PHYSICS 110A : CLASSICAL MECHANICS
HW 2 SOLUTIONS

(1) Taylor 5.2

Here is a sketch of the potential with $A = 1$, $R = 1$, and $S = 1$. From the plot we can see

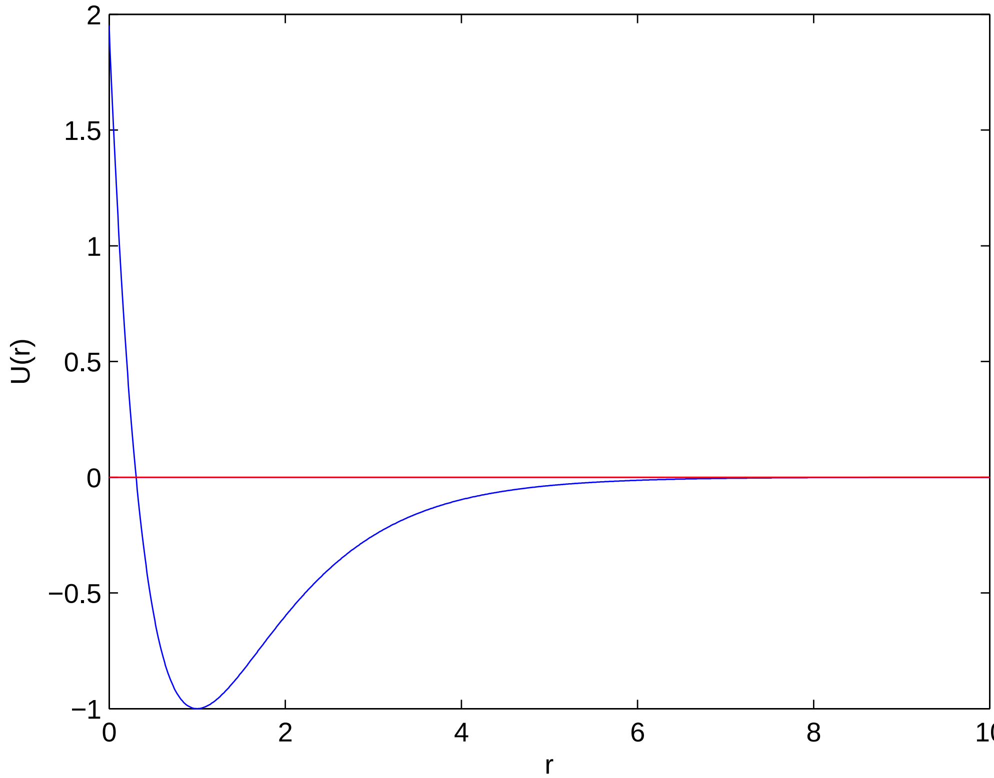

Figure 1: Plot for problem 1.

the minimum of the potential will be at $r = R$. We can also find this by setting the first derivative of $U(r)$ equal to zero.

We have:

$$U'(r) = -\frac{2A}{S} e^{(R-r)/S} \left(e^{(R-r)/S} - 1 \right) = 0.$$

This will be zero when $r = R$. We will call this value r_0.

So now we expand $U(r)$ as a Taylor series around the point r_0:

$$U(r_0 + x) = U(r_0) + U'(r_0)\, x + \frac{1}{2!} U''(r_0)\, x^2 + ...$$

Right away we know the second term will be zero because $U'(r_0)$ is defined to be zero.

Finding the second derivative we have:

$$U''(r) = \frac{2A}{S^2} e^{(R-r)/S} \left(2e^{(R-r)/S} - 1 \right).$$

Plugging in $r_0 = R$ we have:

$$U''(r_0) = \frac{2A}{S^2}.$$

So for small values of x we can say:

$$U(r_0 + x) = -A + \frac{1}{2} \frac{2A}{S^2} x^2 + ...$$

For this potential the k constant is $\frac{2A}{S^2}$.

(2) Taylor 5.13

Similar to problem 1 we have a potential and want to first take the derivative and set it equal to zero to find the potential's minimum:

$$U'(r) = U_0 \left(\frac{1}{R} - \lambda^2 \frac{R}{r^2} \right) = 0.$$

Setting this equal to zero we find the minimum is $r_0 = \lambda R$.

Again we want to express the potential as a Taylor series:

$$U(r_0 + x) = U(r_0) + U'(r_0)\, x + \frac{1}{2!} U''(r_0)\, x^2 + ...$$

Our second derivative of the potential is as follows:

$$U''(r) = \frac{2U_0 \lambda^2 R}{r^3}.$$

And we can write the potential as:

$$U(r_0 + x) = 2U_0 \lambda + \frac{1}{2} \frac{2U_0}{\lambda R^2} x^2 + ...$$

Our expression for the angular frequency is:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2U_0}{m \lambda R^2}}.$$

(3) Taylor 5.38

As in example 5.3 the equation of motion for a driven damped linear oscillator is:

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right]. \tag{1}$$

For us $\omega = 1$, $\beta = .1$, and $\omega_1 = \sqrt{\omega^2 - \beta^2} = .995$.

From equations 5.64 and 5.65 from the text we have:

$A = 2$ and $\delta = \pi/2$.

We have two boundary conditions: $x_0 = 0$ and $v_0 = 6$.

Our job is to calculate the given constants in the equation of motion and then plot the equation of motion.

From equation (1) with $t = 0$ we find:

$$0 = A\cos(-\pi/2) + \left[B_1\cos(0) + B_2\sin(0)\right].$$

Or:

$$B_1 = 0.$$

The velocity function can be found be taking the time derivative of the position function as so:

$$v(t) = -\omega A\sin(\omega t - \delta) + e^{-\beta t}B_2\left(\omega_1\cos(\omega_1 t) - \beta\sin(\omega_1 t)\right).$$

(Where I have dropped the B_1 terms.)

From this at $t = 0$ we have:

$$v_0 = -\omega A\sin(-\pi/2) + B_2\left(\omega_1\cos(0) - \beta\sin(0)\right).$$

And this can be reduced to:

$$B_2 = \frac{v_0 - \omega A}{\omega_1}.$$

Plugging in numbers we get $B_2 \approx 4$ leading to an equation of motion:

$$x(t) = 2\cos(t - \pi/2) + 4e^{-0.1t}\sin(.995t). \tag{2}$$

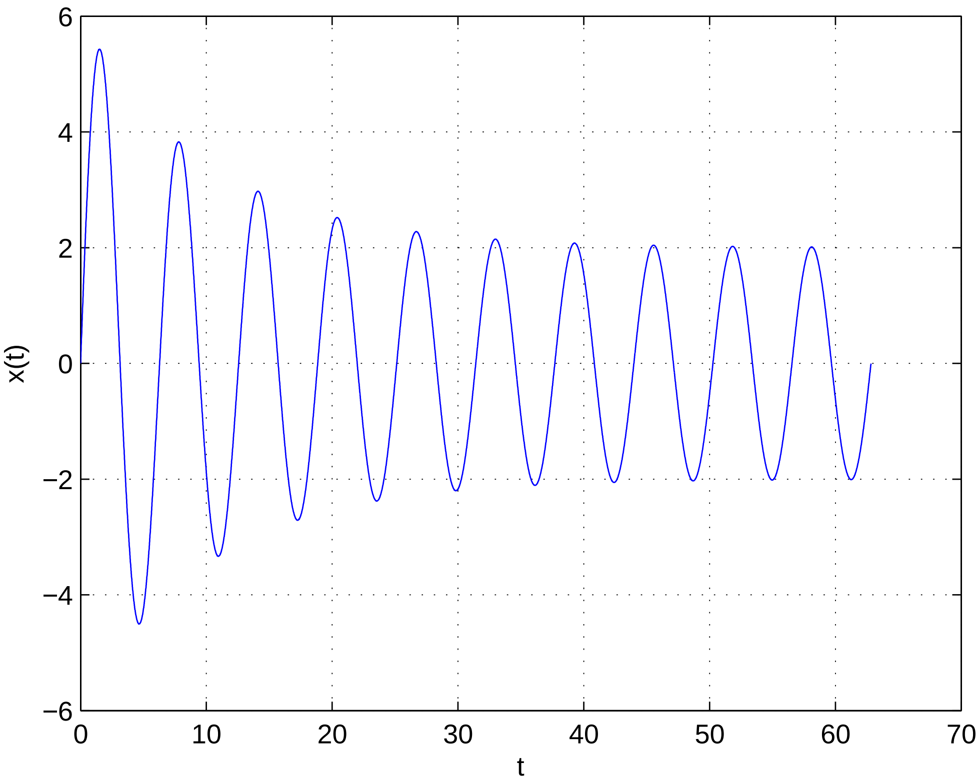

Figure 2: Plot for problem 4.

The function is plotted in figure 2.

(4) Taylor 5.45

(a) First we are to find the time average of the rate $P(t)$ or:

$$\langle P(t) \rangle = \frac{1}{\tau} \int_0^\tau dt P(t).$$

The rate for which a force does work is Fv. So for this force we will have:

$$\langle P(t) \rangle = -\frac{F_0 \omega A}{\tau} \int_0^\tau dt \cos(\omega t) \, sin(\omega t - \delta).$$

Where $v(t) = -\omega A \sin(\omega t - \delta)$. (Note: we get rid of the transient part of the velocity.)

Substituting here we have:

$$\langle P(t)\rangle = -\frac{F_0 \omega A}{2\tau}\int_0^{\tau} dt\left(sin\left(-\delta\right) + sin\left(2\omega t - \delta\right)\right).$$

The time average of a sinusoidal function is zero so we are left with:

$$\langle P(t)\rangle = \frac{F_0 \omega A}{2} sin\left(\delta\right).$$

Note: here we take advantage of $sin\left(-\delta\right) = -sin\left(\delta\right)$.

Now we must substitute for $sin\left(\delta\right)$. From figure 5.14 on page 184 in the text we see:

$$sin\left(\delta\right) = \frac{2\beta\omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}}$$

And comparing with equation 5.64 from the text we see:

$$sin\left(\delta\right) = \frac{2mA\beta\omega}{F_0}$$

Remember $f_0 = F_0/m$. Putting this together we find:

$$\langle P(t)\rangle = m\beta\omega^2 A^2.$$

(b) Now similarly we will do the same for the resistive force which is $F_{\text{res}} = 2m\beta v$.

We can find this as the second term (the friction term) in equation 5.24 from the text, and from equation 5.26 $b = 2\beta m$.

So we will have $P(t) = 2m\beta v^2$ and we have:

$$\langle P(t)\rangle = \frac{2m\beta\omega^2 A^2}{\tau}\int_0^{\tau} dt \sin^2\left(\omega t - \delta\right).$$

But:

$$\frac{1}{\tau}\int_0^{\tau} dt \sin^2\left(\omega t - \delta\right) = \frac{1}{2}.$$

This is true for the square of any sinusoidal function (you may want to check this by substituting for the $\cos^2\left(\omega t - \delta\right)$ as in part (a)).

So we are left with:

$$\langle P(t)\rangle = m\beta\omega^2 A^2.$$

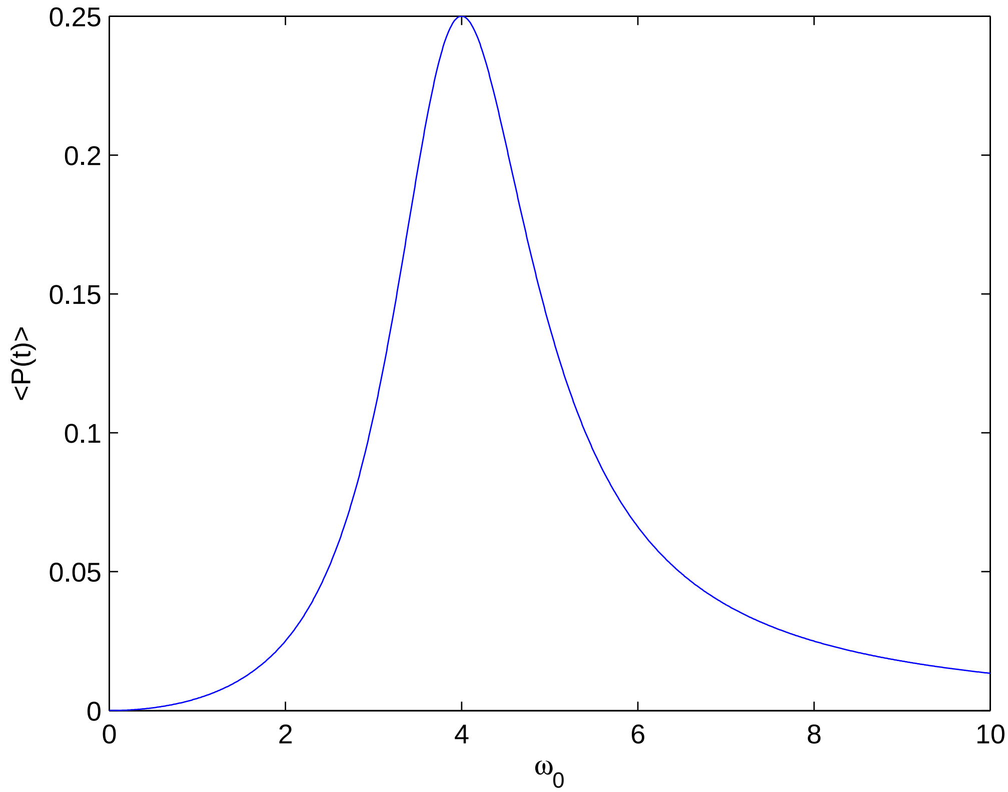

Figure 3: Plot for problem 6.

(c) Writing this out as a function of ω we have:

$$\langle P(t)\rangle = \frac{m\beta\omega^2 f_0^2}{\left(\omega_0^2 - \omega^2\right)^2 + 4\beta^2\omega^2}.$$

Plotting this for $\beta = 1$,$m = 1$, and $\omega_0 = 4$ we see the maximum is at ω_0 as expected. You may also find this by maximizing the function.

(5) Fall 2007 Midterm #1 Question #1

A particle of mass m moves in the one-dimensional potential

$$U(x) = \frac{U_0}{a^4}\left(x^2 - a^2\right)^2 \ . \tag{3}$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.
`[15 points]`

Solution : Clearly the minima lie at $x = \pm a$ and there is a local maximum at $x = 0$.

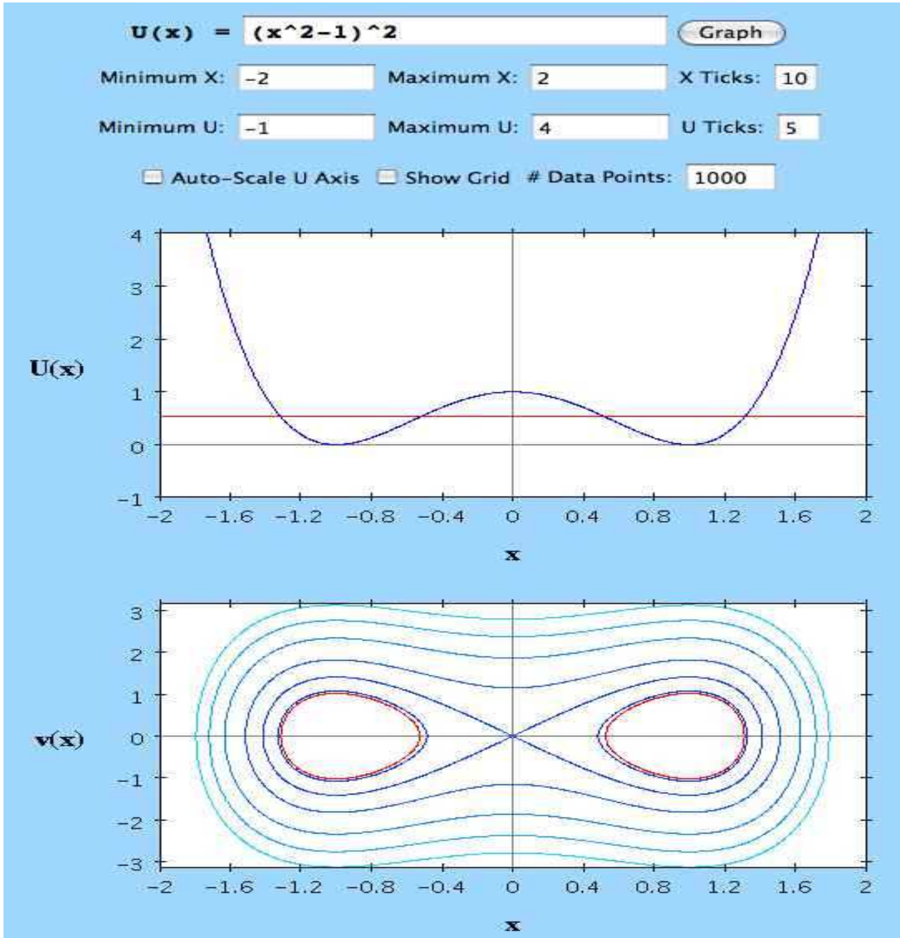

Figure 4: Sketch of the double well potential $U(x) = (U_0/a^4)(x^2 - a^2)^2$, here with distances in units of a, and associated phase curves. The separatrix is the phase curve which runs through the origin. Shown in red is the phase curve for $U = \frac{1}{2}\,U_0$, consisting of two deformed ellipses. For $U = 2\,U_0$, the phase curve is connected, lying outside the separatrix.

(b) Sketch a representative set of phase curves. Be sure to sketch any separatrices which exist, and identify their energies. Also sketch all the phase curves for motions with total energy $E = \frac{1}{2}\,U_0$. Do the same for $E = 2\,U_0$.
`[15 points]`

Solution : See Fig. 4 for the phase curves. Clearly $U(\pm a) = 0$ is the minimum of the potential, and $U(0) = U_0$ is the local maximum and the energy of the separatrix. Thus, $E = \frac{1}{2}\,U_0$ cuts through the potential in both wells, and the phase curves at this energy form two disjoint sets. For $E < U_0$ there are four turning points, at

$$x_{1,<} = -a\sqrt{1 + \sqrt{\frac{E}{U_0}}} \quad , \quad x_{1,>} = -a\sqrt{1 - \sqrt{\frac{E}{U_0}}}$$

and

$$x_{2,<} = a\sqrt{1 - \sqrt{\frac{E}{U_0}}} \quad , \quad x_{2,>} = a\sqrt{1 + \sqrt{\frac{E}{U_0}}}$$

For $E = 2\,U_0$, the energy is above that of the separatrix, and there are only two turning points, $x_{1,<}$ and $x_{2,>}$. The phase curve is then connected.

(c) The phase space dynamics are written as $\dot{\varphi} = \mathbf{V}(\varphi)$, where $\varphi = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. Find the upper and lower components of the vector field $\mathbf{V}$.
[10 points]

Solution :

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{1}{m}\,U'(x) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{4U_0}{a^2}\,x\,(x^2 - a^2) \end{pmatrix} \ . \tag{4}$$

(d) Derive and expression for the period T of the motion when the system exhibits small oscillations about a potential minimum.
[10 points]

Solution : Set $x = \pm a + \eta$ and Taylor expand:

$$U(\pm a + \eta) = \frac{4U_0}{a^2}\,\eta^2 + \mathcal{O}(\eta^3) \ . \tag{5}$$

Equating this with $\frac{1}{2}k\,\eta^2$, we have the effective spring constant $k = 8U_0/a^2$, and the small oscillation frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\,U_0}{ma^2}} \ . \tag{6}$$

The period is $2\pi/\omega_0$.

(6) Fall 2007 Midterm #1 Question #2
An R-L-C circuit is shown in fig. 5. The resistive element is a light bulb. The inductance is $L = 400\,\mu$H; the capacitance is $C = 1\,\mu$F; the resistance is $R = 32\,\Omega$. The voltage $V(t)$ oscillates sinusoidally, with $V(t) = V_0 \cos(\omega t)$, where $V_0 = 4$ V. In this problem, you may neglect all transients; we are interested in the late time, steady state operation of this circuit. Recall the relevant MKS units:

$$1\,\Omega = 1\,\mathrm{V\cdot s\,/\,C} \quad , \quad 1\,\mathrm{F} = 1\,\mathrm{C\,/\,V} \quad , \quad 1\,\mathrm{H} = 1\,\mathrm{V\cdot s^2/\,C} \ .$$

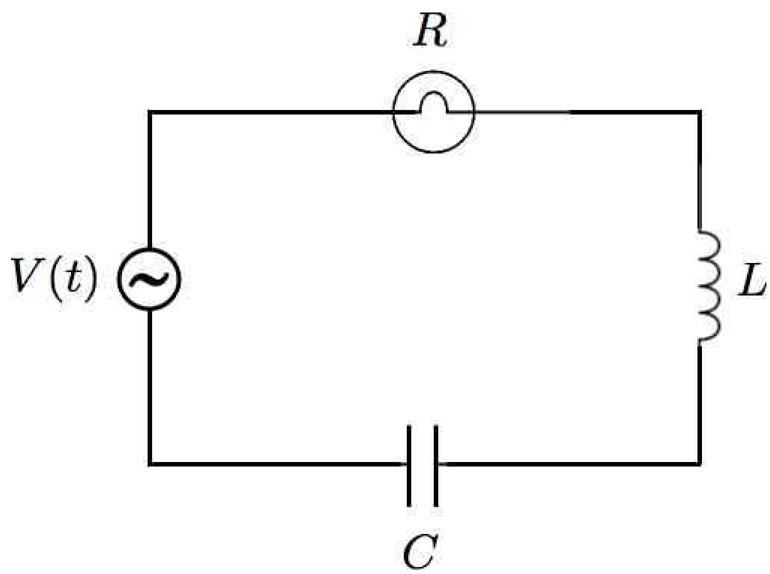

Figure 5: An R-L-C circuit in which the resistive element is a light bulb.

(a) Is this circuit underdamped or overdamped?
[10 points]

Solution : We have

$$\omega_0 = (LC)^{-1/2} = 5 \times 10^4\,\mathrm{s}^{-1} \quad , \quad \beta = \frac{R}{2L} = 4 \times 10^4\,\mathrm{s}^{-1} \ .$$

Thus, $\omega_0^2 > \beta^2$ and the circuit is *underdamped*.

(b) Suppose the bulb will only emit light when the average power dissipated by the bulb is greater than a threshold $P_{\rm th} = \frac{2}{9}\,W$. For fixed $V_0 = 4$ V, find the frequency range for ω over which the bulb emits light. Recall that the instantaneous power dissipated by a resistor is $P_R(t) = I^2(t)R$. (Average this over a cycle to get the average power dissipated.)
[20 points]

Solution : The charge on the capacitor plate obeys the ODE

$$L\,\ddot{Q} + R\,\dot{Q} + \frac{Q}{C} = V(t) \ .$$

The solution is

$$Q(t) = Q_{\text{hom}}(t) + A(\omega)\, \frac{V_0}{L}\, \cos\big(\omega t - \delta(\omega)\big) \ ,$$

with

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\bigg(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\bigg) \ .$$

Thus, ignoring the transients, the power dissipated by the bulb is

$$\begin{aligned} P_R(t) &= \dot{Q}^2(t)\, R \\ &= \omega^2 A^2(\omega)\, \frac{V_0^2 R}{L^2}\, \sin^2\!\big(\omega t - \delta(\omega)\big) \ . \end{aligned}$$

Averaging over a period, we have $\langle\, \sin^2(\omega t - \delta)\,\rangle = \frac{1}{2}$, so

$$\langle\, P_R\,\rangle = \omega^2 A^2(\omega)\, \frac{V_0^2 R}{2L^2} = \frac{V_0^2}{2\,R} \cdot \frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \ .$$

Now $V_0^2/2R = \frac{1}{4}\,\text{W}$. So $P_{\text{th}} = \alpha V_0^2/2R$, with $\alpha = \frac{8}{9}$. We then set $\langle P_R \rangle = P_{\text{th}}$, whence

$$(1 - \alpha) \cdot 4\beta^2\omega^2 = \alpha\, (\omega_0^2 - \omega^2)^2 \ .$$

The solutions are

$$\omega = \pm\sqrt{\frac{1-\alpha}{\alpha}}\; \beta + \sqrt{\bigg(\frac{1-\alpha}{\alpha}\bigg)\beta^2 + \omega_0^2} = \big(3\sqrt{3} \pm \sqrt{2}\big) \times 1000\,\text{s}^{-1} \ .$$

(c) Compare the expressions for the instantaneous power dissipated by the voltage source, $P_V(t)$, and the power dissipated by the resistor $P_R(t) = I^2(t)R$. If $P_V(t) \neq P_R(t)$, where does the power extra power go or come from? What can you say about the averages of P_V and $P_R(t)$ over a cycle? Explain your answer.
`[20 points]`

Solution : The instantaneous power dissipated by the voltage source is

$$\begin{aligned} P_V(t) = V(t)\, I(t) &= -\omega\, A\, \frac{V_0}{L}\, \sin(\omega t - \delta)\, \cos(\omega t) \\ &= \omega\, A\, \frac{V_0}{2L}\, \Big(\sin\delta - \sin(2\omega t - \delta)\Big) \ . \end{aligned}$$

As we have seen, the power dissipated by the bulb is

$$P_R(t) = \omega^2 A^2\, \frac{V_0^2 R}{L^2}\, \sin^2\!(\omega t - \delta) \ .$$

These two quantities are not identical, but they do have identical time averages over one cycle:

$$\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle = \frac{V_0^2}{2R} \cdot 4\beta^2\, \omega^2\, A^2(\omega) \ .$$

Energy conservation means

$$P_V(t) = P_R(t) + \dot{E}(t) \ ,$$

where

$$E(t) = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

is the energy in the inductor and capacitor. Since $Q(t)$ is periodic, the average of $\dot{E}$ over a cycle must vanish, which guarantees $\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle$.