origin. Find the Lagrangian $L(r_1, r_2, \dot{r}_1, \dot{r}_2, t)$.
[5 points]

The Lagrangian is

$$L = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2 + \omega^2\, r_1^2 + \omega^2\, r_2^2\big) - \tfrac{1}{2}k\,(r_1 - a)^2 - \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{1}$$

(b) Derive expressions for all conserved quantities.
[5 points]

The Hamiltonian is conserved. Since the kinetic energy is not homogeneous of degree 2 in the generalized velocities, $H \neq T + U$. Rather,

$$H = \sum_\sigma p_\sigma\, \dot{q}_\sigma - L \tag{2}$$

$$= \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2\big) - \tfrac{1}{2}m\omega^2\big(r_1^2 + r_2^2\big) + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{3}$$

We could define an effective potential

$$U_{\text{eff}}(r_1, r_2) = -\tfrac{1}{2}m\omega^2\big(r_1^2 + r_2^2\big) + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{4}$$

Note the first term, which comes from the kinetic energy, has an interpretation of a fictitious potential which generates a *centrifugal* force.

(c) What equations determine the equilibrium radii r_1^0 and r_2^0? (You do not have to solve these equations.)
[5 points]

The equations of equilibrium are $F_\sigma = 0$. Thus,

$$0 = F_1 = \frac{\partial L}{\partial r_1} = m\omega^2 r_1 - k\,(r_1 - a) + k\,(r_2 - r_1 - a) \tag{5}$$

$$0 = F_2 = \frac{\partial L}{\partial r_2} = m\omega^2 r_2 - k\,(r_2 - r_1 - a)\ . \tag{6}$$

(d) Suppose now that the system is not externally driven, and that the angular coordinate ϕ is a dynamical variable like r_1 and r_2. Find the Lagrangian $L(r_1, r_2, \phi, \dot r_1, \dot r_2, \dot\phi, t)$.
[5 points]

Now we have

$$L = \tfrac{1}{2}m\big(\dot r_1^2 + \dot r_2^2 + r_1^2\,\dot\phi^2 + r_2^2\,\dot\phi^2\big) - \tfrac{1}{2}k\,(r_1 - a)^2 - \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{7}$$

(e) For the system described in part (d), find expressions for all conserved quantities.
[5 points]

There are two conserved quantities. One is p_ϕ, owing to the fact the ϕ is cyclic in the Lagrangian. *I.e.* $\phi \to \phi + \zeta$ is a continuous one-parameter coordinate transformation which leaves L invariant. We have

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m\big(r_1^2 + r_2^2\big)\,\dot\phi\ . \tag{8}$$

The second conserved quantity is the Hamiltonian, which is now $H = T + U$, since T is homogeneous of degree 2 in the generalized velocities. Using conservation of momentum, we can write

$$H = \tfrac{1}{2}m\big(\dot r_1^2 + \dot r_2^2\big) + \frac{p_\phi^2}{2m(r_1^2 + r_2^2)} + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{9}$$

Once again, we can define an effective potential,

$$U_{\text{eff}}(r_1, r_2) = \frac{p_\phi^2}{2m(r_1^2 + r_2^2)} + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ , \tag{10}$$

which is different than the effective potential from part (b). However in both this case and in part (b), we have that the radial coordinates obey the equations of motion

$$m\ddot r_j = -\frac{\partial U_{\text{eff}}}{\partial r_j}\ , \tag{11}$$

for $j = 1, 2$. Note that this equation of motion follows directly from $\dot H = 0$.

[2] A point mass m slides inside a hoop of radius R and mass M, which itself rolls without slipping on a horizontal surface, as depicted in fig. 2.

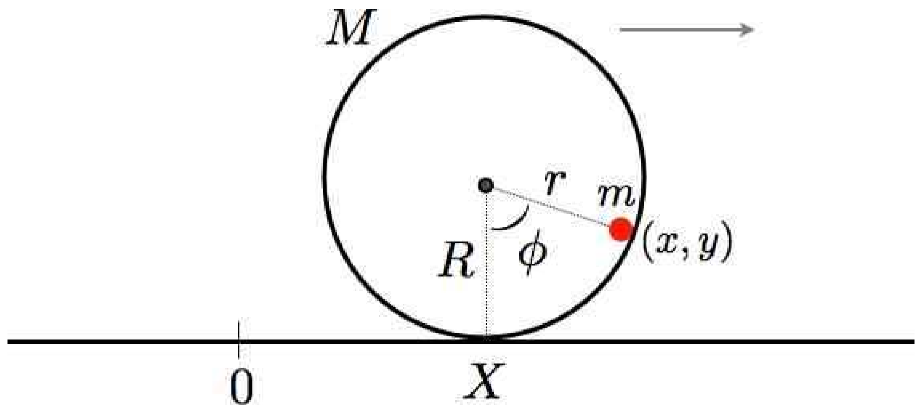

Figure 2: A mass point m rolls inside a hoop of mass M and radius R which rolls without slipping on a horizontal surface.

Choose as general coordinates (X, ϕ, r), where X is the horizontal location of the center of the hoop, ϕ is the angle the mass m makes with respect to the vertical ($\phi = 0$ at the bottom of the hoop), and r is the distance of the mass m from the center of the hoop. Since the mass m slides inside the hoop, there is a constraint:

$$G(X, \phi, r) = r - R = 0 \ .$$

Nota bene: The kinetic energy of the moving hoop, including translational and rotational components (but not including the mass m), is $T_{\text{hoop}} = M\dot{X}^2$ (*i.e.* twice the translational contribution alone).

`(a)` Find the Lagrangian $L(X, \phi, r, \dot{X}, \dot{\phi}, \dot{r}, t)$.
`[5 points]`

The Cartesian coordinates and velocities of the mass m are

$$x = X + r\sin\phi \qquad \dot{x} = \dot{X} + \dot{r}\sin\phi + r\dot{\phi}\cos\phi \tag{12}$$

$$y = R - r\cos\phi \qquad \dot{y} = -\dot{r}\cos\phi + r\dot{\phi}\sin\phi \tag{13}$$

The Lagrangian is then

$$L = \overbrace{(M + \tfrac{1}{2}m)\dot{X}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + m\dot{X}(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)}^{T} - \overbrace{mg(R - r\cos\phi)}^{U} \tag{14}$$

Note that we are not allowed to substitute $r = R$ and hence $\dot{r} = 0$ in the Lagrangian *prior* to obtaining the equations of motion. Only *after* the generalized momenta and forces are computed are we allowed to do so.

(b) Find *all* the generalized momenta p_σ, the generalized forces F_σ, and the forces of constraint Q_σ.
[10 points]

The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{X}\,\sin\phi \tag{15}$$

$$p_X = \frac{\partial L}{\partial \dot{X}} = (2M+m)\dot{X} + m\dot{r}\,\sin\phi + mr\dot{\phi}\,\cos\phi \tag{16}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + mr\dot{X}\,\cos\phi \tag{17}$$

The generalized forces and the forces of constraint are

$$F_r = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + m\dot{X}\dot{\phi}\,\cos\phi + mg\cos\phi \qquad Q_r = \lambda\,\frac{\partial G}{\partial r} = \lambda \tag{18}$$

$$F_X = \frac{\partial L}{\partial X} = 0 \qquad Q_X = \lambda\,\frac{\partial G}{\partial X} = 0 \tag{19}$$

$$F_\phi = \frac{\partial L}{\partial \phi} = m\dot{X}\dot{r}\,\cos\phi - m\dot{X}\dot{\phi}\,\sin\phi - mgr\sin\phi \qquad Q_\phi = \lambda\,\frac{\partial G}{\partial \phi} = 0\ . \tag{20}$$

The equations of motion are

$$\dot{p}_\sigma = F_\sigma + Q_\sigma\ . \tag{21}$$

At this point, we can legitimately invoke the constraint $r = R$ and set $\dot{r} = 0$ in all the p_σ and F_σ.

(c) Derive expressions for all conserved quantities.
[5 points]

There are two conserved quantities, which each derive from continuous invariances of the Lagrangian *which respect the constraint*. The first is the total momentum p_X:

$$F_X = 0 \quad \Longrightarrow \quad P \equiv p_X = \text{constant}\ . \tag{22}$$

The second conserved quantity is the Hamiltonian, which in this problem turns out to be the total energy $E = T + U$. Incidentally, we can use conservation of P to write the energy in terms of the variable ϕ alone. From

$$\dot{X} = \frac{P}{2M+m} - \frac{mR\,\cos\phi}{2M+m}\,\dot{\phi}\ , \tag{23}$$

we obtain

$$\begin{aligned} E &= \tfrac{1}{2}(2M+m)\dot{X}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 + mR\dot{X}\dot{\phi}\,\cos\phi + mgR(1-\cos\phi) \\ &= \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2}mR^2\left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\dot{\phi}^2 + mgR(1-\cos\phi)\ , \end{aligned} \tag{24}$$

where we've defined the dimensionless ratio $\alpha \equiv m/2M$. It is convenient to define the quantity

$$\Omega^2 \equiv \left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\dot\phi^2 + 2\omega_0^2(1-\cos\phi)\ , \tag{25}$$

with $\omega_0 \equiv \sqrt{g/R}$. Clearly Ω^2 is conserved, as it is linearly related to the energy E:

$$E = \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2} m R^2 \Omega^2\ . \tag{26}$$

`(d)` Derive a differential equation of motion involving the coordinate $\phi(t)$ alone. *I.e.* your equation should not involve r, X, or the Lagrange multiplier λ.
`[5 points]`

From conservation of energy,

$$\frac{d(\Omega^2)}{dt} = 0 \quad\Longrightarrow\quad \left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\ddot\phi + \left(\frac{\alpha\,\sin\phi\,\cos\phi}{1+\alpha}\right)\dot\phi^2 + \omega_0^2\,\sin\phi = 0\ , \tag{27}$$

again with $\alpha = m/2M$. Incidentally, one can use these results in eqns. 25 and 27 to eliminate $\dot\phi$ and $\ddot\phi$ in the expression for the constraint force, $Q_r = \lambda = \dot p_r - F_r$. One finds

$$\begin{aligned}\lambda &= -mR\,\frac{\dot\phi^2 + \omega_0^2\,\cos\phi}{1+\alpha\sin^2\!\phi} \\ &= -\frac{mR\,\omega_0^2}{(1+\alpha\sin^2\!\phi)^2}\left\{(1+\alpha)\left(\frac{\Omega^2}{\omega_0^2} - 4\,\sin^2(\tfrac{1}{2}\phi)\right) + (1+\alpha\sin^2\!\phi)\,\cos\phi\right\}\ .\end{aligned} \tag{28}$$

This last equation can be used to determine the angle of detachment, where λ vanishes and the mass m falls off the inside of the hoop. This is because the hoop can only supply a repulsive normal force to the mass m. This was worked out in detail in my lecture notes on constrained systems.

[3] Two objects of masses m_1 and m_2 move under the influence of a central potential $U = k\,\big|\boldsymbol{r}_1 - \boldsymbol{r}_2\big|^{1/4}$.

`(a)` Sketch the effective potential $U_{\rm eff}(r)$ and the phase curves for the radial motion. Identify for which energies the motion is bounded.
`[5 points]`

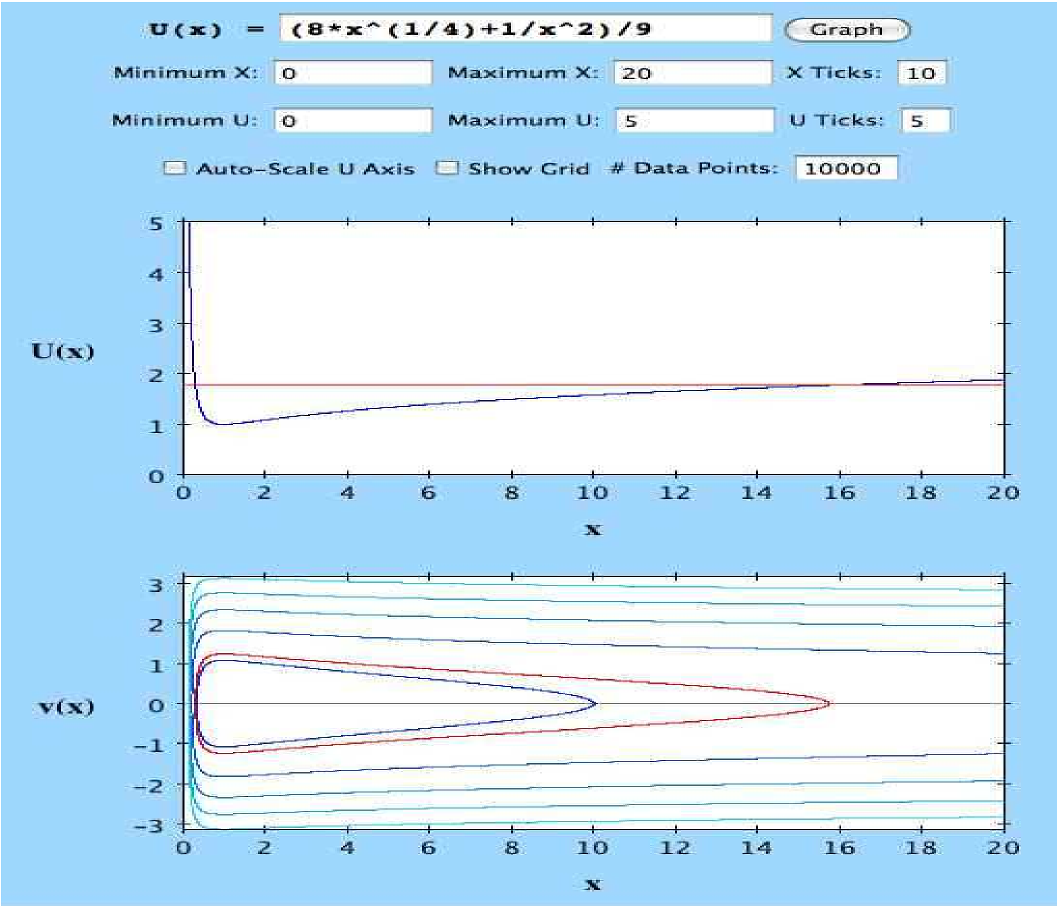

Figure 3: The effective $U_{\rm eff}(r) = E_0\,\mathcal{U}_{\rm eff}(r/r_0)$, where r_0 and E_0 are the radius and energy of the circular orbit.

The effective potential is

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + kr^n \tag{29}$$

with $n = \frac{1}{4}$. In sketching the effective potential, I have rendered it in dimensionless form,

$$U_{\rm eff}(r) = E_0\,\mathcal{U}_{\rm eff}(r/r_0)\ , \tag{30}$$

where $r_0 = (\ell^2/nk\mu)^{(n+2)^{-1}}$ and $E_0 = \big(\frac{1}{2} + \frac{1}{n}\big)\ell^2/\mu r_0^2$, which are obtained from the results of part `(b)`. One then finds

$$\mathcal{U}_{\rm eff}(x) = \frac{n\,x^{-2} + 2\,x^n}{n+2}\ . \tag{31}$$

Although it is not obvious from the detailed sketch in fig. 3, the effective potential does diverge, albeit slowly, for $r \to \infty$. Clearly it also diverges for $r \to 0$. Thus, the relative coordinate motion is bounded for all energies; the allowed energies are $E \geq E_0$.

(b) What is the radius r_0 of the circular orbit? Is it stable or unstable? Why?
[5 points]

For the general power law potential $U(r) = kr^n$, with $nk > 0$ (attractive force), setting $U'_{\text{eff}}(r_0) = 0$ yields

$$-\frac{\ell^2}{\mu r_0^3} + nkr_0^{n-1} = 0 \ . \tag{32}$$

Thus,

$$r_0 = \left(\frac{\ell^2}{nk\mu}\right)^{\frac{1}{n+2}} = \left(\frac{4\ell^2}{k\mu}\right)^{\frac{4}{9}} \ . \tag{33}$$

The orbit $r(t) = r_0$ is stable because the effective potential has a local minimum at $r = r_0$, *i.e.* $U''_{\text{eff}}(r_0) > 0$. This is obvious from inspection of the graph of $U_{\text{eff}}(r)$ but can also be computed explicitly:

$$\begin{aligned} U''_{\text{eff}}(r_0) &= \frac{3\ell^2}{\mu r_0^4} + n(n-1)kr_0^n \\ &= (n+2)\,\frac{\ell^2}{\mu r_0^4} \ . \end{aligned} \tag{34}$$

Thus, provided $n > -2$ we have $U''_{\text{eff}}(r_0) > 0$.

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with $\ell'/\ell = 2$, but $\mu' = \mu$ and $k' = k$. What is the ratio ω'/ω of their frequencies of small radial oscillations?
[5 points]

From the radial coordinate equation $\mu\ddot{r} = -U'_{\text{eff}}(r)$, we expand $r = r_0 + \eta$ and find

$$\mu\ddot{\eta} = -U''_{\text{eff}}(r_0)\,\eta + \mathcal{O}(\eta^2) \ . \tag{35}$$

The radial oscillation frequency is then

$$\omega = (n+2)^{1/2}\,\frac{\ell}{\mu r_0^2} = (n+2)^{1/2}\,n^{\frac{2}{n+2}}\,k^{\frac{2}{n+2}}\,\mu^{-\frac{n}{n+2}}\,\ell^{\frac{n-2}{n+2}} \ . \tag{36}$$

The ℓ dependence is what is key here. Clearly

$$\frac{\omega'}{\omega} = \left(\frac{\ell'}{\ell}\right)^{\frac{n-2}{n+2}} \ . \tag{37}$$

In our case, with $n = \frac{1}{4}$, we have $\omega \propto \ell^{-7/9}$ and thus

$$\frac{\omega'}{\omega} = 2^{-7/9} \ . \tag{38}$$

(d) Find the equation of the shape of the slightly perturbed circular orbit: $r(\phi) = r_0 + \eta(\phi)$. That is, find $\eta(\phi)$. Sketch the shape of the orbit.
[5 points]

We have that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$, with

$$\beta = \frac{\omega}{\dot{\phi}} = \frac{\mu r_0^2}{\ell} \cdot \omega = \sqrt{n+2}\,. \tag{39}$$

With $n = \frac{1}{4}$, we have $\beta = \frac{3}{2}$. Thus, the radial coordinate makes three oscillations for every two rotations. The situation is depicted in fig. 4.

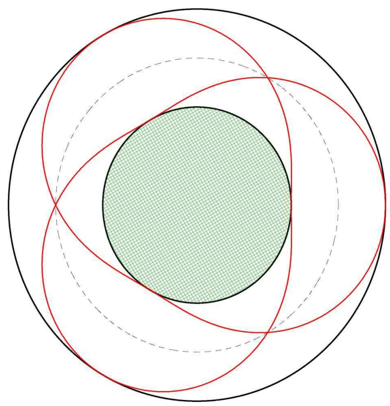

Figure 4: Radial oscillations with $\beta = \frac{3}{2}$.

(e) What value of n would result in a perturbed orbit shaped like that in fig. 5?
[5 points]

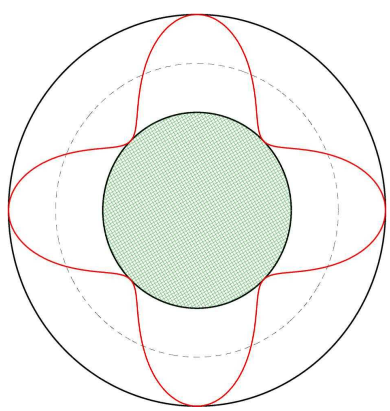

Figure 5: Closed precession in a central potential $U(r) = kr^n$.

Clearly $\beta = \sqrt{n+2} = 4$, in order that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$ executes four complete periods over the interval $\phi \in [0, 2\pi]$. This means $n = 14$.

[4] Two masses and three springs are arranged as shown in fig. 6. You may assume that in equilibrium the springs are all unstretched with length a. The masses and spring constants are simple multiples of fundamental values, *viz.*

$$m_1 = m \quad , \quad m_2 = 4m \quad , \quad k_1 = k \quad , \quad k_2 = 4k \quad , \quad k_3 = 28k \ . \tag{40}$$

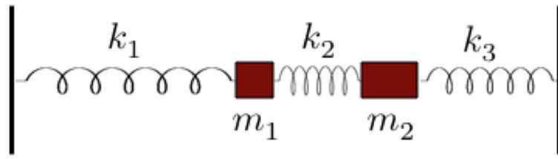

Figure 6: Coupled masses and springs.

`(a)` Find the Lagrangian.
`[5 points]`

Choosing displacements relative to equilibrium as our generalized coordinates, we have

$$T = \tfrac{1}{2}m\,\dot\eta_1^2 + 2m\,\dot\eta_2^2 \tag{41}$$

and

$$U = \tfrac{1}{2}k\,\eta_1^2 + 2k\,(\eta_2 - \eta_1)^2 + 14k\,\eta_2^2 \ . \tag{42}$$

Thus,

$$L = T - U = \tfrac{1}{2}m\,\dot\eta_1^2 + 2m\,\dot\eta_2^2 - \tfrac{1}{2}k\,\eta_1^2 - 2k\,(\eta_2 - \eta_1)^2 - 14k\,\eta_2^2 \ . \tag{43}$$

You are not required to find the equilibrium values of x_1 and x_2. However, suppose all the unstretched spring lengths are a and the total distance between the walls is L. Then, with $x_{1,2}$ being the location of the masses relative to the left wall, we have

$$U = \tfrac{1}{2}k_1\,(x_1 - a)^2 + \tfrac{1}{2}k_2\,(x_2 - x_1 - a)^2 + \tfrac{1}{2}k_3\,(L - x_2 - a)^2 \ . \tag{44}$$

Differentiating with respect to $x_{1,2}$ then yields

$$\frac{\partial U}{\partial x_1} = k_1\,(x_1 - a) - k_2\,(x_2 - x_1 - a) \tag{45}$$

$$\frac{\partial U}{\partial x_2} = k_2\,(x_2 - x_1 - a) - k_3\,(L - x_2 - a) \ . \tag{46}$$

Setting these both to zero, we obtain

$$(k_1 + k_2)\,x_1 - k_2\,x_2 = (k_1 - k_2)\,a \tag{47}$$

$$-k_2\,x_1 + (k_2 + k_3)\,x_2 = (k_2 - k_3)\,a + k_3 L \ . \tag{48}$$

Solving these two inhomogeneous coupled linear equations for $x_{1,2}$ then yields the equilibrium positions. However, we don't need to do this to solve the problem.

(b) Find the T and V matrices.
[5 points]

We have

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot{\eta}_\sigma \partial\dot{\eta}_{\sigma'}} = \begin{pmatrix} m & 0 \\ 0 & 4m \end{pmatrix} \tag{49}$$

and

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\eta_\sigma \partial\eta_{\sigma'}} = \begin{pmatrix} 5k & -4k \\ -4k & 32k \end{pmatrix} . \tag{50}$$

(c) Find the eigenfrequencies ω_1 and ω_2.
[5 points]

We have

$$\begin{aligned} \mathrm{Q}(\omega) \equiv \omega^2\,\mathrm{T} - \mathrm{V} &= \begin{pmatrix} m\omega^2 - 5k & 4k \\ 4k & 4m\omega^2 - 32k \end{pmatrix} \\ &= k \begin{pmatrix} \lambda - 5 & 4 \\ 4 & 4\lambda - 32 \end{pmatrix} , \end{aligned} \tag{51}$$

where $\lambda = \omega^2/\omega_0^2$, with $\omega_0 = \sqrt{k/m}$. Setting $\det \mathrm{Q}(\omega) = 0$ then yields

$$\lambda^2 - 13\lambda + 36 = 0 , \tag{52}$$

the roots of which are $\lambda_- = 4$ and $\lambda_+ = 9$. Thus, the eigenfrequencies are

$$\omega_- = 2\,\omega_0 \qquad , \qquad \omega_+ = 3\,\omega_0 . \tag{53}$$

(d) Find the modal matrix $\mathrm{A}_{\sigma i}$.
[5 points]

To find the normal modes, we set

$$\begin{pmatrix} \lambda_\pm - 5 & 4 \\ 4 & 4\lambda_\pm - 32 \end{pmatrix} \begin{pmatrix} \psi_1^{(\pm)} \\ \psi_2^{(\pm)} \end{pmatrix} = 0 . \tag{54}$$

This yields two linearly dependent equations, from which we can determine only the ratios $\psi_2^{(\pm)}/\psi_1^{(\pm)}$. Plugging in for $\lambda_\pm$, we find

$$\begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = \mathcal{C}_- \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad , \qquad \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{55}$$

We then normalize by demanding $\psi_\sigma^{(i)}\,\mathrm{T}_{\sigma\sigma'}\,\psi_{\sigma'}^{(j)} = \delta_{ij}$. We can practically solve this by inspection:

$$20m\,|\mathcal{C}_-|^2 = 1 \qquad , \qquad 5m\,|\mathcal{C}_+|^2 = 1 . \tag{56}$$

We may now write the modal matrix,

$$\mathrm{A} = \frac{1}{\sqrt{5m}} \begin{pmatrix} 2 & 1 \\ \frac{1}{2} & -1 \end{pmatrix} . \tag{57}$$

(e) Write down the most general solution for the motion of the system.
[5 points]

The most general solution is

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = B_- \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cos(2\omega_0 t + \varphi_-) + B_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(3\omega_0 t + \varphi_+) . \tag{58}$$

Note that there are four constants of integration: $B_\pm$ and $\varphi_\pm$.