PHYSICS 110A : CLASSICAL MECHANICS
HW 8 SOLUTIONS

(1) Taylor 11.14

For our generalized coordinates we will take the angles ϕ_1 and ϕ_2.

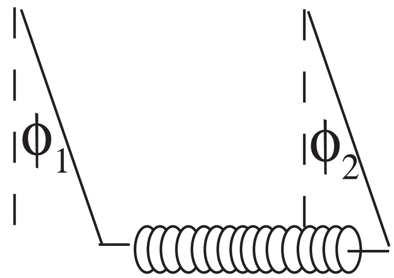

Figure 1: Figure for 11.14.

This leads to a kinetic energy of:

$$T = \frac{1}{2}mL^2[\dot{\phi}_1^2 + \dot{\phi}_2^2].$$

And the potential term will be:

$$U \approx \frac{1}{2}kL^2[\phi_2 - \phi_1]^2 + mgL[2 - \cos\phi_1 - \cos\phi_2].$$

Where we have assumed the springs Δx goes as $L\phi$ since we are dealing with small oscillations. Substituting in for $\cos\phi = 1 - \phi^2/2 + ...$ we get:

$$U \approx \frac{1}{2}kL^2[\phi_2 - \phi_1]^2 + \frac{mgL}{2}[\phi_1^2 - \phi_2^2].$$

From this we build T and V matrices as:

$$T = mL^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And:

$$V = mL^2 \begin{bmatrix} g/L + k/m & -k/m \\ -k/m & g/L + k/m \end{bmatrix}$$

Where we can rewrite as:

$$V = mL^2 \begin{bmatrix} \omega_0^2 + \beta_0^2 & -\beta_0^2 \\ -\beta_0^2 & \omega_0^2 + \beta_0^2 \end{bmatrix}$$

Where $\beta_0^2 = k/m$ and $\omega_0^2 = g/L$.

Using $\det[\omega^2 T - V] = 0$ we find eigenvalues of $\omega_1 = \omega_0$ and $\omega_2 = \sqrt{\omega_0^2 + 2\beta_0^2}$.

These eigenvalues lead to un-normalized eigenvectors of:

$$\Psi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

And:

$$\Psi_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

From these you can see Ψ_1 is the mode where the masses oscillate in phase with each other (this makes sense because if the masses are in phase the spring is not compressed and we see β is not in the expression for ω_1), and Ψ_2 is the mode where the masses oscillates out of phase with each other.

(2) Taylor 11.19

For our generalized coordinates we will take x and ϕ.

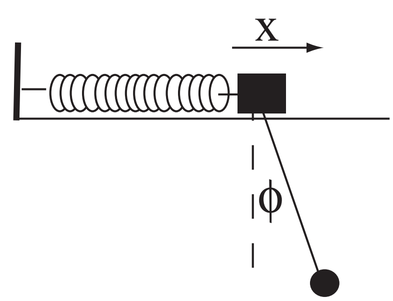

Figure 2: Figure for 11.19.

This leads to a kinetic energy of:

$$T = \frac{1}{2}(m_0 + M)\dot{x}^2 + \frac{1}{2}mL^2\dot{\phi}^2 + ML\dot{\phi}\dot{x}\cos\phi.$$

Where for small angles we have:

$$T \approx \frac{1}{2}(m_0 + M)\dot{x}^2 + \frac{1}{2}ML^2\dot{\phi}^2 + ML\dot{\phi}\dot{x}.$$

The potential term will be:

$$U = \frac{1}{2}kx^2 + MgL[1 - \cos\phi].$$

Where for small angles we have:

$$U \approx \frac{1}{2}kx^2 + \frac{MgL\phi^2}{2}.$$

From this we build T and V matrices as:

$$T = \begin{bmatrix} (m_0 + M) & ML \\ ML & ML^2 \end{bmatrix}$$

And:

$$V = \begin{bmatrix} k & 0 \\ 0 & MgL \end{bmatrix}$$

From the values given for constants we can rewrite as:

$$T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

And:

$$V = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Using $\det[\omega^2 T - V] = 0$ we find eigenvalues of $\omega_1 = \sqrt{2 - \sqrt{2}}$ and $\omega_2 = \sqrt{2 + \sqrt{2}}$.

These eigenvalues lead to un-normalized eigenvectors of:

$$\Psi_1 = \begin{bmatrix} 0.382 \\ 0.541 \end{bmatrix},$$

And:

$$\Psi_2 = \begin{bmatrix} 0.923 \\ -1.30 \end{bmatrix},$$

From these you can see Ψ_1 is the mode where the masses oscillate in phase with each other, and Ψ_2 is the mode where the masses oscillates out of phase with each other.

(3) Taylor 11.29

For our generalized coordinates we will use r and ϕ which mark the location of the center of mass of the rod and α which is the angle of the rod with respect to the horizontal as in figure (3).

So our kinetic energy will be:

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{2}I\dot{\alpha}^2.$$

Plugging in for the moment of inertia of a rod about it's center of mass we have:

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{1}{6}mb^2\dot{\alpha}^2.$$

Now the potential is a bit hairier and we will assume small angles from the outset.

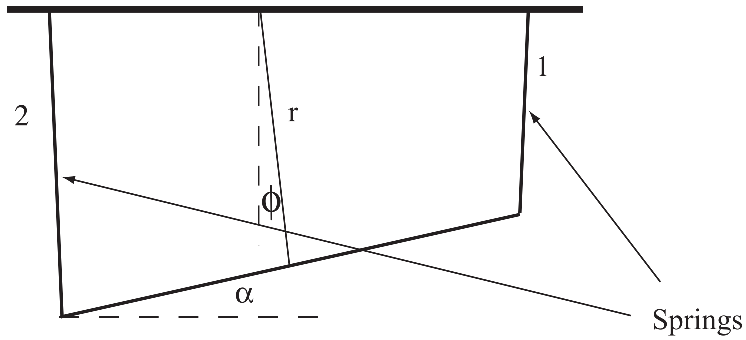

Figure 3: Figure for 11.29.

For small angles we will call the Δx for spring (1) as $L_1 = r + b\alpha$ and Δx for spring (2) as $L_1 = r - b\alpha$.

So the potential due to the springs will be:

$$U = \frac{1}{2}k(r - b\alpha - L_0)^2 + \frac{1}{2}k(r + b\alpha - L_0)^2.$$

Where L_0 is the rest length of the spring.

Now the potential due to gravity is:

$$U = -mgr\cos\phi.$$

So altogether we have:

$$U = \frac{1}{2}k(r - b\alpha - L_0)^2 + \frac{1}{2}k(r + b\alpha - L_0)^2 - mgr\cos\phi.$$

We will make the approximations $cos\phi \approx 1 - \phi^2/2$ and $r = r_0 + \epsilon$. This leads us to:

$$U = -mgr_0 + \frac{1}{2}mgr_0\phi^2 - mg\epsilon + k((r_0 - L_0) + \epsilon)^2 + k(b\alpha^2).$$

Which can be reduced to:

$$U = -mgr_0 + \frac{1}{2}mgr_0\phi^2 - mg\epsilon + k\epsilon^2 + k(r_0 - L_0)^2 + 2k(r_0 - L_0)\epsilon + k(b\alpha^2).$$

Finally we realize that at equilibrium the force up from the springs is equal to gravity. So from Newton's second law we have the relationship:

$$mg = 2k(r_0 - L_0).$$

So the terms linear in ϵ cancel and we have (dropping all constants):

$$U = \frac{1}{2}mgr_0\phi^2 + k\epsilon^2 + k(r_0 - L_0)^2 + k(b\alpha^2).$$

From this we build T and V matrices as:

$$T = \begin{bmatrix} m & 0 & 0 \\ 0 & mr_0^2 & 0 \\ 0 & 0 & \frac{1}{3}mb^2 \end{bmatrix}$$

And:

$$V = kR^2 \begin{bmatrix} 2k & 0 & 0 \\ 0 & mgr_0 & 0 \\ 0 & 0 & 2kb^2 \end{bmatrix}$$

Since these are diagonal $\det[\omega^2 T - V] = 0$ lead us to three equations:

$$\omega^2 m = 2k,$$

$$\omega^2 mr_0^2 = mgr_0,$$

and:

$$\omega^2 \frac{1}{3} mb^2 = 2kb^2,$$

Which lead to eigenvalues of $\omega_1 = \sqrt{\frac{2k}{m}}$ (for the r-coordinate), $\omega_2 = \sqrt{\frac{g}{r_0}}$ (for the ϕ-coordinate), and $\omega_3 = \sqrt{\frac{6k}{m}}$ (for the α-coordinate).

(4) **Taylor 11.31**

For our generalized coordinates we will take the three angles ϕ_1, ϕ_2, and ϕ_3.

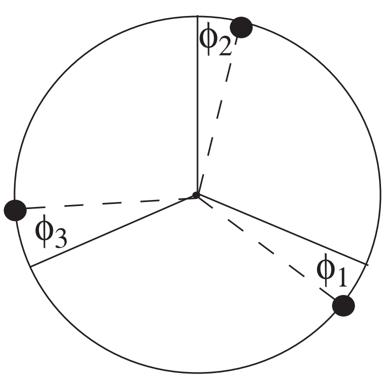

Figure 4: Figure for 11.31.

This leads to a kinetic energy of:

$$T = \frac{1}{2} mR^2 [2\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2].$$

And the potential term will be:

$$U = \frac{1}{2}kR^2[(\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_1)^2].$$

From this we build T and V matrices as:

$$T = mR^2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And:

$$V = kR^2 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Using $\det[\omega^2 T - V] = 0$ we find eigenvalues of $\omega_1 = 0$, $\omega_2 = \sqrt{2}\omega_0$, and $\omega_3 = \sqrt{3}\omega_0$.

These eigenvalues lead to un-normalized eigenvectors of:

$$\Psi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\Psi_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

And:

$$\Psi_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

From these you can see Ψ_1 is the mode where the three masses rotate around at some constant velocity, Ψ_2 is the mode where the first mass oscillates out of phase with the other two masses, and Ψ_3 is the mode where mass 1 doesn't oscillate and the other two masses oscillate out of phase with each other.

Professor Arovas has added some notes in section 10.6.1 of his lecture notes on an alternate technique to solve this problem.