INSTRUCTIONS: Do four problems. You must do problems 1 and 2. Then choose any two of problems 3, 4, and 5.

[1] *Central Forces*

Two particles attract each other via a central force $U(r) = -Cr^{-3/2}$, where C is a constant.

5 (a) Write down and sketch the effective potential $U_{\text{eff}}(r)$.

5 (b) Find the radius r_o of the circular orbit. Is this orbit stable? Why or why not?

5 (c) What is the period of the circular orbit?

5 (d) If the circular orbit is slightly perturbed, the relative coordinate can be written as $r(t) = r_o + \eta(t)$. What is the period of the deviation $\eta(t)$?

5 (e) Find the angular change $\Delta\phi_{\text{p}}$ between successive instances of periapsis. Is periapsis receding (takes more than one complete revolution to achieve periapsis) or advancing (takes less than one complete revolution to achieve periapsis)?

[2] *Lagrangian Dynamics*

The mechanical system depicted below rotates with angular frequency Ω about the central axis. The only degree of freedom is the angle θ.

5 (a) Write down the kinetic energy T, the potential energy U, and the Lagrangian L.

5 (b) What is the equation of motion for θ?

5 (c) Linearize about $\theta = 0$, $\dot{\theta} = 0$ and find the frequency of small oscillations. Show that this motion is stable (the oscillation frequency is real) provided $\Omega^2 \leq \Omega_c^2$. What is the critical frequency Ω_c?

5 (d) For $\Omega^2 > \Omega_c^2$, show that the system exhibits small oscillations when linearized about $\theta = \theta_0$, $\dot{\theta} = 0$. Find θ_0.

5 (e) What is the small oscillation frequency in the case $\Omega^2 > \Omega_c^2$? You may leave your answer in terms of θ_0 and other quantities.
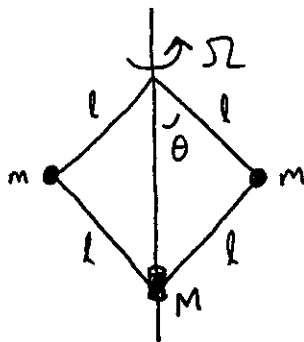
[3] *Driven Linear Systems*

A driven system obeys the equation of motion

$$\left(\frac{d}{dt} + \alpha\right)\left(\frac{d}{dt} + \beta\right)\left(\frac{d}{dt} + \gamma\right) x = f_0 \cos \Omega t \,,$$

where α, β, and γ are all real and positive.

(a) What is the most general solution to the homogeneous equation, $x_h(t)$? *Hint: The equation is already factorized!*

(b) Writing the general inhomogeneous solution as

$$x = A(\Omega)\, f_0 \, \cos\left((\Omega t - \delta(\Omega)) + x_h(t)\right)$$

find expressions for the amplitude $A(\Omega)$ and the phase $\delta(\Omega)$.

(c) Assuming $\alpha \ll \beta \ll \gamma$, sketch $\delta(\Omega)$ and $A(\Omega)$.

(d) For the general inhomogeneous solution, write, but do not solve, three equations which will determine these constants of integration in terms of $x(0)$, $\dot{x}(0)$, and $\ddot{x}(0)$.

[4] *Calculus of Variations*

(a) Derive the equation for a geodesic on an ellipsoidal surface of revolution defined by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

where a and b are constants. Work in cylindrical coordinates (ρ, ϕ, z) and derive a first order differential equation of the form $d\phi = F(\rho)\, d\rho$.

(b) Derive equations for the extremization of the functional

$$S[x(t), y(t), z(t)] = \int_{t_i}^{t_f} dt \left[\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}\dot{y}^2 + \tfrac{1}{2}\dot{z}^2 + x\dot{y} + y\dot{z} + z\dot{x} - U(x, y, z)\right]$$

subject to the constraints

$$x^3\, y\, z = a^5 - x\, y^2\, z^2$$

$$x + y + z = b$$

where a and b are constants. You don't have to solve any equations – just make sure you have as many equations as you have unknowns.

[5] *Phase Space Dynamics*

In the dim and distant past, we explored the logistic equation, $\dot{N} = rN(1 - N/K)$, for population dynamics. Consider now the following model for a *harvested* population:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$$

where H is the harvesting rate.

(a) Show that this equation may be adimensionalized to the form

$$\frac{d\nu}{d\tau} = \nu(1 - \nu) - h$$

and determine ν, τ, and h. *Hint:* $\nu = N/K$.

(b) Locate all fixed points and identify their stability. Show that a *bifurcation* occurs at a certain value of h – for $h > h_c$ there are no longer any fixed points. Find h_c, and plot the vector field $\dot{\nu}$ *versus* ν for the two cases $h < h_c$ and $h > h_c$.

(c) Discuss the long time behavior for the population for $h < h_c$ and $h > h_c$. Can you provide a biological interpretation for each case?

(d) This model has a very silly feature – the population can become negative. An improved model might start with the equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \frac{HN}{L + N}$$

where L is an additional parameter. The harvesting term now goes to zero in the limit $N \to 0$ and $N = 0$ is thus a fixed point for all values of the parameters r, K, H, and L. Under what conditions will $N = 0$ be an attractive fixed point, and under what conditions will it be repulsive? (The full analysis of this improved model is quite interesting.)

The following extra credit problem, basically worthless, is meant for listeners of 88.1 FM.

[*] Miles davis (trumpet) and John Coltrane (tenor saxophone) were two of the five members of one of the greatest quintets in the history of music. Name the other three members, along with their instruments. Incredibly, at the same time there was another great ("west coast") quintet, with Clifford Brown on trumpet and Harold Land/Sonny Rollins on tenor sax. Name the three other members. *Hint: One of the other members had a more famous brother who specialized on the same instrument.* Tragically, Brown and one of the other members died in a car crash on the Pennsylvania turnpike in the summer of 1956, four months before Brown's 26th birthday.
[100 quatloos extra credit]

[1] $U(r) = -Cr^{-3/2}$
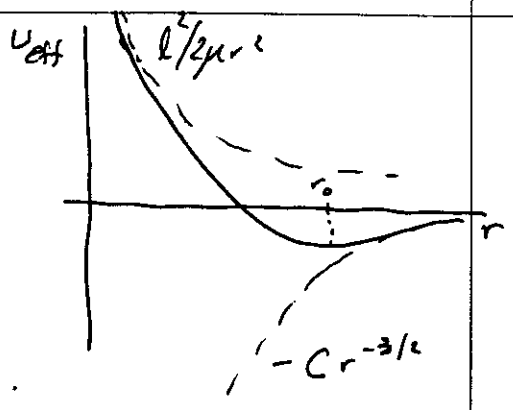
(a) $U_{eff}(r) = \dfrac{l^2}{2\mu r^2} + U(r) = \dfrac{l^2}{2\mu r^2} - \dfrac{C}{r^{3/2}}$

Angular momentum barrier wins as $r \to 0$;
Potential wins as $r \to \infty$. Like kepler.
Clearly there is a single _stable_ circular orbit for each l.

(b) $\dfrac{\partial U_{eff}}{\partial r}\bigg|_{r_0} = -\dfrac{l^2}{\mu r_0^3} + \dfrac{3}{2}\dfrac{C}{r_0^{5/2}} = 0 \;\Rightarrow\; r_0^{1/2} = \dfrac{2l^2}{3\mu C}$

$\underline{r_0 = \dfrac{4}{9}\dfrac{l^4}{\mu^2 C^2}}$; from the sketch $U_{eff}''(r_0) > 0 \Rightarrow$ _stable_

(c) $l = \mu r^2 \dot\phi = \mu r_0^2 \cdot \dfrac{2\pi}{T} \;\Rightarrow\; T = \dfrac{2\pi \mu r_0^2}{l} = \underline{\dfrac{32\pi}{81}\dfrac{l^7}{\mu^3 C^4}}$

(d) $\mu \ddot r = -\dfrac{\partial U_{eff}}{\partial r}$; $r = r_0 + \eta \;\Rightarrow$

$\mu \ddot\eta = -\dfrac{\partial^2 U_{eff}}{\partial r^2}\bigg|_{r_0}\eta + \mathcal{O}(\eta^2) \;\Rightarrow\; \omega^2 = \dfrac{1}{\mu}\dfrac{\partial^2 U_{eff}}{\partial r^2}\bigg|_{r_0}$

$\omega^2 = \dfrac{1}{\mu}\left[\dfrac{3l^2}{\mu r_0^4} - \dfrac{15}{4}\dfrac{C}{r_0^{7/2}}\right] = \dfrac{1}{\mu}\cdot\left[\dfrac{3l^2}{\mu r_0^4} - \dfrac{5}{2r_0}\cdot\left(\dfrac{l^2}{\mu r_0^3}\right)\right]$
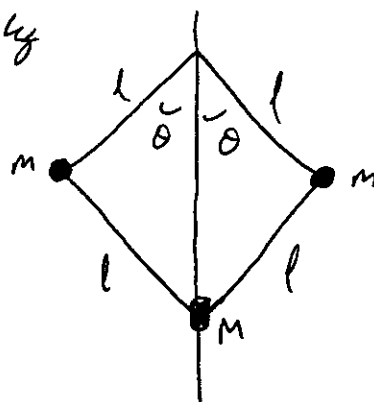
$\curvearrowleft$ use def^n of $r_0 \Rightarrow \dfrac{3}{2}\dfrac{C}{r_0^{5/2}} = \dfrac{l^2}{\mu r_0^3}$

Thus,

$\omega = \dfrac{1}{\sqrt2}\dfrac{l}{\mu r_0^2} = \underline{\dfrac{81}{16\sqrt2}\dfrac{\mu^3 C^4}{l^7}}$; $T' = \dfrac{2\pi}{\omega} = \sqrt2\,T = \dfrac{32\sqrt2\,\pi}{81}\dfrac{l^7}{\mu^3 C^4}$

(e) It takes time $T = \dfrac{2\pi}{\omega}$ for periapsis to recur. During this time ϕ has advanced by $\Delta\phi = \dfrac{2\pi}{\omega}\cdot\dot\phi = \dfrac{2\pi}{\omega}\cdot\dfrac{l}{\mu r_0^2} = 2\pi\sqrt2$.
Thus, periapsis is _receding_ by $2\pi(\sqrt2 - 1)$ each cycle.

[2] Cylindrical coordinates are clearly called for:

(a) $T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \times 2$

$\qquad + \frac{1}{2}M\left(2\dot{z}\right)^2$

where $\rho = \ell\sin\theta$, $z = \ell\cos\theta$ $\Rightarrow$

$\qquad T = m\left(\ell^2\dot{\theta}^2 + \ell^2\Omega^2\sin^2\theta\right) + 2M\ell^2\sin^2\theta\,\dot{\theta}^2$

$\qquad U = -2mg\ell\cos\theta - 2Mg\ell\cos\theta = -2(M+m)g\ell\cos\theta$

$\qquad L = m\ell^2\dot{\theta}^2 + 2M\ell^2\sin^2\theta\,\dot{\theta}^2 + m\ell^2\Omega^2\sin^2\theta + 2(M+m)g\ell\cos\theta$

(b) $\dfrac{\partial L}{\partial\dot{\theta}} = 2m\ell^2\dot{\theta} + 4M\ell^2\sin^2\theta\,\dot{\theta}$

$\dfrac{\partial L}{\partial\theta} = 4M\ell^2\sin\theta\cos\theta\,\dot{\theta}^2 + 2m\ell^2\Omega^2\sin\theta\cos\theta - 2(M+m)g\ell\sin\theta$

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) = 2m\ell^2\ddot{\theta} + 8M\ell^2\sin\theta\cos\theta\,\dot{\theta}^2 + 4M\ell^2\sin^2\theta\,\ddot{\theta}$

Thus, after canceling $4M\ell^2\sin\theta\cos\theta\,\dot{\theta}^2$ common to both sides,

$\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial\dot{\theta}}\right) = \dfrac{\partial L}{\partial\theta} \quad\Rightarrow$

$\qquad 2m\ell^2\ddot{\theta} + 4M\ell^2\sin\theta\cos\theta\,\dot{\theta}^2 + 4M\ell^2\sin^2\theta\,\ddot{\theta} = 2m\ell^2\Omega^2\sin\theta\cos\theta$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - 2(M+m)g\ell\sin\theta$

$\Rightarrow$ $\boxed{\left(1 + \dfrac{2M}{m}\sin^2\theta\right)\ddot{\theta} + \dfrac{2M}{m}\sin\theta\cos\theta\,\dot{\theta}^2 = \left(\Omega^2\cos\theta - \left(1 + \dfrac{M}{m}\right)\dfrac{g}{\ell}\right)\sin\theta}$

(c) Linearize about $\theta = 0$, $\dot\theta = 0 \Rightarrow$

$$\ddot\theta = \left(\Omega^2 - \left(1 + \frac{M}{m}\right)\frac{g}{l}\right)\theta \equiv -\omega^2\theta$$

$$\omega = \sqrt{\left(1 + \frac{M}{m}\right)\frac{g}{l} - \Omega^2} \quad = \text{frequency of small oscillations}$$

Note that ω is real only for $\Omega^2 < \Omega_c^2$, where

$$\Omega_c^2 = \left(1 + \frac{M}{m}\right)\frac{g}{l}$$

(d) If $\Omega^2 > \Omega_c^2$, we set $\theta = \theta_0 + \epsilon$, with

$$\cos\theta_0 = \left(1 + \frac{M}{m}\right)\frac{g}{l}\frac{1}{\Omega^2}$$

Then,

$$\left(1 + \frac{2M}{m}\sin^2\theta_0\right)\ddot\epsilon = \left(\cancel{\Omega^2\cos\theta_0} - \Omega^2\sin\theta_0\,\epsilon + \ldots - \cancel{\left(1 + \frac{M}{m}\right)\frac{g}{l}}\sin\theta_0 + \ldots\right)$$

$$= -\Omega^2\sin^2\theta_0\,\epsilon$$

So,

(e)

$$\omega^2 = \frac{\Omega^2\sin^2\theta_0}{1 + \frac{2M}{m}\sin^2\theta_0} \quad\Rightarrow\quad \omega = \frac{\Omega\sin\theta_0}{\sqrt{1 + \frac{2M}{m}\sin^2\theta_0}}$$

with $\sin\theta_0 = \sqrt{1 - \cos^2\theta_0}$, $\cos\theta_0$ given above.

[3] $$\left(\frac{d}{dt}+\alpha\right)\left(\frac{d}{dt}+\beta\right)\left(\frac{d}{dt}+\gamma\right)x = f_0\cos\Omega t \equiv f(t)$$

(a) Homogeneous solution ($f_0 = 0$) $\Rightarrow$

$$x_h(t) = C_1 e^{-\alpha t} + C_2 e^{-\beta t} + C_3 e^{-\gamma t}$$

(b) Consider $f(t) = f_0 e^{-i\Omega t}$ and write $x(t) = A e^{i\delta} f_0 e^{-i\Omega t}$:
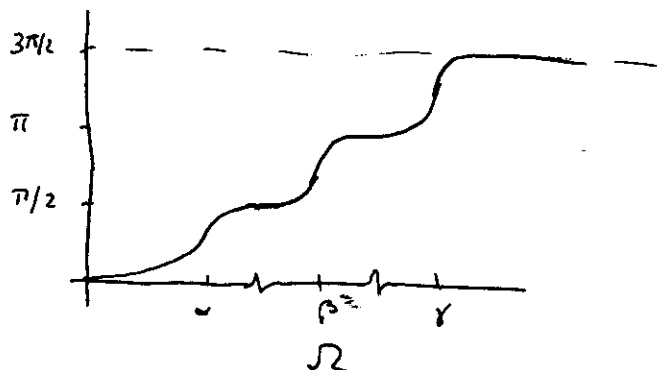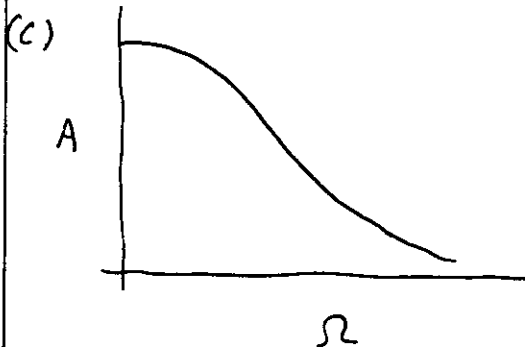
$$\Rightarrow \quad (-i\Omega + \alpha)(-i\Omega + \beta)(-i\Omega + \gamma) A e^{i\delta} = 1$$

$$A e^{i\delta} = \frac{1}{(\alpha - i\Omega)(\beta - i\Omega)(\gamma - i\Omega)} = \frac{e^{i\tan^{-1}\left(\frac{\Omega}{\alpha}\right)}\, e^{i\tan^{-1}\left(\frac{\Omega}{\beta}\right)}\, e^{i\tan^{-1}\left(\frac{\Omega}{\gamma}\right)}}{\sqrt{(\alpha^2+\Omega^2)(\beta^2+\Omega^2)(\gamma^2+\Omega^2)}}$$

So we have

$$A(\Omega) = \frac{1}{\sqrt{(\Omega^2+\alpha^2)(\Omega^2+\beta^2)(\Omega^2+\gamma^2)}}$$

$$\delta(\Omega) = \tan^{-1}\left(\frac{\Omega}{\alpha}\right) + \tan^{-1}\left(\frac{\Omega}{\beta}\right) + \tan^{-1}\left(\frac{\Omega}{\gamma}\right)$$

(c)

(d) $$x(t) = A f_0 \cos(\Omega t - \delta) + x_h(t)$$

$$\begin{cases} x(0) = A f_0 \cos\delta + C_1 + C_2 + C_3 \\ \dot{x}(0) = +\Omega A f_0 \sin\delta - \alpha C_1 - \beta C_2 - \gamma C_3 \\ \ddot{x}(0) = -\Omega^2 A f_0 \cos\delta + \alpha^2 C_1 + \beta^2 C_2 + \gamma^2 C_3 \end{cases}$$

[4] (a) $\rho = x^2 + y^2$, $\phi = \tan^{-1}(y/x)$; $\dfrac{\rho^2}{a^2} + \dfrac{t^2}{b^2} = 1$

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dt^2 \quad ; \quad z\,dz = -\dfrac{b^2}{a^2}\rho\,d\rho$$

$$= d\rho^2 + \rho^2 d\phi^2 + \dfrac{b^4}{a^4}\dfrac{\rho^2}{z^2}d\rho^2 \qquad z^2 = b^2 - \dfrac{b^2}{a^2}\rho^2$$

$$ds = \sqrt{\rho^2\,d\phi^2 + \left[\dfrac{b^2}{a^2}\dfrac{\rho^2}{a^2-\rho^2} + 1\right]d\rho^2}$$

$$D[\phi(\rho)] = \int_{\rho_i}^{\rho_f} d\rho\,\underbrace{\sqrt{\rho^2\left(\dfrac{d\phi}{d\rho}\right)^2 + \left[\dfrac{b^2}{a^2}\dfrac{\rho^2}{a^2-\rho^2} + 1\right]}}_{L\left(\phi,\,\frac{d\phi}{d\rho},\,\rho\right)}$$

$$\dfrac{\partial L}{\partial \phi} = 0 \quad \Rightarrow \quad \dfrac{\partial L}{\partial \phi'} = P = \text{constant}$$

$$P = \dfrac{\rho^2 \phi'}{\sqrt{\rho^2\phi'^2 + \left[\dfrac{b^2}{a^2}\dfrac{\rho^2}{a^2-\rho^2} + 1\right]}} \quad \Rightarrow \quad (P^2\rho^2 - \rho^4)\phi'^2 + \left[\dfrac{b^2}{a^2}\dfrac{\rho^2}{a^2-\rho^2} + 1\right] = 0$$

$$\boxed{\,d\phi = \dfrac{1}{\rho\sqrt{\rho^2 - P^2}}\left[1 + \dfrac{b^2}{a^2}\dfrac{\rho^2}{a^2-\rho^2}\right]^{1/2}d\rho\,}$$

(b) $$S[x(t), y(t), z(t)] = \int_{t_i}^{t_f} dt\left\{\dfrac{\dot{x}^2}{2} + \dfrac{\dot{y}^2}{2} + \dfrac{\dot{z}^2}{2} + x\dot{y} + y\dot{z} + z\dot{x} - U(x,y,z)\right\}$$

$$G_1(x,y,z) = x^3 yz + xy^2 z^2 - a^5 = 0$$
$$G_2(x,y,z) = x + y + z - b = 0$$

$\Big\}$ = 5 equations for $x, y, z, \lambda_1, \lambda_2$

$$\ddot{x} + \dot{z} - \dot{y} + \dfrac{\partial U}{\partial x} = (3x^2 yz + y^2 z^2)\lambda_1 + \lambda_2$$

$$\ddot{y} + \dot{x} - \dot{z} + \dfrac{\partial U}{\partial y} = (x^3 z + 2xyz^2)\lambda_1 + \lambda_2$$

$$\ddot{z} + \dot{y} - \dot{x} + \dfrac{\partial U}{\partial z} = (x^3 y + 2xy^2 z)\lambda_1 + \lambda_2$$

[5] $\dot{N} = \frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - H$

(a) Set $v \equiv N/K$ $\Rightarrow$

$$K \frac{dv}{dt} = rK\, v(1-v) - H$$

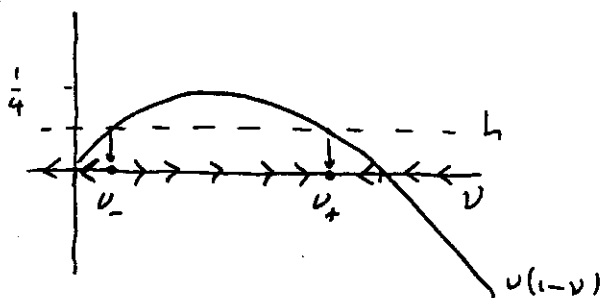Divide by rK $\Rightarrow$ $\frac{1}{r} \frac{dv}{dt} = v(1-v) - \frac{H}{rK}$

So with

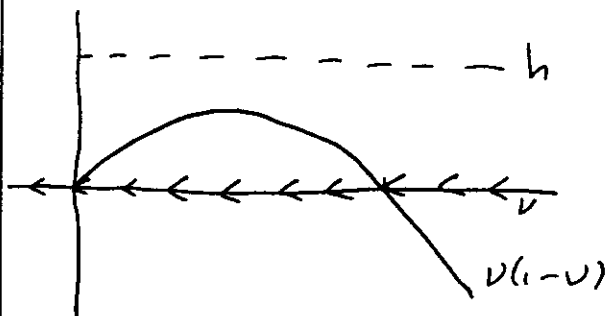$$\bar{\tau} \equiv rt , \quad h \equiv \frac{H}{rK}$$

we have

$$\frac{dv}{d\tau} = v(1-v) - h$$

(b) Consider the function $g(v) = v(1-v)$. Find $g' = 1 - 2v \Rightarrow v = \frac{1}{2}$ is location of maximum $(g'' = -2)$ $\Rightarrow g(\frac{1}{2}) = \frac{1}{4} = g_{max}$. Two cases to consider:

$h < \frac{1}{4} \Rightarrow$ two fixed points at

$v^2 - v + h = 0 \Rightarrow v_\pm = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4h}$

v_- is repulsive, v_+ attractive

$h > \frac{1}{4} \Rightarrow$ no fixed points

$\dot{v} < 0$ for all v

(c) If $h < \frac{1}{4}$, then for $v > v_-$ we will flow to v_+, a stable FP. In this case, the rate of harvesting is not great enough to destroy the FP. Note that $v_+ < 1 \Rightarrow N_+ < K \Rightarrow$ equilibrium population has decreased due to harvesting. For $v < v_-$ we flow to $v = -\infty$ -- this is unphysical! (unbiological?) For $h > \frac{1}{4}$, we harvest too much and the population again decreases, this time for all v. Again we tend to the silly limit $v(\infty) = -\infty$.

(d) $\quad \dfrac{dN}{dt} = rN \left(1 - \dfrac{N}{K}\right) - \dfrac{HN}{L+N}$

$$= \left(r - \frac{H}{L}\right) N + \mathcal{O}(N^2)$$

$$\Rightarrow \quad N = 0 \quad \text{stable (attractive) if} \quad r < \frac{H}{L}$$
$$N = 0 \quad \text{unstable (repulsive) if} \quad r > \frac{H}{L}$$

[*]   Miles Davis (tpt)   Clifford Brown (tpt)
   John Coltrane (ts)   Harold Land/Sonny Rollins (ts)
   Red Garland (piano)   Richie Powell (piano)
   Paul Chambers (bass)   Joe Morrow (bass)
   Philly Joe Jones (drums)   Max Roach (drums)

Richie Powell was Bud Powell's younger brother. Richie Powell, his wife, and Clifford Brown were killed June 26, 1956 in a car crash on the PA turnpike.