PHYSICS 110A : CLASSICAL MECHANICS
HW 1 SOLUTIONS

(2) Taylor 1.46

(a) The equations of motion for the puck are:

$$r = R - vt$$

$$\phi = 0$$

Assuming the puck is launched from the position $\phi = 0$. Technically with the polar coordinates this should only be correct until the puck hits the origin, but let's assume at the origin r turns negative and the angle stays the same.

(b) Now the rotating disc only affects the ϕ direction. Our new equations of motion are:

$$r' = R - vt \quad , \quad \phi' = -\omega t$$

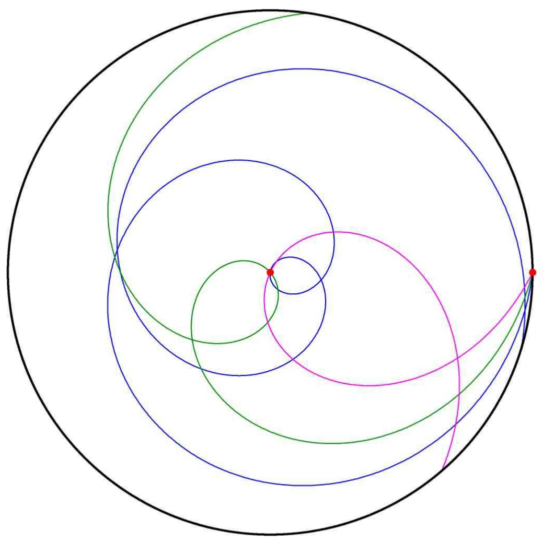

Figure 1: Plots for problem 2b with different $z = \omega R/v$. Blue: z=8, green: z=4, magenta: z=2.

(3) Taylor 2.14

We have from Newton's second law:

$$\frac{dv}{dt} = -\frac{F_0}{m} e^{v/V}$$

Separating this differential equation we come up with:

$$\int_{v_0}^{v} dv' e^{-v'/V} = -\int_{0}^{t} dt' \ \frac{F_0}{m};$$

with the result:

$$V\left[e^{-v/V}\right]_{v_0}^{v} = \frac{F_0\, t}{m} \ .$$

Which reduces to:

$$\left[e^{-v/V} - e^{-v_0/V}\right] = \frac{F_0\, t}{mV} \ .$$

And:

$$e^{-v/V} = \frac{F_0\, t}{mV} + e^{-v_0/V}.$$

And finally:

$$v = -V \ln\left[\frac{F_0\, t}{mV} + e^{-v_0/V}\right]. \qquad (1)$$

(b) By setting equation (1) equal to zero we get,

$$1 = \frac{F_0\, t}{mV} + e^{-v_0/V}.$$

Since $\ln(1) = 0$. Solving this for t we have:

$$t = \frac{mV}{F_0}\left[1 - e^{-v_0/V}\right].$$

(c) Finally from equation (1):

$$\frac{dx}{dt} = -V \ln\left[\frac{F_0\, t}{mV} + e^{-v_0/V}\right].$$

Separating this we have:

$$\int_{x_0}^{x} dx' = -V\int_{0}^{t} dt' \ln\left[\frac{F_0\, t'}{mV} + e^{-v_0/V}\right];$$

and we end up with:

$$x\,(t) = Vt - \frac{V^2 m}{F_0}\left[\left(\frac{F_0 t}{mV} + e^{-v_0/V}\right)\ln\left(\frac{F_0 t}{mV} + e^{-v_0/V}\right) + \frac{v_0}{V}e^{-v_0/V}\right].$$

Plugging in for t we have x_{max}:

$$x_{\max} = \frac{mV^2}{F_0}\left(1 - (1 + \frac{v_0}{V})e^{-v_0/V}\right).$$

(5) **Taylor 3.11**

(a) Similar to text we can show:

$$m\dot{v} = -\dot{m}v_{\text{ex}} - mg.$$

(b) Noting that $\dot{m} = -k$ we have:

$$\dot{v} = \frac{kv_{\text{ex}}}{m_0 - kt} - g.$$

Note: $m_0 - kt$ is the solution of $\dot{m} = -k$.

The equation for $\dot{v}$ is separable and we get:

$$v = \int_0^t dt' \left(\frac{kv_{\text{ex}}}{m_0 - kt'} \right) - gt.$$

This leads to:

$$v = v_{\text{ex}} \ln \left(\frac{m_0}{m_0 - kt} \right) - gt.$$

(c) plugging in numbers we get: ~ 900 m/s.

(d) Depending on the circumstances, as noted in the discussion section, the rocket will either not take off at all, or will have initial negative velocity (e.g. will begin falling downward, or stay on the ground for a rocket at lift-off) and eventually overcome the gravitational force with its thrust.

(6) **Taylor 3.13**

Starting from 3.11 result:

$$v = v_{\text{ex}} \ln\left(\frac{m_o}{m_0 - kt} \right) - gt.$$

This is again separable to find $x(t)$:

$$x(t) - x_0 = v_{\text{ex}} \int_0^t dt' \ln\left(\frac{m_0}{m_0 - kt'} \right) - \frac{gt^2}{2}.$$

We can make this easier by expanding the natural log:

$$x(t) - x_0 = v_{\text{ex}} \int_0^t dt' \Big(\ln(m_0) - \ln\left(m_0 - kt'\right) \Big) - \frac{gt^2}{2}.$$

Then

$$x\,(t) = v_{\rm ex}\left(\ln(m_0)\,t - \frac{1}{k}\Big[-(m_0-kt)\ln(m_0-kt) + (m_0-kt) + m_0\ln(m_0) - m_0\Big]\right) - \frac{gt^2}{2}.$$

Cleaning this up we get:

$$x\,(t) = \frac{v_{\rm ex}}{k}\Big((m_0-kt)\ln(m_0) - (m_0-kt)\ln(m_0-kt) + kt\Big) - \frac{gt^2}{2}.$$

Which can be written as:

$$x\,(t) = v_{\rm ex}\,t - \frac{1}{2}gt^2 - \frac{m(t)\,v_{\rm ex}}{k}\,\ln\!\left(\frac{m_0}{m(t)}\right).$$

Remember that $m(t) = m_0 - kt$.

(7) Taylor 4.2

(a)

$$W = \int_0^1 F_x(x,0)\,dx + \int_0^1 F_y(1,y)\,dy = \int_0^1 x^2 dx + \int_0^1 2y\,dy = \tfrac{4}{3}.$$

(b)

$$W = \int_0^P \Big[F_x(x,y)\,dx + F_y(x,y)\,dy\Big] = \int_0^1 dx\left[F_x(x,x^2) + F_y(x,x^2)\,\frac{dy}{dx}\right]$$

Thus,

$$W = \int_0^1 dx\Big[x^2 + \big(2x\cdot x^2\big)\cdot 2x\Big] = \int_0^1 dx\,\left[x^2 + 4x^4\right] = \tfrac{17}{15}\ .$$

(c)

$$W = \int_0^1 dt\,\left[F_x(x,y)\,\frac{dx}{dt} + F_y(x,y)\,\frac{dy}{dt}\right] = \int_0^1 dt\,\Big[\big(t^3\big)^2\cdot 3t^2 + 2t^3\cdot t^2\cdot 2t\Big] = \int_0^1 dt\,\left[3t^8 + 4t^6\right] = \tfrac{19}{21}$$

(8) Taylor 4.23

To check if these forces are conservative they must satisfy two conditions. (1) They must only depend on position and (2) $\nabla\times\mathbf{F} = 0$. The first is satisfied in each case here and the second is satisfied for (a) and (b). Part (c) has $\nabla\times\mathbf{F} = 2\hat{\mathbf{z}} \neq 0$. A great way to find the potential energy is as follows:

$$U\,(x,y,z) - U\,(0,0,0) = -\int_0^x F_x(x',0,0)\,dx' - \int_0^y F_y(x,y',0)\,dy' - \int_0^z F_z(x,y,z')\,dz'\ .$$

For part (a) this would look like:

$$U(x,y,z) - U(0,0,0) = -k\int_0^x x'\,dx' - 2k\int_0^y y'\,dy' - 3k\int_0^z z'\,dz'.$$

Which leads us to:

$$U(x,y,z) = U_0 - \tfrac{1}{2}kx^2 - ky^2 - \tfrac{3}{2}kz^2.$$

Which when taking the gradient brings us back to our given force. For part (b) we get $U(x,y,z) = U_0 - kxy$ through similar means.

(9) Taylor 4.36

(a) The potential may be written:

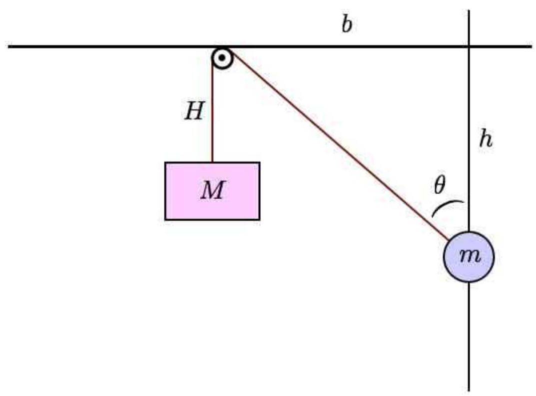

Figure 2: Figure for problem 9.

$$U = -MgH - mgh.$$

Substituting $h = b/\tan\theta$ and $H = (l - b/\sin\theta)$ we have:

$$U = -Mg\left(l - \frac{b}{\sin\theta}\right) - mg\frac{b}{\tan\theta},$$

where l is the length of the string. This can be rearranged as:

$$U = -Mgl + gb\left(\frac{M - m\cos\theta}{\sin\theta}\right) \equiv -Mgl + Mgb\,u(\theta)$$

where

$$u(\theta) = \frac{1 - (m/M)\cos\theta}{\sin\theta}$$

Taking the derivative with respect to θ we have:

$$\frac{\partial U}{\partial\theta} = \frac{gb}{\sin^2\theta}\left(M\cos\theta - m\right).$$

Setting this equal to zero leads to $\theta = \cos^{-1}(m/M)$. Obviously in order to have a solution we must have $m/M < 1$. However, there is another constraint we must acknowledge. From the figure above it can be seen there is a minimum value for θ: $\theta_{\min} = \sin^{-1}(b/\ell)$. So we see we have three conditions (see also fig. 3):

(1) If $\frac{m}{M} < \left(1 - \frac{b^2}{l^2}\right)^{1/2}$, then there is an equilibrium at angle $\theta^* = \cos^{-1}(m/M)$.

(2) If $\left(1 - \frac{b^2}{l^2}\right)^{1/2} < \frac{m}{M} < 1$, then the angle θ^* for which $u(\theta)$ is a minimum lies in the forbidden region $\theta^* < \theta_{\min}$. The lowest energy configuration then occurs at the endpoint of the interval $\theta \in \left[\theta_{\min}, \frac{\pi}{2}\right]$, $i.e. \theta = \theta_{\min} = \sin^{-1}(b/l)$.

(3) If $m > M$ there is no solution to $u'(\theta) = 0$. The equilibrium configuration is, as in case (2), at $\theta = \theta_{\min} = \sin^{-1}(b/l)$.

Note that this problem can be solved with Physics 1 logic. The tension in the string must support both masses. For the mass m, the vertical component of the tension balances gravity, hence $T\cos\theta = mg$. For the mass M, the tension is directed vertically and $T = Mg$. Eliminating T, we have $\theta = \theta^* = \cos^{-1}(m/M)$. If $\theta^* < \theta_{\min}$, then the mass M is pulled all the way up to the top. The tension in the rope still exceeds the gravitational force on the mass M. But now since the mass abuts the pulley assembly, normal forces from the upper surface are in play, and augment gravity (pushing downward) to balance the tension in the rope.

(10) Taylor 4.41

Starting with the potential $U = kr^n$ we can take the gradient to find the force:

$$\mathbf{F} = -\vec{\nabla} U = -nkr^{n-1}\hat{r}.$$

Now equating the magnitude of this force to mass times the centripetal acceleration (for circular motion) we have:

$$\frac{mv^2}{r} = nkr^{n-1}.$$

Manipulating this we find:

$$\frac{1}{2}mv^2 = \frac{1}{2}nkr^n = \frac{nU}{2}$$

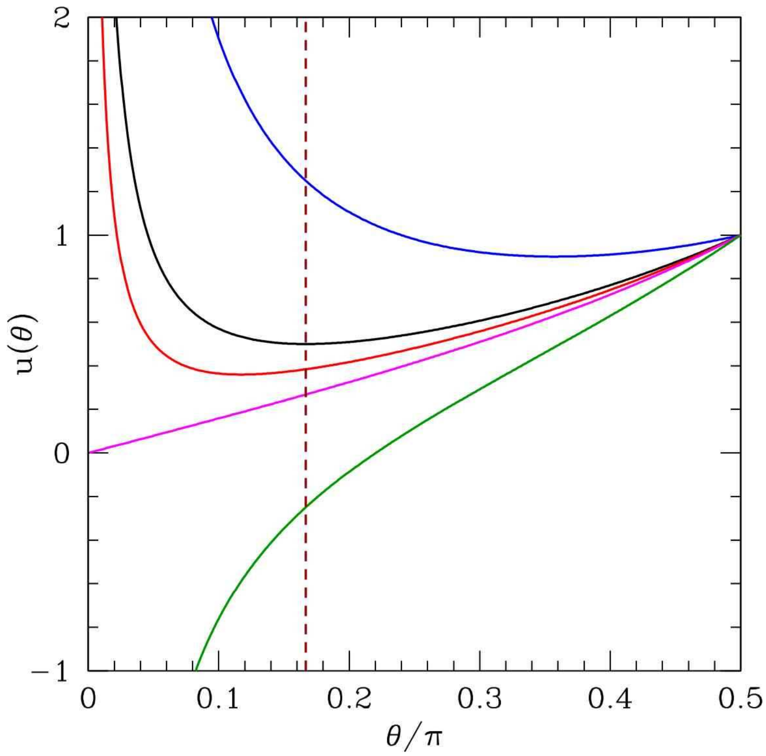

Figure 3: Plot of $u(\theta)$ (Taylor 4.36) for different values of m/M with $b/l = \frac{1}{2}$. Green: $m/M > 1$; pink: $m/M = 1$; red: $\cos\theta_{\rm min} < m/M < 1$; black: $m/M = \cos\theta_{\rm min}$; blue: $m/M < \cos\theta_{\rm min}$. The dashed line shows the location of $\theta_{\rm min} = \sin^{-1}(\frac{b}{l}) = \frac{\pi}{6}$.