PHYSICS 110A : CLASSICAL MECHANICS
MIDTERM EXAM #2

[1] Two blocks connected by a spring of spring constant k are free to slide frictionlessly along a horizontal surface, as shown in Fig. 1. The unstretched length of the spring is a.

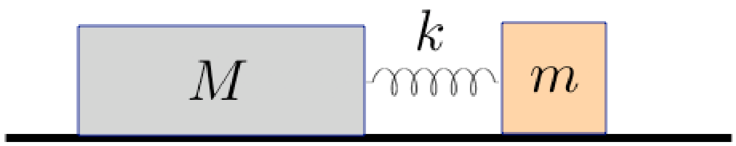

Figure 1: Two masses connected by a spring sliding horizontally along a frictionless surface.

(a) Identify a set of generalized coordinates and write the Lagrangian.
`[15 points]`

Solution : As generalized coordinates I choose X and u, where X is the position of the right edge of the block of mass M, and $X + u + a$ is the position of the left edge of the block of mass m, where a is the unstretched length of the spring. Thus, the extension of the spring is u. The Lagrangian is then

$$\begin{aligned} L &= \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2} m (\dot{X} + \dot{u})^2 - \tfrac{1}{2} k u^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2} m \dot{u}^2 + m \dot{X}\dot{u} - \tfrac{1}{2} k u^2 \ . \end{aligned} \tag{1}$$

(b) Find the equations of motion.
`[15 points]`

Solution : The canonical momenta are

$$p_X \equiv \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{u} \quad , \quad p_u \equiv \frac{\partial L}{\partial \dot{u}} = m(\dot{X} + \dot{u}) \ . \tag{2}$$

The corresponding equations of motion are then

$$\dot{p}_X = F_X = \frac{\partial L}{\partial X} \qquad \Rightarrow \qquad (M+m)\ddot{X} + m\ddot{u} = 0 \tag{3}$$

$$\dot{p}_u = F_u = \frac{\partial L}{\partial u} \qquad \Rightarrow \qquad m(\ddot{X} + \ddot{u}) = -ku \ . \tag{4}$$

(c) Find all conserved quantities.
`[10 points]`

Solution : There are two conserved quantities. One is p_X itself, as is evident from the fact that L is cyclic in X. This is the conserved 'charge' Λ associated with the continuous symmetry $X \to X + \zeta$. *i.e.* $\Lambda = p_X$. The other conserved quantity is the Hamiltonian H, since L is cyclic in t. Furthermore, because the kinetic energy is homogeneous of degree two in the generalized velocities, we have that $H = E$, with

$$E = T + U = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2} m\dot{u}^2 + m\dot{X}\dot{u} + \tfrac{1}{2} k u^2 \ . \tag{5}$$

It is possible to eliminate $\dot{X}$, using the conservation of Λ:

$$\dot{X} = \frac{\Lambda - m\dot{u}}{M+m} \ . \tag{6}$$

This allows us to write

$$E = \frac{\Lambda^2}{2(M+m)} + \frac{Mm\,\dot{u}^2}{2(M+m)} + \tfrac{1}{2}ku^2 \ . \tag{7}$$

(d) Find a complete solution to the equations of motion. As there are two degrees of freedom, your solution should involve 4 constants of integration. You need not match initial conditions, and you need not choose the quantities in part (c) to be among the constants. `[10 points]`

Solution : Using conservation of Λ, we may write $\ddot{X}$ in terms of $\ddot{x}$, in which case

$$\frac{Mm}{M+m}\,\ddot{u} = -ku \quad \Rightarrow \quad u(t) = A\cos(\varOmega t) + B\sin(\varOmega t) \ , \tag{8}$$

where

$$\varOmega = \sqrt{\frac{(M+m)k}{Mm}} \ . \tag{9}$$

For the X motion, we integrate eqn. 6 above, obtaining

$$X(t) = X_0 + \frac{\Lambda t}{M+m} - \frac{m}{M+m}\Big(A\cos(\varOmega t) - A + B\sin(\varOmega t)\Big) \ . \tag{10}$$

There are thus four constants: X_0, Λ, A, and B. Note that conservation of energy says

$$E = \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}k(A^2+B^2) \ . \tag{11}$$

Alternate solution : We could choose X as the position of the left block and x as the position of the right block. In this case,

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}k(x - X - b)^2 \ . \tag{12}$$

Here, b includes the unstretched length a of the spring, but may also include the size of the blocks if, say, X and x are measured relative to the blocks' midpoints. The canonical momenta are

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \quad , \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \ . \tag{13}$$

The equations of motion are then

$$\dot{p}_X = F_X = \frac{\partial L}{\partial X} \qquad \Rightarrow \qquad M\ddot{X} = k(x - X - b) \tag{14}$$

$$\dot{p}_x = F_x = \frac{\partial L}{\partial x} \qquad \Rightarrow \qquad m\ddot{x} = -k(x - X - b) \ . \tag{15}$$

The one-parameter family which leaves L invariant is $X \to X + \zeta$ and $x \to x + \zeta$, *i.e.* simultaneous and identical displacement of both of the generalized coordinates. Then

$$\Lambda = M\dot{X} + m\dot{x} \ , \tag{16}$$

which is simply the x-component of the total momentum. Again, the energy is conserved:

$$E = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}k\,(x - X - b)^2 \ . \tag{17}$$

We can combine the equations of motion to yield

$$Mm\,\frac{d^2}{dt^2}\,\big(x - X - b\big) = -k\,(M + m)\big(x - X - b\big) \ , \tag{18}$$

which yields

$$x(t) - X(t) = b + A\cos(\varOmega t) + B\sin(\varOmega t) \ , \tag{19}$$

From the conservation of Λ, we have

$$M X(t) + m\,x(t) = \Lambda t + C \ , \tag{20}$$

were C is another constant. Thus, we have the motion of the system in terms of four constants: A, B, Λ, and C:

$$X(t) = -\tfrac{m}{M+m}\big(b + A\cos(\varOmega t) + B\sin(\varOmega t)\big) + \frac{\Lambda t + C}{M + m} \tag{21}$$

$$x(t) = \tfrac{M}{M+m}\big(b + A\cos(\varOmega t) + B\sin(\varOmega t)\big) + \frac{\Lambda t + C}{M + m} \ . \tag{22}$$

[2] A uniformly dense ladder of mass m and length 2ℓ leans against a block of mass M, as shown in Fig. 2. Choose as generalized coordinates the horizontal position X of the right end of the block, the angle θ the ladder makes with respect to the floor, and the coordinates (x, y) of the ladder's center-of-mass. These four generalized coordinates are not all independent, but instead are related by a certain set of constraints.

Recall that the kinetic energy of the ladder can be written as a sum $T_{\rm CM} + T_{\rm rot}$, where $T_{\rm CM} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ is the kinetic energy of the center-of-mass motion, and $T_{\rm rot} = \frac{1}{2}I\dot{\theta}^2$, where I is the moment of inertial. For a uniformly dense ladder of length 2ℓ, $I = \frac{1}{3}m\ell^2$.

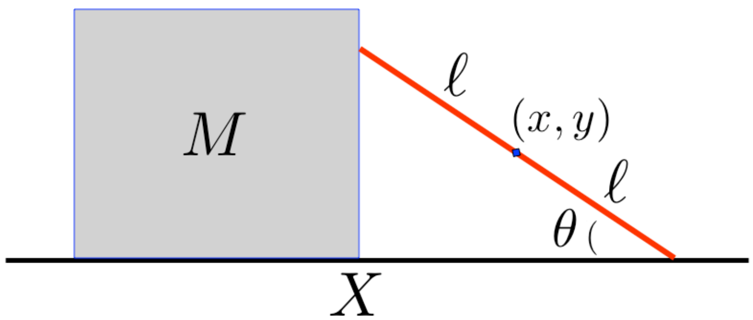

Figure 2: A ladder of length 2ℓ leaning against a massive block. All surfaces are frictionless..

(a) Write down the Lagrangian for this system in terms of the coordinates X, θ, x, y, and their time derivatives.
`[10 points]`

Solution : We have $L = T - U$, hence

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \ . \tag{23}$$

(b) Write down all the equations of constraint.
`[10 points]`

Solution : There are two constraints, corresponding to contact between the ladder and the block, and contact between the ladder and the horizontal surface:

$$G_1(X, \theta, x, y) = x - \ell\cos\theta - X = 0 \tag{24}$$

$$G_2(X, \theta, x, y) = y - \ell\sin\theta = 0 \ . \tag{25}$$

(c) Write down all the equations of motion.
`[10 points]`

Solution : Two Lagrange multipliers, λ_1 and λ_2, are introduced to effect the constraints. We have for each generalized coordinate q_σ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \equiv Q_\sigma \ , \tag{26}$$

where there are $k = 2$ constraints. We therefore have

$$M\ddot{X} = -\lambda_1 \tag{27}$$

$$m\ddot{x} = +\lambda_1 \tag{28}$$

$$m\ddot{y} = -mg + \lambda_2 \tag{29}$$

$$I\ddot{\theta} = \ell \sin\theta \, \lambda_1 - \ell \cos\theta \, \lambda_2 \ . \tag{30}$$

These four equations of motion are supplemented by the two constraint equations, yielding six equations in the six unknowns $\{X, \theta, x, y, \lambda_1, \lambda_2\}$.

(d) Find all conserved quantities.
[10 points]

Solution : The Lagrangian and all the constraints are invariant under the transformation

$$X \to X + \zeta \quad , \quad x \to x + \zeta \quad , \quad y \to y \quad , \quad \theta \to \theta \ . \tag{31}$$

The associated conserved 'charge' is

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = M\dot{X} + m\dot{x} \ . \tag{32}$$

Using the first constraint to eliminate x in terms of X and θ, we may write this as

$$\Lambda = (M + m)\dot{X} - m\ell \sin\theta \, \dot{\theta} \ . \tag{33}$$

The second conserved quantity is the total energy E. This follows because the Lagrangian and all the constraints are independent of t, and because the kinetic energy is homogeneous of degree two in the generalized velocities. Thus,

$$E = \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} I\dot{\theta}^2 + mgy \tag{34}$$

$$= \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}\Big(I + m\ell^2 - \tfrac{m}{M+m}\, m\ell^2 \sin^2\theta\Big)\,\dot{\theta}^2 + mg\ell \sin\theta \ , \tag{35}$$

where the second line is obtained by using the constraint equations to eliminate x and y in terms of X and θ.

(e) What is the condition that the ladder detaches from the block? You do not have to solve for the angle of detachment! Express the detachment condition in terms of any quantities you find convenient.
[10 points]

Solution : The condition for detachment from the block is simply $\lambda_1 = 0$, *i.e.* the normal force vanishes.

Further analysis : It is instructive to work this out in detail (though this level of analysis was not required for the exam). If we eliminate x and y in terms of X and θ, we find

$$x = X + \ell\cos\theta \qquad\qquad y = \ell\sin\theta \tag{36}$$

$$\dot{x} = \dot{X} - \ell\sin\theta\,\dot{\theta} \qquad\qquad \dot{y} = \ell\cos\theta\,\dot{\theta} \tag{37}$$

$$\ddot{x} = \ddot{X} - \ell\sin\theta\,\ddot{\theta} - \ell\cos\theta\,\dot{\theta}^2 \qquad\qquad \ddot{y} = \ell\cos\theta\,\ddot{\theta} - \ell\sin\theta\,\dot{\theta}^2\ . \tag{38}$$

We can now write

$$\lambda_1 = m\ddot{x} = m\ddot{X} - m\ell\sin\theta\,\ddot{\theta} - m\ell\cos\theta\,\dot{\theta}^2 = -M\ddot{X}\ , \tag{39}$$

which gives

$$(M+m)\ddot{X} = m\ell\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big)\ , \tag{40}$$

and hence

$$Q_x = \lambda_1 = -\frac{Mm}{m+m}\,\ell\,\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big)\ . \tag{41}$$

We also have

$$\begin{aligned} Q_y = \lambda_2 &= mg + m\ddot{y} \\ &= mg + m\ell\big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\big)\ . \end{aligned} \tag{42}$$

We now need an equation relating $\ddot{\theta}$ and $\dot{\theta}$. This comes from the last of the equations of motion:

$$\begin{aligned} I\ddot{\theta} &= \ell\sin\theta\,\lambda_1 - \ell\cos\theta\lambda_2 \\ &= -\tfrac{Mm}{M+m}\,\ell^2\big(\sin^2\!\theta\,\ddot{\theta} + \sin\theta\cos\theta\,\dot{\theta}^2\big) - mg\ell\cos\theta - m\ell^2\big(\cos^2\!\theta\,\ddot{\theta} - \sin\theta\cos\theta\,\dot{\theta}^2\big) \\ &= -mg\ell\cos\theta - m\ell^2\Big(1 - \tfrac{m}{M+m}\,\sin^2\!\theta\Big)\,\ddot{\theta} + \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2\ . \end{aligned} \tag{43}$$

Collecting terms proportional to $\ddot{\theta}$, we obtain

$$\big(I + m\ell^2 - \tfrac{m}{M+m}\,\sin^2\!\theta\big)\,\ddot{\theta} = \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 - mg\ell\,\cos\theta\ . \tag{44}$$

We are now ready to demand $Q_x = \lambda_1 = 0$, which entails

$$\ddot{\theta} = -\frac{\cos\theta}{\sin\theta}\,\dot{\theta}^2\ . \tag{45}$$

Substituting this into eqn. 44, we obtain

$$\big(I + m\ell^2\big)\,\dot{\theta}^2 = mg\ell\,\sin\theta\ . \tag{46}$$

Finally, we substitute this into eqn. 35 to obtain an equation for the detachment angle, θ^*

$$E - \frac{\Lambda^2}{2(M+m)} = \left(3 - \frac{m}{M+m}\cdot\frac{m\ell^2}{I+m\ell^2}\,\sin^2\!\theta^*\right)\cdot\tfrac{1}{2}mg\ell\,\sin\theta^*\ . \tag{47}$$

If our initial conditions are that the system starts from rest[1] with an angle of inclination θ_0, then the detachment condition becomes

$$\begin{aligned} \sin\theta_0 &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\left(\tfrac{m}{M+m}\right)\left(\tfrac{m\ell^2}{I+m\ell^2}\right)\sin^3\theta^* \\ &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\,\alpha^{-1}\,\sin^3\theta^* \ , \end{aligned} \tag{48}$$

where

$$\alpha \equiv \left(1+\frac{M}{m}\right)\left(1+\frac{I}{m\ell^2}\right) \ . \tag{49}$$

Note that $\alpha \geq 1$, and that when $M/m = \infty$[2], we recover $\theta^* = \sin^{-1}\left(\frac{2}{3}\sin\theta_0\right)$. For finite α, the ladder detaches at a larger value of θ^*. A sketch of θ^* *versus* θ_0 is provided in Fig. 3. Note that, provided $\alpha \geq 1$, detachment always occurs for some unique value θ^* for each θ_0.

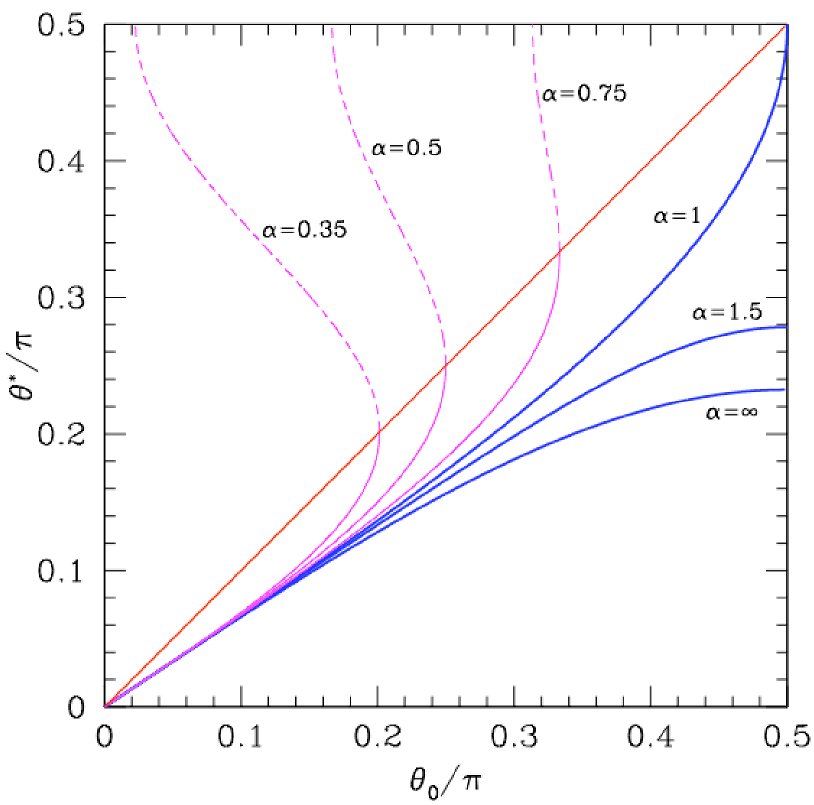

Figure 3: Plot of θ^* *versus* θ_0 for the ladder-block problem (eqn. 48). Allowed solutions, shown in blue, have $\alpha \geq 1$, and thus $\theta^* \leq \theta_0$. Unphysical solutions, with $\alpha < 1$, are shown in magenta. The line $\theta^* = \theta_0$ is shown in red.

[1]'Rest' means that the initial velocities are $\dot{X} = 0$ and $\dot{\theta} = 0$, and hence $\Lambda = 0$ as well.

[2]I must satisfy $I \leq m\ell^2$.