

Physics 110A: Midterm Exam #1

[1] A particle of mass m moves in one dimension under the influence of the potential

$$U(x) = \frac{U_0}{a^6} x^2 (x^2 - a^2)^2 ,$$

with $U_0 > 0$.

- (a) Sketch $U(x)$, and find the location of all minima and maxima.
- (b) Sketch the phase curves for a particle with energy $E_1 = 0$, $E_2 = \frac{1}{8}U_0$, $E_3 = \frac{1}{4}U_0$.
- (c) Sketch the separatrix and determine its associated energy.
- (d) Compute the frequency of oscillation when x is close to a .

[2] A damped oscillator obeys the usual second order homogeneous differential equation

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0 ,$$

with $\beta = 5$ Hz and $\omega_0 = 4$ Hz.

- (a) Write down the solution $x(t)$ given initial conditions $x(0) = 0$ and $\dot{x}(0) = -1$ m/s.

Next assume the oscillator is forced:

$$\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = f(t) ,$$

with $f(t) = f_0 \cos(\Omega t)$. The forcing amplitude is $f_0 = \sqrt{10}$ m/s², and the forcing frequency is $\Omega = 6$ Hz. Assume that a long time has passed so that any transient homogeneous oscillations have damped away.

- (b) What is the time delay Δt between the maximum of the force $f(t)$ and the maximum of the response $x(t)$?
- (c) What is the amplitude of the forced oscillations?

PHYSICS 110A MIDTERM #1 SOLUTIONS

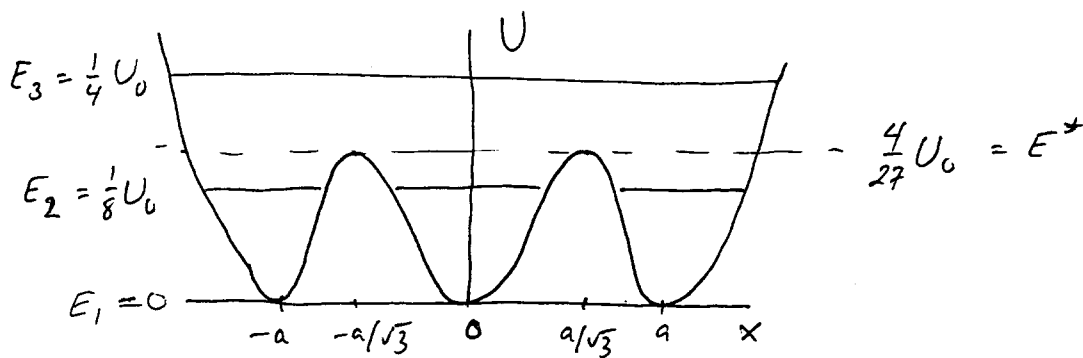
[1] $U(x) = \frac{U_0}{a^6} x^2 (x^2 - a^2)^2$

(a) Note $U(x) \geq 0$ and $U(0) = U(\pm a) = 0$. Differentiating

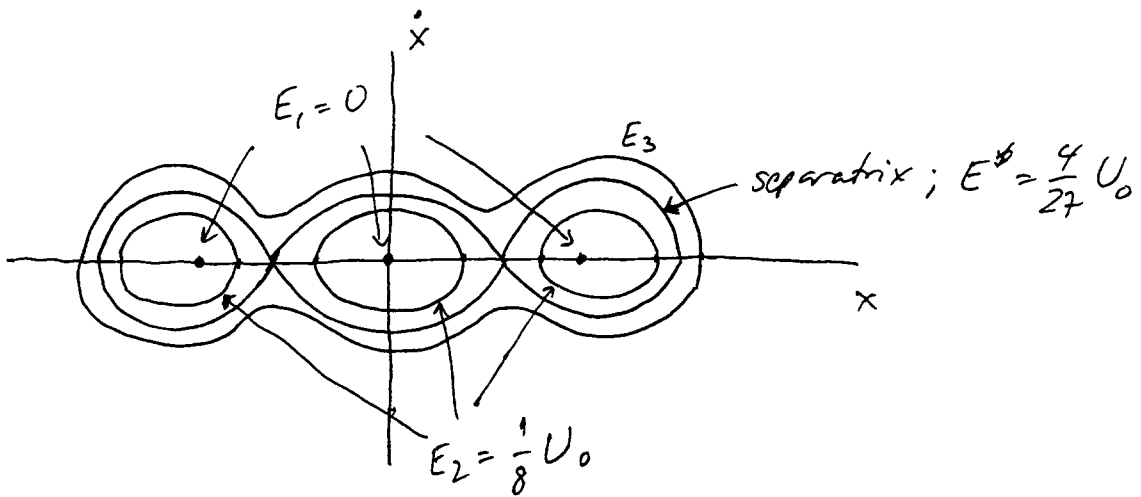
gives $U'(x) = U_0 \cdot \left(\frac{3x^4}{a^4} - \frac{4x^2}{a^2} + 1 \right) \cdot \frac{2x}{a}$, so

$U'(x) = 0 \Rightarrow x = 0, (x/a)^2 = \frac{4 \pm \sqrt{16-12}}{6} = 1, \frac{1}{3}$ i.e. $\frac{x}{a} = \pm \frac{1}{\sqrt{3}}, \pm 1$.

$x = \pm \frac{a}{\sqrt{3}}$ are then local maxima. Note $U(\pm \frac{a}{\sqrt{3}}) = \frac{4}{27} U_0$. Thus,



(b)



(c) $E^* = \frac{4}{27} U_0 = U(\pm \frac{a}{\sqrt{3}})$.

(d) $U(x) = U_0 \cdot \left\{ \frac{x^6}{a^6} - \frac{2x^4}{a^4} + \frac{x^2}{a^2} \right\} \Rightarrow U''(x) = \frac{U_0}{a^2} \cdot \left\{ \frac{30x^4}{a^4} - \frac{24x^2}{a^2} + 2 \right\}$

$U''(a) = \frac{8U_0}{a^2} \Rightarrow \omega_{osc} = \sqrt{\frac{8U_0}{ma^2}}$

[2] (a) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$; $\beta = 5 \text{ Hz}$, $\omega_0 = 4 \text{ Hz}$

General solⁿ : $x_h = C_+ e^{-i\omega_+ t} + C_- e^{-i\omega_- t}$

with $\omega_{\pm} = -i\beta \pm \sqrt{\omega_0^2 - \beta^2} = -5i \text{ Hz} \pm \sqrt{16 - 25} \text{ Hz}$
 $= -2i \text{ Hz}, -8i \text{ Hz}$

Hence, with t measured in seconds,

$$x(t) = C_+ e^{-2t} + C_- e^{-8t}$$

$$\dot{x}(t) = -2C_+ e^{-2t} - 8C_- e^{-8t}$$

Match initial conditions :

$$x(0) = 0 = C_+ + C_- \Rightarrow C_- = -C_+$$

$$\dot{x}(0) = -1 = -2C_+ - 8C_- = +6C_+$$

Solution : $C_+ = -\frac{1}{6}$, $C_- = +\frac{1}{6}$.

$$x(t) = -\frac{1}{6} e^{-2t} + \frac{1}{6} e^{-8t}$$

x in meters
 t in seconds

(b) $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) = f_0 \cos(\Omega t)$; $f_0 = \sqrt{10} \frac{\text{m}}{\text{s}^2}$, $\Omega = 6 \text{ Hz}$

$$\Rightarrow x(t) = x_h(t) + A(\Omega) f_0 \cos(\Omega t - \delta(\Omega))$$

$$A(\Omega) = [(\omega_0^2 - \Omega^2)^2 + 4\beta^2 \Omega^2]^{-1/2} ; \delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right)$$

Compute the phase lag $\delta = \tan^{-1}\left(\frac{2 \cdot 5 \cdot 6}{4^2 - 6^2}\right) = \tan^{-1}(-3) \approx 1.893$

To compute Δt , set $\Omega \cdot \Delta t - \delta(\Omega) = 0$, so

$$\Delta t = \frac{\delta(\Omega)}{\Omega} = \frac{\tan^{-1}(-3)}{6 \text{ Hz}} = 0.315 \text{ sec}$$

(c) $A = [(16 - 36)^2 + 4 \cdot 5^2 \cdot 6^2]^{-1/2} \text{ s}^2 = [400 + 3600]^{-1/2} \text{ s}^2 = (4000)^{-1/2} \text{ s}^2$

$$A = \frac{1}{20\sqrt{10}} \text{ s}^2 \Rightarrow X_{\max} = \frac{1}{20\sqrt{10}} \text{ s}^2 \cdot \sqrt{10} \frac{\text{m}}{\text{s}^2} = \frac{1}{20} \text{ m} = 5 \text{ cm}$$

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