**PHYSICS 110A : CLASSICAL MECHANICS**
**FALL 2010 FINAL EXAMINATION**

**(1)** A point mass $m_1$ slides frictionlessly along a curve $y = f(x)$, as depicted in Fig. 1. Affixed to the mass is a rigid rod of length $\ell$, at the other end of which is a second point mass $m_2$. The entire apparatus moves under the influence of gravity. Choose as generalized coordinates the set $\{x, y, \theta\}$, where $(x, y)$ are the Cartesian coordinates of the mass $m_1$, and $\theta$ is the angle shown in the figure. Treat the condition $y = f(x)$ as a constraint.

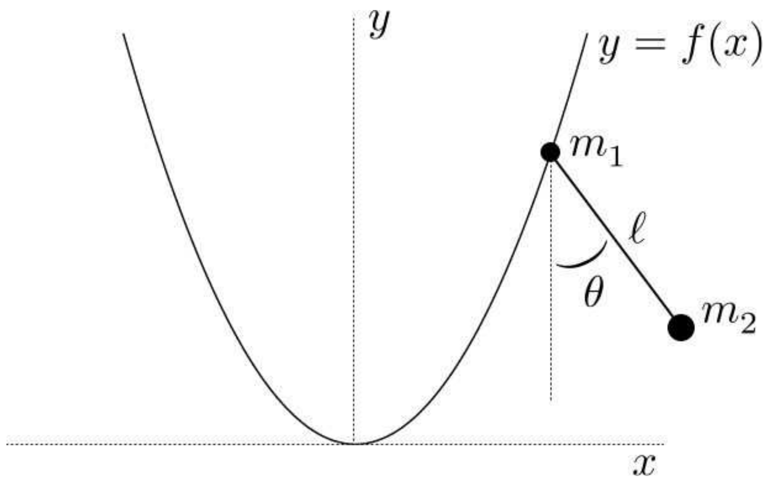


Figure 1: A mass point $m_1$ moves frictionlessly along the curve $y = f(x)$. Affixed to this mass is a rigid rod of length $\ell$ at the end of which is a second point mass $m_2$.

(a) Find the Lagrangian $L(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, t)$. `[5 points]`

**Solution** : The coordinates of the mass $m_2$ are $(x + \ell\sin\theta\,,\ y - \ell\cos\theta)$. The Lagrangian is

$$L = \tfrac{1}{2}m_1\big(\dot{x}^2 + \dot{y}^2\big) + \tfrac{1}{2}m_2\,\big(\dot{x}_2^2 + \dot{y}_2^2\big) - m_1 g y - m_2 g y_2$$

$$= \tfrac{1}{2}\,(m_1 + m_2)\,\big(\dot{x}^2 + \dot{y}^2\big) + \tfrac{1}{2}m_2\,\ell^2\,\dot{\theta}^2 + m_2\ell\,(\dot{x}\cos\theta + \dot{y}\sin\theta)\,\dot{\theta} - (m_1 + m_2)gy + m_2 g\ell\cos\theta\ ,$$

which follows from $x_2 = x + \ell\sin\theta$ and $y_2 = y - \ell\cos\theta$, after taking time derivatives and squaring.

(b) Find the momenta $p_x$, $p_y$, and $p_\theta$. `[6 points]`

**Solution** : We have

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\,\dot{x} + m_2\,\ell\,\dot{\theta}\cos\theta$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = (m_1 + m_2)\,\dot{y} + m_2\,\ell\,\dot{\theta}\sin\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m_2\,\ell^2\,\dot{\theta} + m_2\,\ell\,(\dot{x}\cos\theta + \dot{y}\sin\theta)\ .$$

(c) Find the forces $F_x$, $F_y$, and $F_\theta$. `[3 points]`

**Solution** : We have

$$
\begin{aligned}
F_x &= \frac{\partial L}{\partial x} = 0 \\
F_y &= \frac{\partial L}{\partial y} = -\,(m_1 + m_2)\, g \\
F_\theta &= \frac{\partial L}{\partial \theta} = m_2 \ell\, (-\dot{x} \sin\theta + \dot{y} \cos\theta)\, \dot{\theta} - m_2 g \ell \sin\theta \ .
\end{aligned}
$$

(d) Find the forces of constraint $Q_x$, $Q_y$, and $Q_\theta$. `[3 points]`

**Solution** : We have one constraint, $G(x, y, \theta, t) = y - f(x) = 0$, hence

$$
\begin{aligned}
Q_x &= \lambda\, \frac{\partial G}{\partial x} = -\lambda f'(x) \\
Q_y &= \lambda\, \frac{\partial G}{\partial y} = \lambda \\
Q_\theta &= \lambda\, \frac{\partial G}{\partial \theta} = 0 \ .
\end{aligned}
$$

(e) Find the equations of motion in terms of $x$, $y$, $\theta$, their first and second time derivatives, and the Lagrange multiplier $\lambda$. `[6 points]`

**Solution** : The general form of the equations of motion is

$$
\dot{p}_\sigma = F_\sigma + Q_\sigma \ ,
$$

hence

$$
(m_1 + m_2)\, \ddot{x} - m_2\, \ell\, \dot{\theta}^2 \sin\theta + m_2\, \ell\, \ddot{\theta} \cos\theta = -\lambda f'(x)
$$

$$
(m_1 + m_2)\, \ddot{y} + m_2\, \ell\, \dot{\theta}^2 \cos\theta + m_2\, \ell\, \ddot{\theta} \sin\theta = -\,(m_1 + m_2)\, g + \lambda
$$

$$
m_2\, \ell\, \ddot{\theta} + m_2 \ell\, (\ddot{x} \cos\theta + \ddot{y} \sin\theta) = -m_2 g \ell \sin\theta \ .
$$

These three equations in the four unknowns $(x, y, \theta, \lambda)$ are supplemented by a fourth equation, which is the equation of constraint $y - f(x) = 0$.

(f) What is conserved for this system? `[2 points]`

**Solution** : The only conserved quantity is the Hamiltonian, which is the total energy $H = E = T + U$.

**(2)** Treat the system described in problem (1) without the constraint formalism, using generalized coordinates $x$ and $\theta$. Assume $f(x) = f(-x)$ is a symmetric function with a single minimum at $x = 0$, and that $f''(0) > 0$.

(a) Find the Lagrangian $L(x, \theta, \dot{x}, \dot{\theta}, t)$. `[5 points]`

**Solution** : The coordinates of the mass $m_2$ are $(x + \ell \sin\theta \,,\, y - \ell\cos\theta)$. The Lagrangian is

$$
\begin{aligned}
L &= \tfrac{1}{2} m_1 \left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2\right) - m_1 g y - m_2 g y_2 \\
&= \tfrac{1}{2}\,(m_1 + m_2)\left(1 + f'(x)^2\right)\dot{x}^2 + \tfrac{1}{2} m_2\,\ell^2\,\dot{\theta}^2 + m_2 \ell \left(\cos\theta + f'(x)\sin\theta\right)\dot{x}\,\dot{\theta} - (m_1 + m_2) g f(x) + m_2\, g\ell\cos\theta\ .
\end{aligned}
$$

(b) Find the equilibrium values $(x^*, \theta^*)$ and the T and V matrices. `[5 points]`

**Solution** : Clearly $(x^*, \theta^*) = (0, 0)$. We then have for T,

$$
\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot{q}_\sigma\,\partial\dot{q}_{\sigma'}}\bigg|_{q^*} = \begin{pmatrix} m_1 + m_2 & m_2\,\ell \\ m_2\,\ell & m_2\,\ell^2 \end{pmatrix} .
$$

Note that $f'(0) = 0$ since $f(x) = f(-x)$ is an even function. The V matrix is

$$
\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q^*} = \begin{pmatrix} (m_1 + m_2)\,g\,f''(0) & 0 \\ 0 & m_2\,g\,\ell^2 \end{pmatrix} .
$$

(c) Consider the case $f(x) = x^2/2b$. Define $\Omega_0 = \sqrt{g/b}$ and $\Omega_1 = \sqrt{g/\ell}$. Find a general expression for the normal mode frequencies $\omega_\pm$. Then consider the case where $m_1 = 21m$, $m_2 = 4m$, $\Omega_0 = 3\Omega$, and $\Omega_1 = 5\Omega$. Find $\omega_\pm$. `[10 points]`

**Solution** : We have

$$
\omega^2\mathrm{T} - \mathrm{V} = \begin{pmatrix} (m_1 + m_2)\left(\omega^2 - \Omega_0^2\right) & m_2\,\ell\,\omega^2 \\ m_2\,\ell\,\omega^2 & m_2\,\ell^2\left(\omega^2 - \Omega_1^2\right) \end{pmatrix} .
$$

The characteristic polynomial is

$$
\begin{aligned}
P(\omega^2) = \det\left(\omega^2\mathrm{T} - \mathrm{V}\right) &= m_2\,(m_1 + m_2)\,\ell^2 \left\{ \left(\omega^2 - \Omega_0^2\right)\left(\omega^2 - \Omega_1^2\right) - \frac{m_2}{m_1 + m_2}\,\omega^4 \right\} \\
&= m_2\,(m_1 + m_2)\,\ell^2 \left\{ \frac{m_1}{m_1 + m_2}\,\omega^4 - \left(\Omega_0^2 + \Omega_1^2\right)\omega^2 + \Omega_0^2\,\Omega_1^2 \right\} .
\end{aligned}
$$

Thus, the normal mode frequencies are

$$
P(\omega^2 = 0) \quad\Rightarrow\quad \omega_\pm^2 = \left(1 + \frac{m_2}{m_1}\right)\left(\frac{\Omega_0^2 + \Omega_1^2}{2}\right) \pm \left(1 + \frac{m_2}{m_1}\right)\sqrt{\left(\frac{\Omega_0^2 - \Omega_1^2}{2}\right)^2 + \frac{m_2\,\Omega_0^2\,\Omega_1^2}{m_1 + m_2}}\ .
$$

Using the given values for $m_{1,2}$ and $\Omega_{0,1}$, we find

$$\omega_- = \frac{5\,\Omega}{\sqrt{3}} \quad , \quad \omega_+ = \frac{15\,\Omega}{\sqrt{7}} \ .$$

(d) Find the eigenvectors $\boldsymbol{\psi}^{(\pm)}$. *You do not have to normalize them.* `[5 points]`

**Solution** : From the equation $\big(\omega_\pm^2 \mathrm{T} - \mathrm{V}\big)\,\boldsymbol{\psi}^{(\pm)} = 0$, we take the top component and find

$$(m_1 + m_2)\,\big(\omega_\pm^2 - \Omega_0^2\big)\,\psi_1^{(\pm)} + m_2\,\ell\,\omega_\pm^2\,\psi_2^{(\pm)} = 0 \ ,$$

we have

$$\frac{\ell\,\psi_2^{(\pm)}}{\psi_1^{(\pm)}} = \left(1 + \frac{m_1}{m_2}\right)\left(\frac{\Omega_0^2}{\omega_\pm^2} - 1\right) .$$

From this, find

$$\boldsymbol{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 25\,\ell \\ 2 \end{pmatrix} \qquad , \qquad \boldsymbol{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 25\,\ell \\ -18 \end{pmatrix} \ .$$

**(3)** Two particles of masses $m_1$ and $m_2$ interact via the central potential

$$U(\boldsymbol{r}_1, \boldsymbol{r}_2) = -U_0 \left(\frac{a}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}\right)^{1/2} .$$

(a) Find and sketch the effective potential $U_{\text{eff}}(r)$. Sketch the phase curves in the $(r, \dot{r})$ plane. Identify any separatrices and find their energies. `[5 points]`

**Solution** : The effective potential is

$$U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - U_0 \left(\frac{a}{r}\right)^{1/2} .$$

It is useful to defined the dimensionless length $\rho \equiv r/a$ and the dimensionless effective potential $\widetilde{U}_{\text{eff}}(\rho) \equiv U_{\text{eff}}(a\rho)/U_0$. Then

$$\widetilde{U}_{\text{eff}}(\rho) = \frac{\gamma^2}{2\rho^2} - \frac{1}{\rho^{1/2}} \ ,$$

where $\gamma = \ell/a\sqrt{\mu U_0}$ is a dimensionless quantity proportional to $\ell$. In the sketch in the top panel of fig. 2, we have taken $\gamma = 1$. Defining the dimensionless energy $\widetilde{E} = E/U_0$ and the dimensionless velocity $\tilde{v} = \sqrt{U_0/\mu}$, we have

$$\widetilde{E} = \frac{v^2}{2\tilde{v}^2} + \widetilde{U}_{\text{eff}}(\rho) \ .$$

The phase curves are shown in the lower panel of fig. 2.

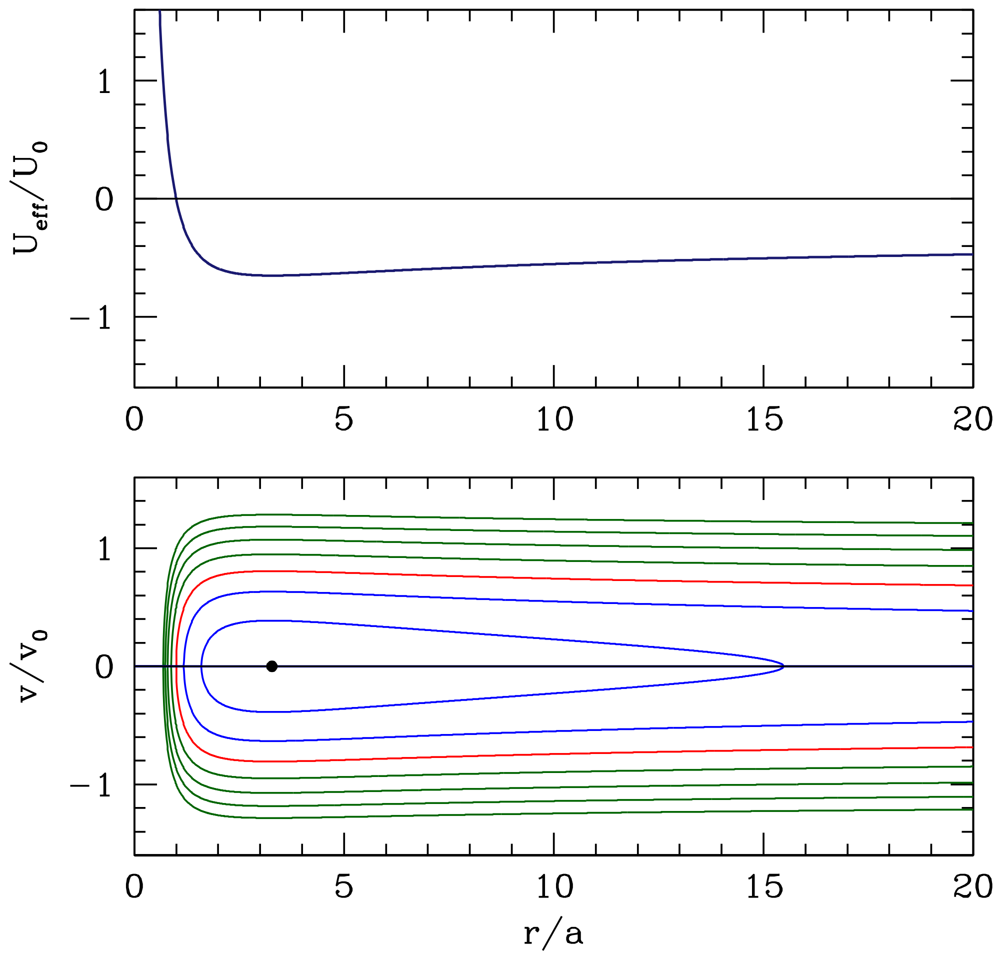


Figure 2: Effective potential $U_{\rm eff}(r)$ and phase curves for the potential $U(r) = -U_0\,(a/r)^{1/2}$. Blue curves, for which $E < 0$, are bounded by two turning points. Green curves, for which $E > 0$, are unbounded. The separatrix, which lies at energy $E = 0$ and which is marginally unbound, is shown in red. The black dot shows the location of the circular orbit.

(b) Find the radius $r_0$ of the circular orbit as a function of the angular momentum $\ell$ and other constants. `[5 points]`

**Solution** : We set $U'_{\rm eff}(r) = 0$ and obtain the equation

$$-\frac{\ell^2}{\mu r_0^3} + \frac{U_0\,a^{1/2}}{2r_0^{3/2}} = 0 \quad \Rightarrow \quad r_0 = \left(\frac{2\ell^2}{\mu U_0 \sqrt{a}}\right)^{2/3} .$$

(c) Writing $r(t) = r_0 + \eta(t)$, find the linearized equations of motion for $\eta(t)$. Find the frequency $\omega$ of the radial oscillations. [5 points]

**Solution** : We have

$$\mu\,\ddot{\eta} = -U''_{\text{eff}}(r_0)\,\eta + \mathcal{O}(\eta^2)\ .$$

Now

$$U''_{\text{eff}}(r_0) = \frac{3\ell^2}{\mu r_0^4} - \frac{3U_0\,a^{1/2}}{4r_0^{3/2}} = \frac{3\ell^2}{2\mu r_0^4} \qquad \Rightarrow \qquad \omega = \sqrt{\frac{3}{2}} \cdot \frac{\ell}{\mu r_0^2}\ .$$

(d) What is the shape of the nearly circular orbits? What is the shape $r(\phi)$ of the nearly circular orbits? Are those orbits closed? Why or why not? [5 points]

**Solution** : We have $\eta'' = -\beta^2\,\eta + \mathcal{O}(\eta^2)$, with

$$\beta = \frac{\omega}{\dot{\phi}} = \frac{\mu\,r_0^2\,\omega}{\ell} = \sqrt{\frac{3}{2}}\ .$$

The shape is

$$r(\phi) = r_0 + \eta_0 \cos(\beta\phi + \delta_0)\ ,$$

where $\eta_0$ and $\delta_0$ are constants determined by initial conditions. Since $\beta$ is not a rational number, the almost circular orbit is not closed.

(e) What is the ratio of the escape velocity at $r_0$ to the orbital velocity at $r_0$? [5 points]

**Solution** : The orbital velocity is given by

$$\ell = \mu\,r_0\,v_{\text{orb}} \quad \Rightarrow \quad v_{\text{orb}} = \frac{\ell}{\mu r_0}\ .$$

The escape velocity is the velocity necessary to achieve a marginally unbound orbit given the relative coordinate is $r$, *i.e.* the velocity such that the (relative coordinate) energy is $E = 0$. Thus,

$$E = 0 = \tfrac{1}{2}\mu v_{\text{esc}}^2 - U_0\bigg(\frac{a}{r_0}\bigg)^{1/2} \quad \Rightarrow \quad v_{\text{esc}} = \bigg(\frac{2U_0}{\mu}\bigg)^{1/2}\bigg(\frac{a}{r_0}\bigg)^{1/4} = \frac{2\ell}{\mu r_0} = 2v_{\text{orb}}\ .$$

**(4)** Provide brief but substantial answers to the following questions.

(a) For a system with kinetic and potential energies

$$T = \tfrac{1}{2}m\big(\dot{x}^2 + \omega^2 x^2\big) \qquad , \qquad U = U(x)\ ,$$

find the Hamiltonian. Under what conditions is $H = T + U$? [5 points]

**Solution** : We have $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and

$$H = p\dot{x} - L = \frac{p^2}{2m} - \tfrac{1}{2}m\omega^2 x^2 + U(x)\ .$$

Note that $H \neq T + U$. In order for $H$ to be equal to $T + U$, the kinetic energy $T$ must be a homogeneous function of degree $k = 2$ in the generalized velocities, and the potential energy $U$ must be a homogeneous function of degree $k = 0$ in the generalized velocities.

(b) For central force motion, what is the definition of a bounded orbit? What is a closed orbit? What are the conditions for a circular orbit, and under what conditions is a circular orbit stable with respect to small perturbations? Under what conditions is an almost circular orbit closed? [5 points]

**Solution** : A bounded orbit is an orbit whose radial motion is bounded by two turning points: $r_{\rm min} \leq r(t) \leq r_{\rm max}$. A closed orbit is one which eventually retraces itself. Circular orbits exist for $r$ values which extremize the effective potential $U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r)$, which is to say $r_0^3 U'(r_0) = \frac{\ell^2}{\mu}$. For almost circular orbits, write $r = r_0 + \eta$, and the linearized equations of motion are $\mu \ddot{\eta} = -U''_{\rm eff}(r_0)\, \eta$. An almost circular orbit is thus stable provided $U''_{\rm eff}(r_0) > 0$. Almost circular orbits which are stable execute radial oscillations with frequency $\omega = \sqrt{U''_{\rm eff}(r_0)/\mu}$. An almost circular orbit is closed when $\beta \equiv \omega/\dot{\phi}$ is a rational number, where $\dot{\phi} = \ell/\mu r_0^2$.

(c) Write down an example of a Lagrangian for a system with two generalized coordinates (and no constraints), and which yields two and only two conserved quantities. [5 points]

**Solution** : Lots of possible examples. Here's one:

$$L = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) - U(y) \ ,$$

where $U(y)$ is an arbitrary non-constant function of its argument. The conserved quantities here are $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $H = E = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}\right)^2 + U(y)$.

(d) Consider the functional

$$F[y(x)] = \int\limits_{-\infty}^{\infty} dx \left[ \tfrac{1}{2} a \left(\frac{d^2 y}{dx^2}\right)^2 + \tfrac{1}{2} b \left(\frac{dy}{dx}\right)^2 + \tfrac{1}{2} c y^2 - j(x)\, y(x) \right]$$

What is the differential equation which extremizes $F[y(x)]$? [5 points]

**Solution** : Setting $\delta F = 0$ we obtain

$$\frac{d^2}{dx^2}\left(\frac{\partial L}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) + \frac{\partial L}{\partial y} = 0 \ ,$$

where $L = \frac{1}{2} a (y'')^2 + \frac{1}{2} b (y')^2 + \frac{1}{2} c y^2 - jy$ is the integrand of $F$. Thus, we obtain

$$a y'''' - b y'' + c y = j(x) \ .$$

(e) Consider an equilateral triangle composed of three point masses connected by three springs which moves in a horizontal plane. How many normal modes of oscillation are there? Some of the normal modes involve no restoring force. Can you identify the type of motion for these three 'zero modes'? `[5 points]`

**Solution** : For each of the three point masses, there are two associated generalized coordinates: $x$ and $y$. Thus, there are six generalized coordinates and therefore six normal modes. Three of these are zero modes, and are associated with uniform translation in the $x$ direction, uniform translation in the $y$ direction, and uniform rotation in the $(x, y)$ plane.