PHYSICS 110A : CLASSICAL MECHANICS
MIDTERM EXAM #1 SOLUTIONS

[1] A particle of mass m moves in the one-dimensional potential

$$U(x) = k\left(x^2 - a^2\right) e^{-x/a} \ . \tag{1}$$

(a) What are the dimensions of the constants k and a?

(b) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.

(c) Sketch a representative set of phase curves. Identify any and all fix points, find their energies, and classify them as either stable or unstable equilibria. Find the energy of each and every separatrix.

(d) Find the frequency of small oscillations about the minimum of $U(x)$.

Solution:

(a) Since $\left[U\right] = \mathrm{ML}^2/\mathrm{T}^2$ (energy), we have $\left[k\right] = \mathrm{M}/\mathrm{T}^2$ and $\left[a\right] = \mathrm{L}$.

(b) Clearly $U(x)$ diverges to $+\infty$ for $x \to -\infty$, and $U(x) \to 0$ for $x \to +\infty$. Setting $U'(x) = 0$, we obtain the equation

$$U'(x) = k\left(2x - \frac{x^2 - a^2}{a}\right) e^{-x/a} = 0 \ . \tag{2}$$

For finite x there are two solutions:

$$x_\pm = \left(1 \pm \sqrt{2}\right) a \ . \tag{3}$$

From the sketch, shown in fig. 1, it is clear that x_- is a global minimum and x_+ is a local maximum.

(c) A set of phase curves is shown in fig. 2. There are two fixed points for finite x, located at $x = x_\pm$. The point x_- is a local minimum for $U(x)$, corresponding to a stable equilibrium. The energy for this fixed point is

$$E_- = U(x_-) = -2\left(\sqrt{2} - 1\right) \exp\left(\sqrt{2} - 1\right) ka^2 \approx -1.254\, ka^2 \ . \tag{4}$$

The point x_+ is a local maximum for $U(x)$, corresponding to the only separatrix, depicted as a red curve in fig. 2. The energy of the separatrix is

$$E_+ = U(x_+) = 2\left(\sqrt{2} + 1\right) \exp\left(-\sqrt{2} - 1\right) ka^2 \approx 0.4318\, ka^2 \ . \tag{5}$$

(d) The frequency of small oscillations about the stable equilibrium $x = x_-$ is

$$\omega = \sqrt{\frac{U''(x_-)}{m}} \ . \tag{6}$$

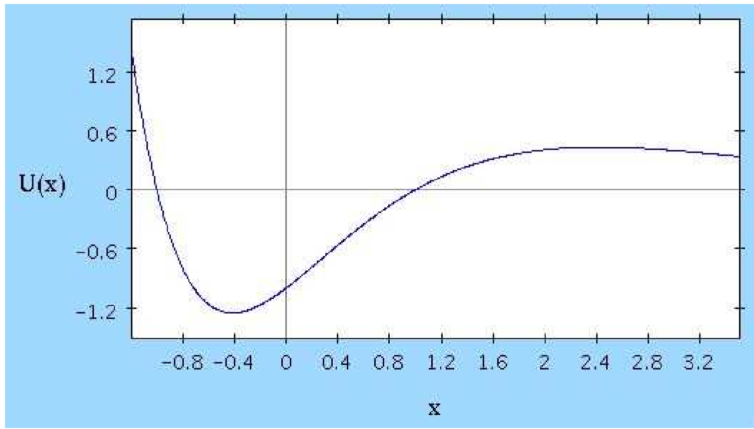

Figure 1: The potential $U(x)$. Distances are here measured in units of a, and the potential in units of ka^2.

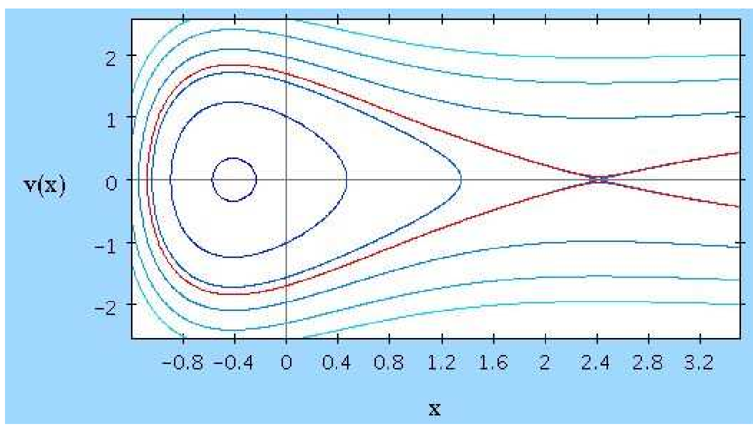

Figure 2: Phase curves for the potential $U(x)$. The separatrix, at energy E_+, is shown in red.

Taking the derivative of $U'(x)$ above, we have

$$\begin{aligned} U''(x) &= k\left(1 - \frac{4x}{a} + \frac{x^2}{a^2}\right) e^{-x/a} \\ &= 2k\left(1 - \frac{x}{a}\right) e^{-x/a} - \frac{U'(x)}{a} \ . \end{aligned} \tag{7}$$

Thus,

$$\omega = 2^{3/4} \exp\left(\frac{\sqrt{2}-1}{2}\right) \sqrt{\frac{k}{m}} \approx 2.069 \sqrt{\frac{k}{m}} \ . \tag{8}$$

[2] Consider the electrical circuit depicted in fig. 3. The inductance is $L = 1\,\mathrm{mH}$ and the capacitances are $C_1 = 100\,\mu\mathrm{F}$ and $C_2 = 150\,\mu\mathrm{F}$. The system is forced by a time-dependent voltage source $V(t) = V_0\,\cos(\Omega t)$, where $V_0 = 8\,\mathrm{mV}$ and $\Omega = 10^3\,\mathrm{s}^{-1}$. The charge Q_1 on the upper plate of capacitor C_1 is found to lead the voltage source $V(t)$ (*i.e.* the difference in potential between the upper and lower termini of the source in the figure) by a phase angle $\delta = \frac{\pi}{4}$. Recall the relevant MKS units:

$$1\,\Omega = 1\,\mathrm{V}\cdot\mathrm{s}\,/\,\mathrm{C} \quad , \quad 1\,\mathrm{F} = 1\,\mathrm{C}\,/\,\mathrm{V} \quad , \quad 1\,\mathrm{H} = 1\,\mathrm{V}\cdot\mathrm{s}^2/\,\mathrm{C}\ .$$

(a) The voltage drops across the two capacitors are the same. Use this fact to express Q_1 in terms of the total charge $Q = Q_1 + Q_2$. Do the same for Q_2.

(b) Write town the equation of motion for $Q(t)$.

(c) What is the value of the resistance R?

(d) Find the current $I(t)$ flowing through the resistor. Your expression should involve no unknown quantities other than the time variable t.

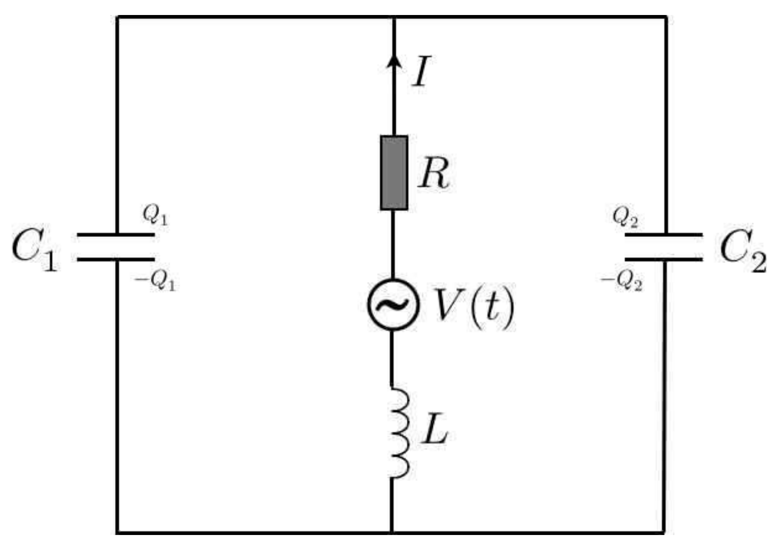

Figure 3: The circuit for problem 2.

Solution:

(a) The voltage drop across the capacitors (from top to bottom) is $Q_1/C_1 = Q_2/C_2$. Thus,

$$Q_1 = \frac{C_1\,Q}{C_1 + C_2} \qquad , \qquad Q_2 = \frac{C_2\,Q}{C_1 + C_2}\ , \tag{9}$$

where $Q = Q_1 + Q_2$.

(b) The voltage drop along the resistor, voltage source, and inductor is (from top to bottom) $RI - V(t) + L\dot{I}$. Equating this with the voltage drop across either capacitor gives the equation

$$L\,\ddot{Q} + R\,\dot{Q} + \frac{Q}{C} = V(t)\ , \tag{10}$$

where $C = C_1 + C_2 = 250\,\mu\text{F}$, which happens to be the effective capacitance for C_1 and C_2 in parallel. Note that $I = \dot{Q}$.

(c) Dividing by L, we have

$$\ddot{Q} + 2\beta\,\dot{Q} + \omega_0^2\,Q = \frac{V(t)}{L} \ , \tag{11}$$

where $\beta = R/2L$ is as yet unknown (since we are not given R), and $\omega_0 = 1/\sqrt{LC} = 2000\,\text{s}^{-1}$. The solution for $Q(t)$ is

$$Q(t) = \frac{V_0}{L} \cdot A\cos(\Omega t - \delta) \ , \tag{12}$$

where

$$A = \frac{1}{\sqrt{(\Omega^2 - \omega_0^2)^2 + 4\beta^2\Omega^2}} \qquad , \qquad \delta = \tan^{-1}\!\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right) . \tag{13}$$

We are told that $\delta = \frac{\pi}{4}$, thus $\tan\delta = 1$, so we must have

$$\beta = \frac{\omega_0^2 - \Omega^2}{2\Omega} = \omega_0 \cdot \frac{1 - (\Omega/\omega_0)^2}{2\Omega/\omega_0} = 1500\,\text{s}^{-1} \ , \tag{14}$$

since $\Omega/\omega_0 = \frac{1}{2}$. We can now solve for R:

$$R = 2\beta L = 3000\,\text{s}^{-1} \cdot 1\,\text{mH} = 3\,\Omega \ . \tag{15}$$

(d) We have

$$I(t) = \dot{Q}(t) = -\frac{\Omega\,V_0}{L} \cdot A\sin(\Omega t - \delta) \ . \tag{16}$$

Note that

$$A = \frac{1}{\omega_0^2} \cdot \frac{1}{\sqrt{\left[1 - \left(\frac{\Omega}{\omega_0}\right)^2\right]^2 + \left[\frac{2\beta}{\omega_0} \cdot \frac{\Omega}{\omega_0}\right]^2}} \tag{17}$$

and $1/L\omega_0^2 = C$. Now $\Omega/\omega_0 = \frac{1}{2}$ and $2\beta/\omega_0 = \frac{3}{2}$, so we find $A = \frac{2}{3}\sqrt{2}$, and

$$I(t) = \tfrac{4}{3}\sqrt{2}\,\sin\left(t\,[\text{ms}] + \tfrac{3\pi}{4}\right)\text{mA} \ , \tag{18}$$

where $t\,[\text{ms}] = \Omega t$ is the time in units of milliseconds.