PHYSICS 110A : CLASSICAL MECHANICS
HW 7 SOLUTIONS

(1) Taylor 8.13

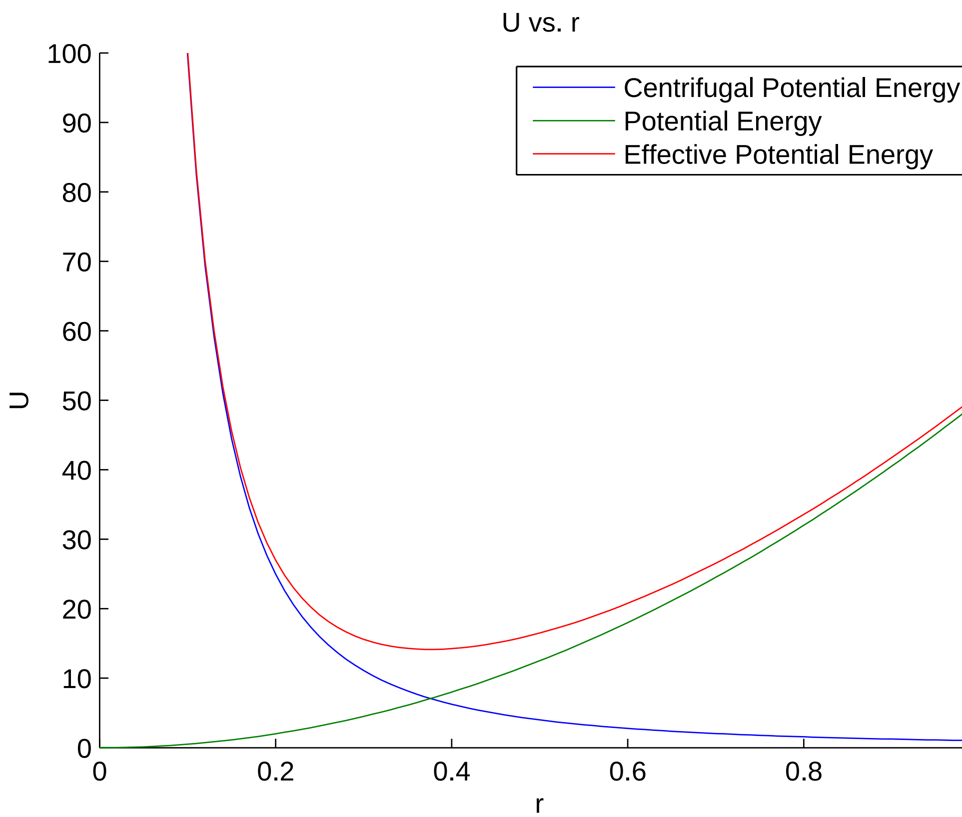

Figure 1: Plot of U_{eff} vs. r for $U = \frac{1}{2}kr^2$ where $\frac{k\mu}{l^2} = 50$.

The effective potential will be:

$$U_{eff} = \frac{l^2}{2\mu r^2} + \frac{1}{2}kr^2.$$

This is plotted in figure (1). In order to find the circular orbit we set $\ddot{r} = 0$ which gives us $\frac{dU_{eff}}{dr} = 0$. For this we find:

$$U'_{eff} = -\frac{l^2}{\mu r^3} + kr = 0. \tag{1}$$

Or:

$$\frac{l^2}{\mu r^3} = kr.$$

Which leads to:

$$r_0 = \sqrt[4]{\frac{l^2}{\mu k}}.$$

To find the Taylor expansion we want to find:

$$U_{eff}(r) = U_{eff}(r_0) + U'_{eff}(r_0)(r - r_0) + \frac{1}{2}U''_{eff}(r_0)(r - r_0)^2 + ...$$

So we have:

$$U_{eff}(r_0) = \frac{l^2}{2\mu r_0^2} + \frac{1}{2}kr_0^2.$$

If we plug in:

$$r_0^2 = \frac{l}{\sqrt{\mu k}}.$$

We have:

$$U_{eff}(r_0) = \sqrt{\frac{k}{\mu}}l.$$

By definition (equation 1) $U'_{eff}(r_0) = 0$, so we need to find the second derivative:

$$U''_{eff}(r_0) = \frac{3l^2}{\mu r_0^4} + k.$$

If we plug in:

$$r_0^4 = \frac{l^2}{\mu k}.$$

We have:

$$U''_{eff}(r_0) = 4k.$$

So our Taylor expansion is:

$$U_{eff}(r) \approx \sqrt{\frac{k}{\mu}}l + \frac{1}{2}4k(r - r_0)^2 + ...$$

Equation 8.29 in the text gives the equation of motion:

$$\mu\ddot{r} = -\frac{dU_{eff}(r)}{dr}.$$

With this our equation of motion will be:

$$\ddot{r} = -\frac{4k}{\mu}(r - r_0).$$

Which if we take $r = r_0 + \epsilon(t)$ gives us:

$$\ddot{\epsilon} = -\frac{4k}{\mu}\epsilon.$$

So this leads to a oscillator frequency of:

$$\omega = \sqrt{\frac{4k}{\mu}}.$$

(2) Taylor 8.14

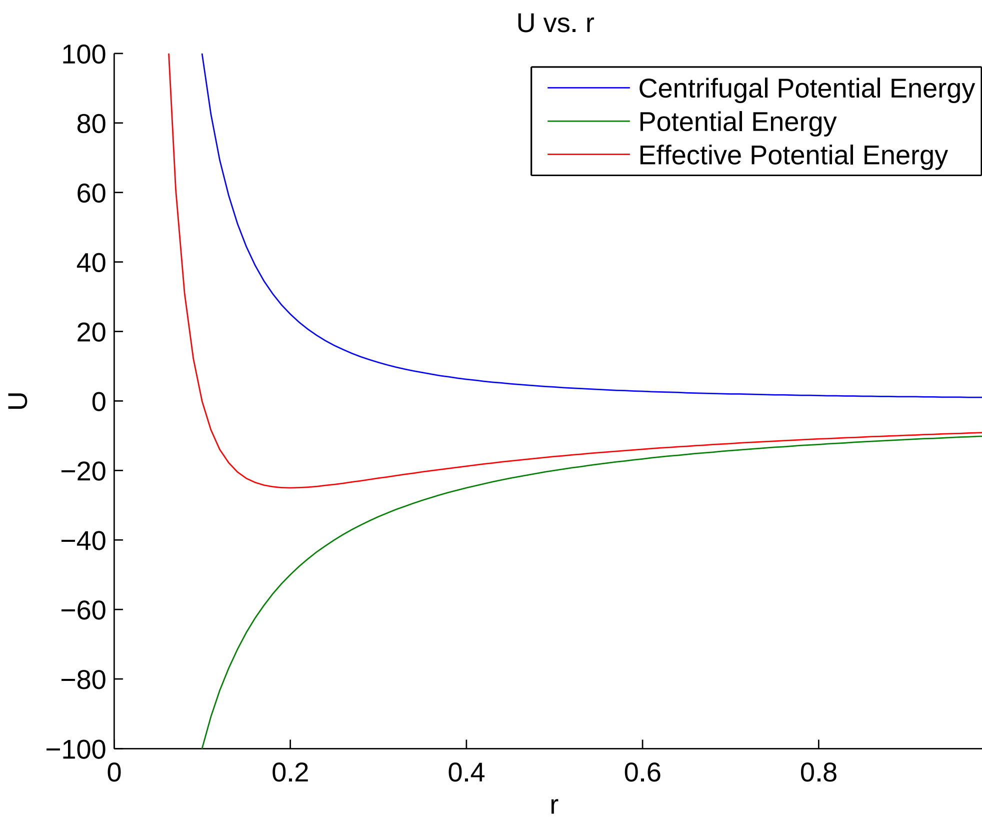

Figure 2: Plot of U_{eff} vs. r for $U = \frac{k}{r}$ where $\frac{k\mu}{l^2} = -10$.

The effective potential will be:

$$U_{eff} = \frac{l^2}{2\mu r^2} + kr^n.$$

This is plotted in figures (1), (2), and (3) for values $n = 2, -1$, and -3, respectively. (Note: since $kn > 0$ if $n < 0$ $k < 0$ as well). In order to find the circular orbit we set $\ddot{r} = 0$ which gives us $\frac{dU_{eff}}{dr} = 0$. For this we find:

$$U'_{eff} = -\frac{l^2}{\mu r^3} + nkr^{n-1} = 0. \tag{2}$$

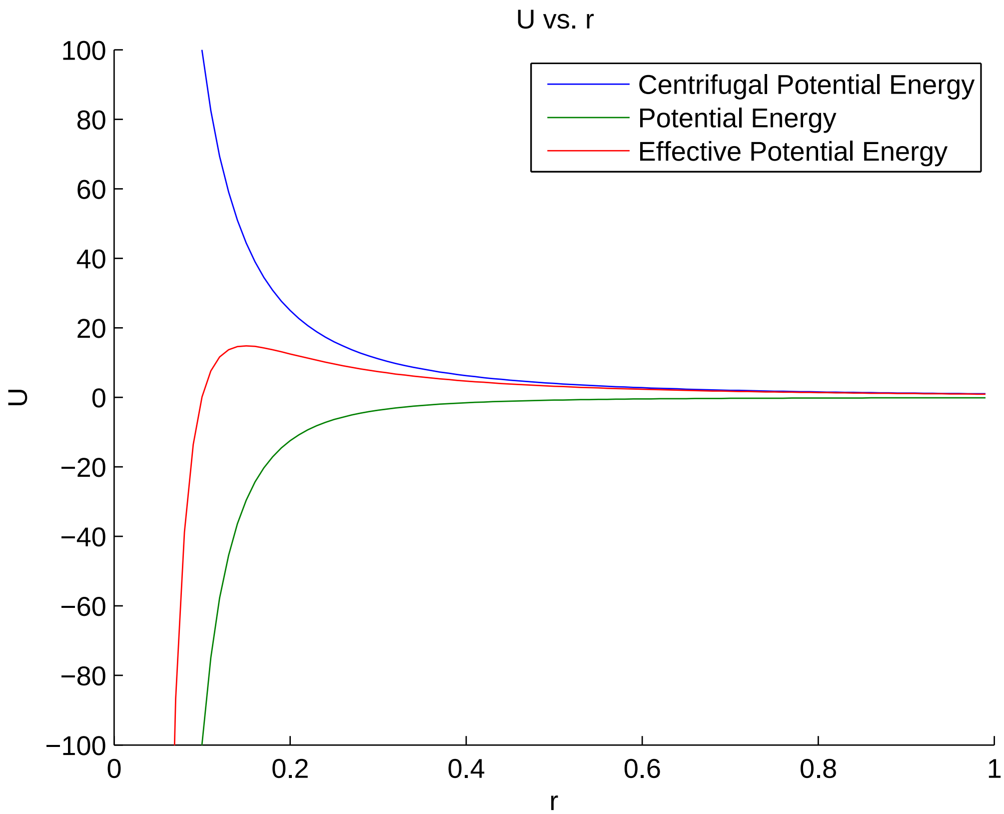

Figure 3: Plot of U_{eff} vs. r for $U = \frac{k}{r^3}$ where $\frac{k\mu}{l^2} = -0.1$.

Or:

$$\frac{l^2}{\mu r^3} = nkr^{n-1}.$$

Which leads to:

$$r_0 = \sqrt[n+2]{\frac{l^2}{n\mu k}}.$$

To determine which are stable orbits we need a second derivative test. We will have a stable equilibrium when $U''_{eff}(r_0) > 0$.

So:

$$U''_{eff}(r_0) = \frac{3l^2}{\mu r_0^4} + n(n-1)kr_0^{n-2}. \tag{3}$$

Now from above we have:

$$r_0 = \sqrt[n+2]{\frac{l^2}{n\mu k}}.$$

Or:

$$r_0^{n+2} = \frac{l^2}{n\mu k}.$$

Or:
$$r_0^{n-2} = \frac{l^2}{n\mu k r_0^4}.$$
Where I divided both sides by r_0^4.

Inserting this into equation (3) we have:
$$U''_{eff}(r_0) = \frac{3l^2}{\mu r_0^4} + n(n-1)k\frac{l^2}{n\mu k r_0^4}.$$
Or:
$$U''_{eff}(r_0) = \frac{3l^2}{\mu r_0^4} + (n-1)\frac{l^2}{\mu r_0^4}.$$
Or finally:
$$U''_{eff}(r_0) = \frac{(n+2)l^2}{\mu r_0^4}. \tag{4}$$
Where we see $U''_{eff}(r_0) > 0$ for all $n > -2$. This is in agreement with our plots. The plots of $n = 2$ and $n = -1$ have a stable minimum point where the plot of $n = -3$ does not.

We can find the period by finding the angular frequency from the Taylor expansion:
$$U_{eff}(r) = U_{eff}(r_0) + U'_{eff}(r_0)(r - r_0) + \frac{1}{2}U''_{eff}(r_0)(r - r_0)^2 + ...$$
with equations (2) and (4) we have:
$$U_{eff}(r) \approx U_{eff}(r_0) + \frac{1}{2}\frac{(n+2)l^2}{\mu r_0^4}(r - r_0)^2.$$
Equation 8.29 in the text gives the equation of motion:
$$\mu\ddot{r} = -\frac{dU_{eff}(r)}{dr}.$$
With this our equation of motion will be:
$$\ddot{r} = -\frac{(n+2)l^2}{\mu^2 r_0^4}(r - r_0).$$
Which if we take $r = r_0 + \epsilon(t)$ gives us:
$$\ddot{\epsilon} = -\frac{(n+2)l^2}{\mu^2 r_0^4}\epsilon.$$
So this leads to a oscillator frequency of:
$$\omega = \frac{\sqrt{n+2}l}{\mu r_0^2}.$$

So we see:

$$\tau_{osc} = \frac{2\pi\mu r_0^2}{\sqrt{n+2l}} = \frac{\tau_{orb}}{\sqrt{n+2}}.$$

(Note: check the first sentence of the Professor's notes in section 9.4.4 for the definition of τ_{orb}. An easy way to see this is to note that the time to go around a circle is

$$\tau_{orb} = \frac{2\pi r_0}{v} = \frac{2\pi r_0}{r_0\dot{\phi}} = \frac{2\pi r_0}{r_0(\ell/\mu r_0^2)} = \frac{2\pi\mu r_0^2}{\ell}$$

where we have used the circumference of the circle divided by the constant tangential speed about the circle ($v = r\omega$) to determine the period, or time to circumscribe the circle).

Now if $\sqrt{n+2}$ is rational we have:

$$\frac{\tau_{osc}}{\tau_{orb}} = \frac{A}{B}.$$

Where A and B are integers. This implies that the orbit will indeed repeat itself, if we repeat the precessing orbit at least B times, therefore being a closed orbit.

(3) Taylor 8.17

If $G = \mathbf{r}\cdot\mathbf{p}$ then:

$$\dot{G} = \dot{\mathbf{r}}\cdot\mathbf{p} + \mathbf{r}\cdot\dot{\mathbf{p}}.$$

We can write this a little different:

$$\dot{G} = \mathbf{v}\cdot\mathbf{p} + \mathbf{r}\cdot\mathbf{F}.$$

Integrating both sides over t we have:

$$\int_0^t dt'\dot{G} = \int_0^t dt'\left[\mathbf{v}\cdot\mathbf{p} + \mathbf{r}\cdot\mathbf{F}\right].$$

Which can be rewritten as:

$$G(t) - G(0) = 2\int_0^t dt'\frac{1}{2}mv^2 + \int_0^t dt'\mathbf{F}\cdot\mathbf{r}.$$

Dividing both sides by t we have:

$$\frac{G(t)-G(0)}{t} = \frac{1}{t}\left[2\int_0^t dt'T + \int_0^t dt'\mathbf{F}\cdot\mathbf{r}\right].$$

Or:

$$\frac{G(t)-G(0)}{t} = 2 < T > + < \mathbf{F}\cdot\mathbf{r} > .$$

Then we have:

$$0 = 2 < T > + < -nkr^{n-1}\hat{\mathbf{r}}\cdot\mathbf{r} > .$$

This is true since $G(t) = \mathbf{r} \cdot \mathbf{p}$ is always finite, since in a given orbit $\mathbf{r}$ has a maximum finite value and $\mathbf{p}$ also has a maximum finite value, so $\mathbf{r} \cdot \mathbf{p}$ is always finite, though oscillating. We can then rewrite this as

$$0 = 2 < T > -n < kr^n > .$$

And:

$$0 = 2 < T > -n < U > .$$

So we have:

$$< T >= n < U > /2.$$

(4) Taylor 8.19

Here we have equations for r_{min} and r_{max} for an ellipse:

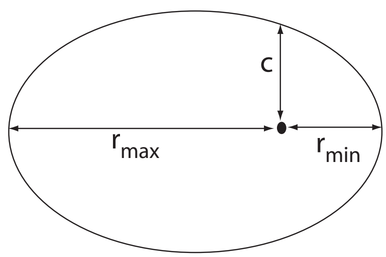

Figure 4: Figure for 8.19.

$$r_{max} = \frac{c}{1-\epsilon},$$

and:

$$r_{min} = \frac{c}{1+\epsilon},$$

Now remember r_{min} = 6400 km + 300 km = 6700 km, and r_{max} = 6400 km + 3000 km = 9400 km. Where 6400 km is the radius of the Earth.

So solving the above two equations for ϵ we get $\epsilon = 0.17$.

At this point it's easy to plug back in for c = 7802 km. Subtracting the radius of the Earth we get d = 1400 km which is the satellites distance to the surface of the Earth when it crosses the y-axis.

(5) Taylor 8.29

By the virial theorem we see that for a circular orbit under the influence of a power law potential $U = kr^n$:

$$< T >= -n < U > /2.$$

Which since the gravitational potential has $n = 1$ our kinetic energy is:

$$T = -U/2.$$

Where I dropped the average sign.

So our total energy would be:

$$E = -U_0/2 + U_0.$$

Now if the sun lost half of it's mass the potential energy would drop by a half, but the kinetic would not change. So we would have:

$$E = -U_0/2 + U_0/2 = 0.$$

We know that for $E = 0$ we have $\epsilon = 1$ and a parabolic orbit, so the earth would eventually leave the sun.

(6) Taylor 8.35

This is similar to example 8.6 in the text except run backwards. Initially we have

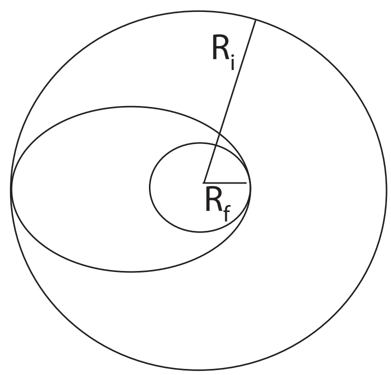

Figure 5: Figure for 8.35.

$r_{max} = r_{max}$ for the initial circular orbit R_i and the elliptical path which will transfer the craft between circular orbits.

Our equation will be:

$$\frac{c_1}{1-\epsilon_1} = \frac{c_2}{1-\epsilon_2}.$$

However $\epsilon_1 = 0$ because it's a circular orbit so we have:

$$c_1 = \frac{c_2}{1-\epsilon_2}. \tag{5}$$

Now the relation between c constants is:

$$c_1 = \lambda^2 c_2. \tag{6}$$

Again this is derived from:

$$v_1 = \lambda v_2,$$

and the fact that $v \propto l$ and $l^2 \propto c$.

Solving for ϵ_2 we get:

$$\epsilon_2 = 1 - \lambda^2.$$

Now we also want the r_{min} of this ellipse to match with our final radius R_f. For this to happen we need:

$$c_3 = \frac{c_2}{1+\epsilon_2}.$$

Or:

$$R_f = \frac{\lambda^2 R_i}{1+\epsilon_2}.$$

If we plug in our value for ϵ_2 we get:

$$\lambda = \sqrt{\frac{2R_f}{R_i + R_f}} = \sqrt{\frac{2}{5}}.$$

For the second thrust we want to switch from the elliptical orbit into a circular one. So we want to have the same r_{min} and we will have a relationship between c constants of:

$$c_3 = \lambda'^2 c_2. \tag{7}$$

So in order to have the same r_min we have:

$$c_3 = \frac{c_2}{1+\epsilon_2}. \tag{8}$$

Solving these for the thrust factor we get:

$$\lambda = \frac{1}{2-\lambda^2} = \sqrt{\frac{5}{8}}.$$

Similar to the example we use angular momentum to solve for the overall gain in speed:

$$v_3 = \lambda' \frac{v_2(per)}{v_2(apo)} \lambda v_1 = \frac{v_1}{2}.$$