Chapter 14

Continuum Mechanics

14.1 Strings

Consider a string of linear mass density $\mu(x)$ under tension $\tau(x)$.[1] Let the string move in a plane, such that its shape is described by a smooth function $y(x)$, the vertical displacement of the string at horizontal position x, as depicted in fig. 14.1. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. Consider a differential element of the string extending from x to $x+dx$. The change in length relative to the unstretched ($y=0$) configuration is

$$d\ell = \sqrt{dx^2+dy^2} - dx = \frac{1}{2}\bigg(\frac{\partial y}{\partial x}\bigg)^{\!2} dx + \mathcal{O}\big(dx^2\big) \ . \tag{14.1}$$

The differential potential energy is then

$$dU = \tau(x)\, d\ell = \tfrac{1}{2}\,\tau(x) \left(\frac{\partial y}{\partial x}\right)^{\!2} dx \ . \tag{14.2}$$

The differential kinetic energy is simply

$$dT = \tfrac{1}{2}\,\mu(x) \left(\frac{\partial y}{\partial t}\right)^{\!2} dx \ . \tag{14.3}$$

We can then write

$$L = \int\! dx\, \mathcal{L} \ , \tag{14.4}$$

where the *Lagrangian density* $\mathcal{L}$ is

$$\mathcal{L}(y,\dot y,y';x,t) = \tfrac{1}{2}\,\mu(x)\bigg(\frac{\partial y}{\partial t}\bigg)^{\!2} - \tfrac{1}{2}\,\tau(x)\bigg(\frac{\partial y}{\partial x}\bigg)^{\!2} \ . \tag{14.5}$$

[1] As an example of a string with a position-dependent tension, consider a string of length ℓ freely suspended from one end at $z=0$ in a gravitational field. The tension is then $\tau(z) = \mu g\,(\ell - z)$.

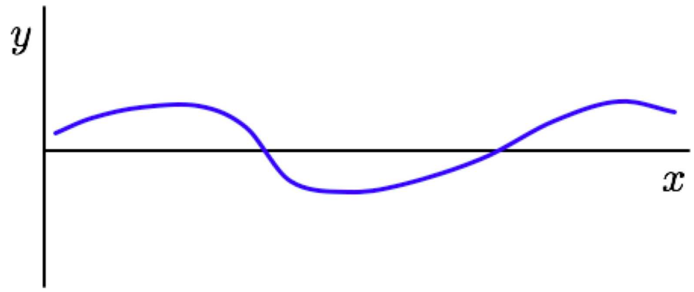

Figure 14.1: A string is described by the vertical displacement field $y(x,t)$.

The action for the string is now a double integral,

$$S = \int_{t_a}^{t_b} \!dt \int_{x_a}^{x_b} \!dx \; \mathcal{L}(y, \dot{y}, y'; x, t) \ , \tag{14.6}$$

where $y(x,t)$ is the vertical displacement field. Typically, we have $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2$. The first variation of S is

$$\delta S = \int_{x_a}^{x_b} \!dx \int_{t_a}^{t_b} \!dt \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\right] \delta y \tag{14.7}$$

$$+ \int_{x_a}^{x_b} \!dx \left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\, \delta y \right]_{t=t_a}^{t=t_b} + \int_{t_a}^{t_b} \!dt \left[\frac{\partial \mathcal{L}}{\partial y'}\, \delta y \right]_{x=x_b}^{x=x_a} \ , \tag{14.8}$$

which simply recapitulates the general result from eqn. 14.203. There are two boundary terms, one of which is an integral over time and the other an integral over space. The first boundary term vanishes provided $\delta y(x,t_a) = \delta y(x,t_b) = 0$. The second boundary term vanishes provided $\tau(x)\, y'(x)\, \delta y(x) = 0$ at $x = x_a$ and $x = x_b$, for all t. Assuming $\tau(x)$ does not vanish, this can happen in one of two ways: at each endpoint either $y(x)$ is fixed or $y'(x)$ vanishes.

Assuming that either $y(x)$ is fixed or $y'(x) = 0$ at the endpoints $x = x_a$ and $x = x_b$, the Euler-Lagrange equations for the string are obtained by setting $\delta S = 0$:

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= \frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) \\ &= \frac{\partial}{\partial x}\left[\tau(x)\, \frac{\partial y}{\partial x}\right] - \mu(x)\, \frac{\partial^2 y}{\partial t^2} \ , \end{aligned} \tag{14.9}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. When $\tau(x) = \tau$ and $\mu(x) = \mu$ are both constants, we obtain the Helmholtz equation,

$$\frac{1}{c^2}\, \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \ , \tag{14.10}$$

which is the wave equation for the string, where $c = \sqrt{\tau/\mu}$ has dimensions of velocity. We will now see that c is the speed of wave propagation on the string.

14.2 d'Alembert's Solution to the Wave Equation

Let us define two new variables,

$$u \equiv x - ct \qquad , \qquad v \equiv x + ct \ . \tag{14.11}$$

We then have

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\,\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\,\frac{\partial}{\partial v} \qquad = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{14.12}$$

$$\frac{1}{c}\,\frac{\partial}{\partial t} = \frac{1}{c}\,\frac{\partial u}{\partial t}\,\frac{\partial}{\partial u} + \frac{1}{c}\,\frac{\partial v}{\partial t}\,\frac{\partial}{\partial v} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \ . \tag{14.13}$$

Thus,

$$\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -4\,\frac{\partial^2}{\partial u\,\partial v} \ . \tag{14.14}$$

Thus, the wave equation may be solved:

$$\frac{\partial^2 y}{\partial u\,\partial v} = 0 \qquad \Longrightarrow \qquad y(u,v) = f(u) + g(v) \ , \tag{14.15}$$

where $f(u)$ and $g(v)$ are arbitrary functions. For the moment, we work with an infinite string, so we have no spatial boundary conditions to satisfy. Note that $f(u)$ describes a right-moving disturbance, and $g(v)$ describes a left-moving disturbance:

$$y(x,t) = f(x - ct) + g(x + ct) \ . \tag{14.16}$$

We do, however, have boundary conditions in time. At $t = 0$, the configuration of the string is given by $y(x,0)$, and its instantaneous vertical velocity is $\dot{y}(x,0)$. We then have

$$y(x,0) = f(x) + g(x) \tag{14.17}$$

$$\dot{y}(x,0) = -c\,f'(x) + c\,g'(x) \ , \tag{14.18}$$

hence

$$f'(x) = \tfrac{1}{2}\,y'(x,0) - \tfrac{1}{2c}\,\dot{y}(x,0) \tag{14.19}$$

$$g'(x) = \tfrac{1}{2}\,y'(x,0) + \tfrac{1}{2c}\,\dot{y}(x,0) \ , \tag{14.20}$$

and integrating we obtain the right and left moving components

$$f(\xi) = \tfrac{1}{2}\, y(\xi,0) - \tfrac{1}{2c}\int\limits_0^{\xi} d\xi'\; \dot y(\xi',0) - \mathcal{C} \tag{14.21}$$

$$g(\xi) = \tfrac{1}{2}\, y(\xi,0) + \tfrac{1}{2c}\int\limits_0^{\xi} d\xi'\; \dot y(\xi',0) + \mathcal{C}\ , \tag{14.22}$$

where $\mathcal{C}$ is an arbitrary constant. Adding these together, we obtain the full solution

$$y(x,t) = \tfrac{1}{2}\Big[y(x-ct,0) + y(x+ct,0)\Big] + \tfrac{1}{2c}\int\limits_{x-ct}^{x+ct} d\xi\; \dot y(\xi,0)\ , \tag{14.23}$$

valid for all times.

14.2.1 Energy density and energy current

The Hamiltonian density for a string is

$$\mathcal{H} = \mathit{\Pi}\,\dot y - \mathcal{L}\ , \tag{14.24}$$

where

$$\mathit{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot y} = \mu\,\dot y \tag{14.25}$$

is the momentum density. Thus,

$$\mathcal{H} = \frac{\mathit{\Pi}^2}{2\mu} + \tfrac{1}{2}\tau\, y'^2\ . \tag{14.26}$$

Expressed in terms of $\dot y$ rather than $\mathit{\Pi}$, this is the energy density $\mathcal{E}$,

$$\mathcal{E} = \tfrac{1}{2}\mu\,\dot y^2 + \tfrac{1}{2}\tau\, y'^2\ . \tag{14.27}$$

We now evaluate $\dot{\mathcal{E}}$ for a solution to the equations of motion:

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} &= \mu\,\frac{\partial y}{\partial t}\,\frac{\partial^2 y}{\partial t^2} + \tau\,\frac{\partial y}{\partial x}\,\frac{\partial^2 y}{\partial t\,\partial x}\\ &= \tau\,\frac{\partial y}{\partial t}\,\frac{\partial}{\partial x}\bigg(\tau\,\frac{\partial y}{\partial x}\bigg) + \tau\,\frac{\partial y}{\partial x}\,\frac{\partial^2 y}{\partial t\,\partial x}\\ &= \frac{\partial}{\partial x}\bigg[\tau\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t}\bigg] \equiv -\frac{\partial j_{\mathcal{E}}}{\partial x}\ ,\end{aligned} \tag{14.28}$$

where the *energy current density* (or energy flux) is

$$j_{\mathcal{E}} = -\tau\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t}\ . \tag{14.29}$$

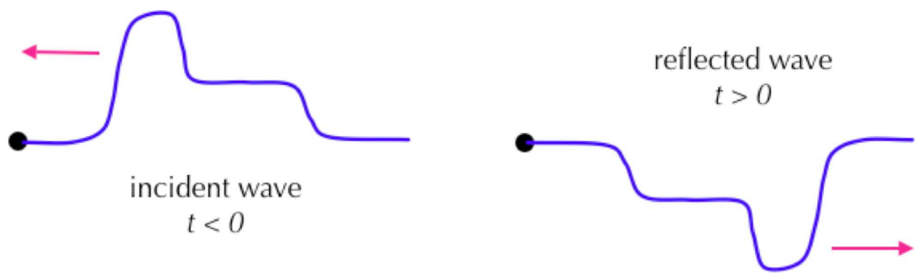

Figure 14.2: Reflection of a pulse at an interface at $x = 0$, with $y(0,t) = 0$.

We therefore have that solutions of the equation of motion also obey the energy *continuity equation*

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j_{\mathcal{E}}}{\partial x} = 0 \ . \tag{14.30}$$

Let us integrate the above equation between points x_1 and x_2. We obtain

$$\frac{\partial}{\partial t} \int\limits_{x_1}^{x_2} dx \, \mathcal{E}(x,t) = -\int\limits_{x_1}^{x_2} dx \, \frac{\partial j_{\mathcal{E}}(x,t)}{\partial x} = j_{\mathcal{E}}(x_1,t) - j_{\mathcal{E}}(x_2,t) \ , \tag{14.31}$$

which says that the time rate of change of the energy contained in the interval $\big[x_1, x_2\big]$ is equal to the difference between the entering and exiting energy flux.

When $\tau(x) = \tau$ and $\mu(x) = \mu$, we have

$$y(x,t) = f(x - ct) + g(x + ct) \tag{14.32}$$

and we find

$$\mathcal{E}(x,t) = \tau\big[f'(x - ct)\big]^2 + \tau\big[g'(x + ct)\big]^2 \tag{14.33}$$

$$j_{\mathcal{E}}(x,t) = c\tau\big[f'(x - ct)\big]^2 - c\tau\big[g'(x + ct)\big]^2 \ , \tag{14.34}$$

which are each sums over right-moving and left-moving contributions.

14.2.2 Reflection at an interface

Consider a semi-infinite string on the interval $\big[0, \infty\big]$, with $y(0,t) = 0$. We can still invoke d'Alembert's solution, $y(x,t) = f(x - ct) + g(x + ct)$, but we must demand

$$y(0,t) = f(-ct) + g(ct) = 0 \quad \Rightarrow \quad f(\xi) = -g(-\xi) \ . \tag{14.35}$$

Thus,

$$y(x,t) = g(ct + x) - g(ct - x) \ . \tag{14.36}$$

Now suppose $g(\xi)$ describes a pulse, and is nonzero only within a neighborhood of $\xi = 0$. For large negative values of t, the right-moving part, $-g(ct - x)$, is negligible everywhere,

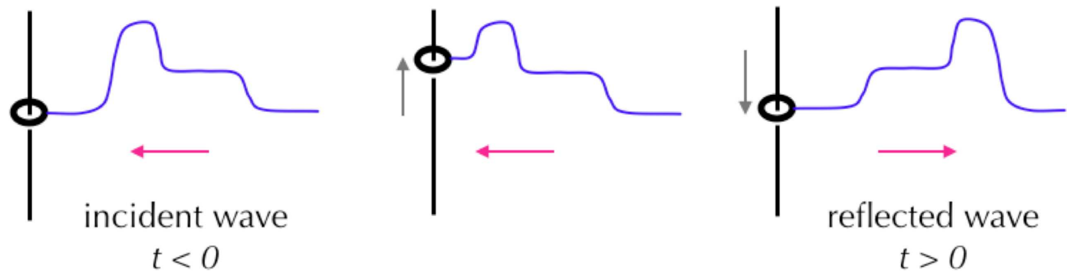

Figure 14.3: Reflection of a pulse at an interface at $x = 0$, with $y'(0,t) = 0$.

since $x > 0$ means that the argument $ct - x$ is always large and negative. On the other hand, the left moving part $g(ct + x)$ is nonzero for $x \approx -ct > 0$. Thus, for $t < 0$ we have a left-moving pulse incident from the right. For $t > 0$, the situation is reversed, and the left-moving component is negligible, and we have a right moving reflected wave. However, the minus sign in eqn. 14.35 means that the reflected wave is *inverted.*

If instead of fixing the endpoint at $x = 0$ we attach this end of the string to a massless ring which frictionlessly slides up and down a vertical post, then we must have $y'(0,t) = 0$, else there is a finite vertical force on the massless ring, resulting in infinite acceleration. We again write $y(x,t) = f(x - ct) + g(x + ct)$, and we invoke

$$y'(0,t) = f'(-ct) + g'(ct) \quad \Rightarrow \quad f'(\xi) = -g'(-\xi) \ , \tag{14.37}$$

which, upon integration, yields $f(\xi) = g(-\xi)$, and therefore

$$y(x,t) = g(ct + x) + g(ct - x) \ . \tag{14.38}$$

The reflected pulse is now 'right-side up', in contrast to the situation with a fixed endpoint.

14.2.3 Mass point on a string

Next, consider the case depicted in Fig. 14.4, where a point mass m is affixed to an infinite string at $x = 0$. Let us suppose that at large negative values of t, a right moving wave $f(ct - x)$ is incident from the left. The full solution may then be written as a sum of incident, reflected, and transmitted waves:

$$x < 0 \quad : \quad y(x,t) = f(ct - x) + g(ct + x) \tag{14.39}$$

$$x > 0 \quad : \quad y(x,t) = h(ct - x) \ . \tag{14.40}$$

At $x = 0$, we invoke Newton's second Law, $F = ma$:

$$m\,\ddot{y}(0,t) = \tau\, y'(0^+,t) - \tau\, y'(0^-,t) \ . \tag{14.41}$$

Any discontinuity in the derivative $y'(x,t)$ at $x = 0$ results in an acceleration of the point mass. Note that

$$y'(0^-,t) = -f'(ct) + g'(ct) \qquad , \qquad y'(0^+,t) = -h'(ct) \ . \tag{14.42}$$

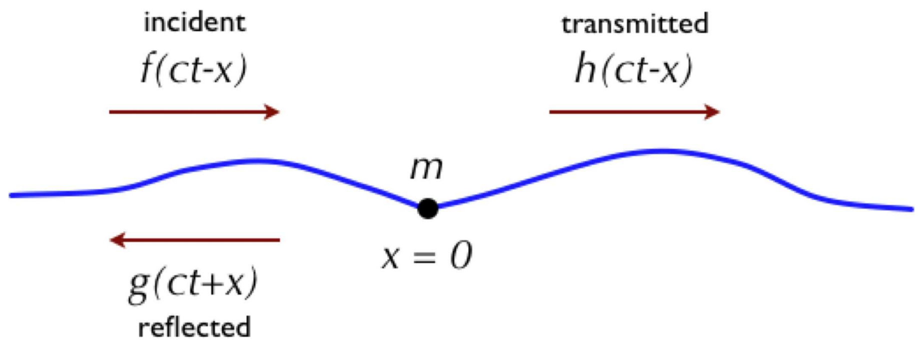

Figure 14.4: Reflection and transmission at an impurity. A point mass m is affixed to an infinite string at $x = 0$.

Further invoking continuity at $x = 0$, *i.e.* $y(0^-, t) = y(0^+, t)$, we have

$$h(\xi) = f(\xi) + g(\xi) \ , \tag{14.43}$$

and eqn. 14.41 becomes

$$g''(\xi) + \frac{2\tau}{mc^2}\, g'(\xi) = -f''(\xi) \ . \tag{14.44}$$

We solve this equation by Fourier analysis:

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad , \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi} \ . \tag{14.45}$$

Defining $\kappa \equiv 2\tau/mc^2 = 2\mu/m$, we have

$$\big[-k^2 + i\kappa k \big]\, \hat{g}(k) = k^2\, \hat{f}(k) \ . \tag{14.46}$$

We then have

$$\hat{g}(k) = -\frac{k}{k - i\kappa}\, \hat{f}(k) \equiv r(k)\, \hat{f}(k) \tag{14.47}$$

$$\hat{h}(k) = \frac{-i\kappa}{k - i\kappa}\, \hat{f}(k) \equiv t(k)\, \hat{f}(k) \ , \tag{14.48}$$

where $r(k)$ and $t(k)$ are the reflection and transmission amplitudes, respectively. Note that

$$t(k) = 1 + r(k) \ . \tag{14.49}$$

In real space, we have

$$h(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, \hat{f}(k)\, e^{ik\xi} \tag{14.50}$$

$$= \int\limits_{-\infty}^{\infty} d\xi' \left[\int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')} \right] f(\xi') \tag{14.51}$$

$$\equiv \int\limits_{-\infty}^{\infty} d\xi'\, \mathcal{T}(\xi-\xi')\, f(\xi') \ , \tag{14.52}$$

where

$$\mathcal{T}(\xi-\xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')} \ , \tag{14.53}$$

is the transmission kernel in real space. For our example with $r(k) = -i\kappa/(k - i\kappa)$, the integral is done easily using the method of contour integration:

$$\mathcal{T}(\xi-\xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{-i\kappa}{k-i\kappa}\, e^{ik(\xi-\xi')} = \kappa\, e^{-\kappa(\xi-\xi')}\, \Theta(\xi-\xi') \ . \tag{14.54}$$

Therefore,

$$h(\xi) = \kappa \int\limits_{-\infty}^{\xi} d\xi'\, e^{-\kappa(\xi-\xi')}\, f(\xi') \ , \tag{14.55}$$

and of course $g(\xi) = h(\xi) - f(\xi)$. Note that $m = \infty$ means $\kappa = 0$, in which case $r(k) = -1$ and $t(k) = 0$. Thus we recover the inversion of the pulse shape under reflection found earlier.

For example, let the incident pulse shape be $f(\xi) = b\,\Theta\big(a - |\xi|\big)$. Then

$$\begin{aligned} h(\xi) &= \kappa \int\limits_{-\infty}^{\xi} d\xi'\, e^{-\kappa(\xi-\xi')}\, b\,\Theta(a-\xi')\,\Theta(a+\xi') \\ &= b\, e^{-\kappa\xi} \Big[e^{\kappa \min(a,\xi)} - e^{-\kappa a} \Big]\, \Theta(\xi+a) \ . \end{aligned} \tag{14.56}$$

Taking cases,

$$h(\xi) = \begin{cases} 0 & \text{if } \xi < -a \\ b\left(1 - e^{-\kappa(a+\xi)}\right) & \text{if } -a < \xi < a \\ 2b\, e^{-\kappa\xi}\, \sinh(\kappa a) & \text{if } \xi > a \ . \end{cases} \tag{14.57}$$

In Fig. 14.5 we show the reflection and transmission of this square pulse for two different values of κa.

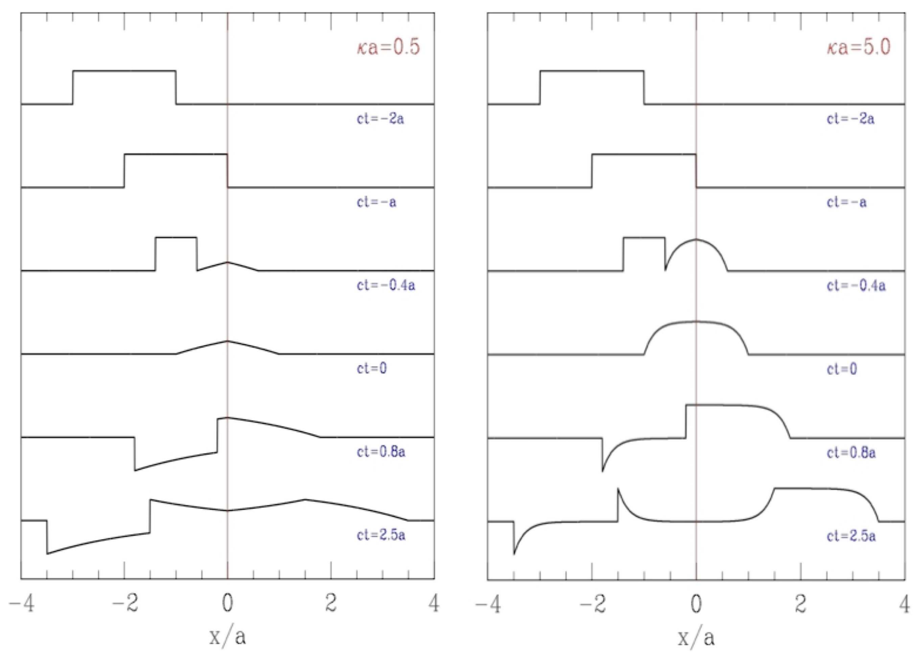

Figure 14.5: Reflection and transmission of a square wave pulse by a point mass at $x = 0$ The configuration of the string is shown for six different times, for $\kappa a = 0.5$ (left panel) and $\kappa a = 5.0$ (right panel). Note that the $\kappa a = 0.5$ case, which corresponds to a large mass $m = 2\mu/\kappa$, results in strong reflection with inversion, and weak transmission. For large κ, corresponding to small mass m, the reflection is weak and the transmission is strong.

14.2.4 Interface between strings of different mass density

Consider the situation in fig. 14.6, where the string for $x < 0$ is of density $\mu_{\rm L}$ and for $x > 0$ is of density $\mu_{\rm R}$. The d'Alembert solution in the two regions, with an incoming wave from the left, is

$$x < 0: \quad y(x,t) = f(c_{\rm L}t - x) + g(c_{\rm L}t + x) \tag{14.58}$$

$$x > 0: \quad y(x,t) = h(c_{\rm R}t - x)\ . \tag{14.59}$$

At $x = 0$ we have

$$f(c_{\rm L}t) + g(c_{\rm L}t) = h(c_{\rm R}t) \tag{14.60}$$

$$-f'(c_{\rm L}t) + g'(c_{\rm L}t) = -h'(c_{\rm R}t)\ , \tag{14.61}$$

where the second equation follows from $\tau\, y'(0^+,t) = \tau\, y'(0^-,t)$, so there is no finite vertical force on the infinitesimal interval bounding $x = 0$, which contains infinitesimal mass. Defining $\alpha \equiv c_{\rm R}/c_{\rm L}$, we integrate the second of these equations and have

$$f(\xi) + g(\xi) = h(\alpha\,\xi) \tag{14.62}$$

$$f(\xi) - g(\xi) = \alpha^{-1}\, h(\alpha\,\xi)\ . \tag{14.63}$$

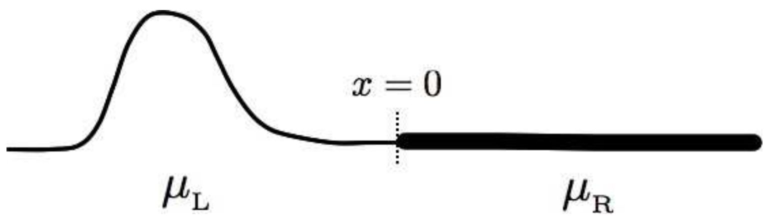

Figure 14.6: String formed from two semi-infinite regions of different densities..

Note that $y(\pm\infty, 0) = 0$ fixes the constant of integration. The solution is then

$$g(\xi) = \frac{\alpha - 1}{\alpha + 1}\, f(\xi) \tag{14.64}$$

$$h(\xi) = \frac{2\alpha}{\alpha + 1}\, f(\xi/\alpha)\ . \tag{14.65}$$

Thus,

$$x < 0: \quad y(x,t) = f\big(c_\mathrm{L} t - x\big) + \left(\frac{\alpha - 1}{\alpha + 1}\right) f\big(c_\mathrm{L} t + x\big) \tag{14.66}$$

$$x > 0: \quad y(x,t) = \frac{2\alpha}{\alpha + 1}\, f\big((c_\mathrm{R} t - x)/\alpha\big)\ . \tag{14.67}$$

It is instructive to compute the total energy in the string. For large negative values of the time t, the entire disturbance is confined to the region $x < 0$. The energy is

$$E(-\infty) = \tau \int\limits_{-\infty}^{\infty} d\xi\, \big[f'(\xi)\big]^2\ . \tag{14.68}$$

For large positive times, the wave consists of the left-moving reflected $g(\xi)$ component in the region $x < 0$ and the right-moving transmitted component $h(\xi)$ in the region $x > 0$. The energy in the reflected wave is

$$E_\mathrm{L}(+\infty) = \tau \left(\frac{\alpha - 1}{\alpha + 1}\right)^2 \int\limits_{-\infty}^{\infty} d\xi\, \big[f'(\xi)\big]^2\ . \tag{14.69}$$

For the transmitted portion, we use

$$y'(x > 0, t) = \frac{2}{\alpha + 1}\, f'\big((c_\mathrm{R} t - x)/\alpha\big) \tag{14.70}$$

to obtain

$$
\begin{aligned}
E_{\mathrm{R}}(\infty) &= \frac{4\tau}{(\alpha+1)^2} \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi/\alpha)\big]^2 \\
&= \frac{4\alpha\tau}{(\alpha+1)^2} \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi)\big]^2 \ . \qquad (14.71)
\end{aligned}
$$

Thus, $E_{\mathrm{L}}(\infty) + E_{\mathrm{R}}(\infty) = E(-\infty)$, and energy is conserved.

14.3 Finite Strings : Bernoulli's Solution

Suppose $x_a = 0$ and $x_b = L$ are the boundaries of the string, where $y(0,t) = y(L,t) = 0$. Again we write

$$
y(x,t) = f(x - ct) + g(x + ct) \ . \qquad (14.72)
$$

Applying the boundary condition at $x_a = 0$ gives, as earlier,

$$
y(x,t) = g(ct + x) - g(ct - x) \ . \qquad (14.73)
$$

Next, we apply the boundary condition at $x_b = L$, which results in

$$
g(ct + L) - g(ct - L) = 0 \quad \Longrightarrow \quad g(\xi) = g(\xi + 2L) \ . \qquad (14.74)
$$

Thus, $g(\xi)$ is periodic, with period $2L$. Any such function may be written as a Fourier sum,

$$
g(\xi) = \sum_{n=1}^{\infty} \bigg\{ \mathcal{A}_n \cos\!\left(\frac{n\pi\xi}{L}\right) + \mathcal{B}_n \sin\!\left(\frac{n\pi\xi}{L}\right) \bigg\} \ . \qquad (14.75)
$$

The full solution for $y(x,t)$ is then

$$
\begin{aligned}
y(x,t) &= g(ct + x) - g(ct - x) \\
&= \left(\frac{2}{\mu L}\right)^{1/2} \sum_{n=1}^{\infty} \sin\!\left(\frac{n\pi x}{L}\right) \bigg\{ A_n \cos\!\left(\frac{n\pi ct}{L}\right) + B_n \sin\!\left(\frac{n\pi ct}{L}\right) \bigg\} \ , \qquad (14.76)
\end{aligned}
$$

where $A_n = \sqrt{2\mu L}\, \mathcal{B}_n$ and $B_n = -\sqrt{2\mu L}\, \mathcal{A}_n$. This is known as Bernoulli's solution.

We define the functions

$$
\psi_n(x) \equiv \left(\frac{2}{\mu L}\right)^{1/2} \sin\!\left(\frac{n\pi x}{L}\right) \ . \qquad (14.77)
$$

We also write

$$
k_n \equiv \frac{n\pi x}{L} \quad , \quad \omega_n \equiv \frac{n\pi c}{L} \quad , \quad n = 1, 2, 3, \ldots, \infty \ . \qquad (14.78)
$$

Thus, $\psi_n(x) = \sqrt{2/\mu L}\,\sin(k_n x)$ has $(n+1)$ nodes at $x = jL/n$, for $j \in \{0, \ldots, n\}$. Note that

$$\langle\, \psi_m \,|\, \psi_n \,\rangle \equiv \int\limits_0^L \! dx\, \mu\, \psi_m(x)\, \psi_n(x) = \delta_{mn} \ . \tag{14.79}$$

Furthermore, this basis is complete:

$$\mu \sum_{n=1}^{\infty} \psi_n(x)\, \psi_n(x') = \delta(x - x') \ . \tag{14.80}$$

Our general solution is thus equivalent to

$$y(x,0) = \sum_{n=1}^{\infty} A_n\, \psi_n(x) \tag{14.81}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\, B_n\, \psi_n(x) \ . \tag{14.82}$$

The Fourier coefficients $\{A_n, B_n\}$ may be extracted from the initial data using the orthonormality of the basis functions and their associated resolution of unity:

$$A_n = \int\limits_0^L \! dx\, \mu\, \psi_n(x)\, y(x,0) \tag{14.83}$$

$$B_n = \frac{L}{n\pi c} \int\limits_0^L \! dx\, \mu\, \psi_n(x)\, \dot{y}(x,0) \ . \tag{14.84}$$

As an example, suppose our initial configuration is a triangle, with

$$y(x,0) = \begin{cases} \frac{2b}{L}\, x & \text{if } 0 \le x \le \frac{1}{2}L \\ \\ \frac{2b}{L}\,(L-x) & \text{if} \frac{1}{2}L \le x \le L \ , \end{cases} \tag{14.85}$$

and $\dot{y}(x,0) = 0$. Then $B_n = 0$ for all n, while

$$\begin{aligned} A_n &= \left(\frac{2\mu}{L}\right)^{1/2} \cdot \frac{2b}{L} \left\{ \int\limits_0^{L/2} \! dx\, x\, \sin\!\left(\frac{n\pi x}{L}\right) + \int\limits_{L/2}^{L} \! dx\, (L-x)\, \sin\!\left(\frac{n\pi x}{L}\right) \right\} \\ &= (2\mu L)^{1/2} \cdot \frac{4b}{n^2 \pi^2}\, \sin\!\left(\tfrac{1}{2} n\pi\right) \delta_{n,\mathrm{odd}} \ , \end{aligned} \tag{14.86}$$

after changing variables to $x = L\theta/n\pi$ and using $\theta \sin\theta\, d\theta = d\big(\sin\theta - \theta\, \cos\theta\big)$. Another way to write this is to separately give the results for even and odd coefficients:

$$A_{2k} = 0 \qquad , \qquad A_{2k+1} = \frac{4b}{\pi^2}\, (2\mu L)^{1/2} \cdot \frac{(-1)^k}{(2k+1)^2} \ . \tag{14.87}$$

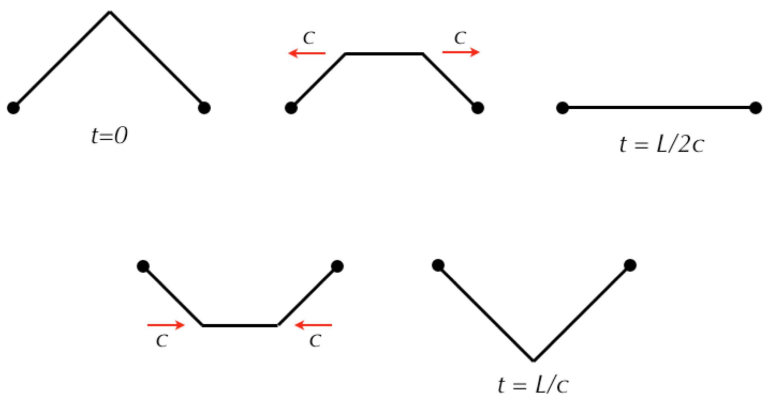

Figure 14.7: Evolution of a string with fixed ends starting from an isosceles triangle shape.

Note that each $\psi_{2k}(x) = -\psi_{2k}(L-x)$ is antisymmetric about the midpoint $x = \frac{1}{2}L$, for all k. Since our initial conditions are that $y(x,0)$ is symmetric about $x = \frac{1}{2}L$, none of the even order eigenfunctions can enter into the expansion, precisely as we have found. The d'Alembert solution to this problem is particularly simple and is shown in Fig. 14.7. Note that $g(x) = \frac{1}{2}y(x,0)$ must be extended to the entire real line. We know that $g(x) = g(x+2L)$ is periodic with spatial period $2L$, but how to we extend $g(x)$ from the interval $\big[0,L\big]$ to the interval $\big[-L,0\big]$? To do this, we use $y(x,0) = g(x) - g(-x)$, which says that $g(x)$ must be *antisymmetric*, *i.e.* $g(x) = -g(-x)$. Equivalently, $\dot{y}(x,0) = cg'(x) - cg'(-x) = 0$, which integrates to $g(x) = -g(-x)$.

14.4 Sturm-Liouville Theory

Consider the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\,\mu(x)\,\dot{y}^2 - \tfrac{1}{2}\,\tau(x)\,y'^2 - \tfrac{1}{2}\,v(x)\,y^2 \ . \tag{14.88}$$

The last term is new and has the physical interpretation of a harmonic potential which attracts the string to the line $y = 0$. The Euler-Lagrange equations are then

$$-\frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] + v(x)\,y = -\mu(x)\,\frac{\partial^2 y}{\partial t^2} \ . \tag{14.89}$$

This equation is invariant under time translation. Thus, if $y(x,t)$ is a solution, then so is $y(x,t+t_0)$, for any t_0. This means that the solutions can be chosen to be eigenstates of the operator ∂_t, which is to say $y(x,t) = \psi(x)\,e^{-i\omega t}$. Because the coefficients are real, both y and y^* are solutions, and taking linear combinations we have

$$y(x,t) = \psi(x)\,\cos(\omega t + \phi) \ . \tag{14.90}$$

Plugging this into eqn. 14.89, we obtain

$$-\frac{d}{dx}\Big[\tau(x)\,\psi'(x)\Big] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \ . \tag{14.91}$$

This is the Sturm-Liouville equation. There are four types of boundary conditions that we shall consider:

1. Fixed endpoint: $\psi(x) = 0$, where $x = x_{a,b}$.
2. Natural: $\tau(x)\,\psi'(x) = 0$, where $x = x_{a,b}$.
3. Periodic: $\psi(x) = \psi(x + L)$, where $L = x_b - x_a$.
4. Mixed homogeneous: $\alpha\,\psi(x) + \beta\,\psi'(x) = 0$, where $x = x_{a,b}$.

The Sturm-Liouville equation is an eigenvalue equation. The eigenfunctions $\{\psi_n(x)\}$ satisfy

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_n'(x)\Big] + v(x)\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x)\ . \tag{14.92}$$

Now suppose we a second solution $\psi_m(x)$, satisfying

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_m'(x)\Big] + v(x)\,\psi_m(x) = \omega_m^2\,\mu(x)\,\psi_m(x)\ . \tag{14.93}$$

Now multiply (14.92)* by $\psi_m(x)$ and (14.93) by $\psi_n^*(x)$ and subtract, yielding

$$\psi_n^*\,\frac{d}{dx}\Big[\tau\,\psi_m'\Big] - \psi_m\,\frac{d}{dx}\Big[\tau\,\psi'^*_n\Big] = \big(\omega_n^{*\,2} - \omega_m^2\big)\,\mu\,\psi_m\,\psi_n^* \tag{14.94}$$

$$= \frac{d}{dx}\Big[\tau\,\psi_n^*\,\psi_m' - \tau\,\psi_m\,\psi'^*_n\Big]\ . \tag{14.95}$$

We integrate this equation over the length of the string, to get

$$\begin{aligned}\big(\omega_n^{*\,2} - \omega_m^2\big)\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi_n^*(x)\,\psi_m(x) &= \Big[\tau(x)\,\psi_n^*(x)\,\psi_m'(x) - \tau(x)\,\psi_m(x)\,\psi'^*_n(x)\Big]_{x=x_a}^{x=x_b}\\ &= 0\ .\end{aligned} \tag{14.96}$$

The RHS vanishes for any of the four types of boundary conditions articulated above.

Thus, we have

$$\big(\omega_n^{*\,2} - \omega_m^2\big)\,\big\langle\,\psi_n\,\big|\,\psi_m\,\big\rangle = 0\ , \tag{14.97}$$

where the inner product is defined as

$$\big\langle\,\psi\,\big|\,\phi\,\big\rangle \equiv \int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi^*(x)\,\phi(x)\ . \tag{14.98}$$

Note that the distribution $\mu(x)$ is non-negative definite. Setting $m = n$, we have $\big\langle\,\psi_n\,\big|\,\psi_n\,\big\rangle \geq 0$, and hence $\omega_n^{*\,2} = \omega_n^2$, which says that $\omega_n^2 \in \mathbb{R}$. When $\omega_m^2 \neq \omega_n^2$, the eigenfunctions are orthogonal with respect to the above inner product. In the case of degeneracies, we may invoke the Gram-Schmidt procedure, which orthogonalizes the eigenfunctions within a given degenerate subspace. Since the Sturm-Liouville equation is linear, we may normalize the eigenfunctions, taking

$$\langle \psi_m \,|\, \psi_n \rangle = \delta_{mn}. \tag{14.99}$$

Finally, since the coefficients in the Sturm-Liouville equation are all real, we can and henceforth do choose the eigenfunctions themselves to be real.

Another important result, which we will not prove here, is the *completeness* of the eigenfunction basis. Completeness means

$$\mu(x) \sum_n \psi_n^*(x)\, \psi_n(x') = \delta(x - x') \ . \tag{14.100}$$

Thus, any function can be expanded in the eigenbasis, *viz.*

$$\phi(x) = \sum_n C_n\, \psi_n(x) \qquad , \qquad C_n = \langle \psi_n \,|\, \phi \rangle \ . \tag{14.101}$$

14.4.1 Variational method

Consider the functional

$$\omega^2\big[\psi(x)\big] = \frac{\frac{1}{2}\int\limits_{x_a}^{x_b} dx \, \Big\{ \tau(x)\, \psi'^2(x) + v(x)\, \psi^2(x) \Big\}}{\frac{1}{2}\int\limits_{x_a}^{x_b} dx\, \mu(x)\, \psi^2(x)} \equiv \frac{\mathcal{N}}{\mathcal{D}} \ . \tag{14.102}$$

The variation is

$$\begin{aligned} \delta\omega^2 &= \frac{\delta\mathcal{N}}{\mathcal{D}} - \frac{\mathcal{N}\,\delta\mathcal{D}}{\mathcal{D}^2} \\ &= \frac{\delta\mathcal{N} - \omega^2\,\delta\mathcal{D}}{\mathcal{D}} \ . \end{aligned} \tag{14.103}$$

Thus,

$$\delta\omega^2 = 0 \qquad \Longrightarrow \qquad \delta\mathcal{N} = \omega^2\,\delta\mathcal{D} \ , \tag{14.104}$$

which says

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{d\psi(x)}{dx}\bigg] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \ , \tag{14.105}$$

which is the Sturm-Lioiuville equation. In obtaining this equation, we have dropped a boundary term, which is correct provided

$$\Big[\tau(x)\,\psi'(x)\,\psi(x)\Big]_{x=x_a}^{x=x_b} = 0 \ . \tag{14.106}$$

This condition is satisfied for any of the first three classes of boundary conditions: $\psi = 0$ (fixed endpoint), $\tau\,\psi' = 0$ (natural), or $\psi(x_a) = \psi(x_b)$, $\psi'(x_a) = \psi'(x_b)$ (periodic). For the fourth class of boundary conditions, $\alpha\psi + \beta\psi' = 0$ (mixed homogeneous), the Sturm-Liouville equation may still be derived, provided one uses a slightly different functional,

$$\omega^2\big[\psi(x)\big] = \frac{\widetilde{\mathcal{N}}}{\mathcal{D}} \qquad \text{with} \qquad \widetilde{\mathcal{N}} = \mathcal{N} + \frac{\alpha}{2\beta}\Big[\tau\big(x_b\big)\,\psi^2\big(x_b\big) - \tau\big(x_a\big)\,\psi^2\big(x_a\big)\Big] \ , \tag{14.107}$$

since then

$$\begin{aligned}\delta\widetilde{\mathcal{N}} - \widetilde{\mathcal{N}}\,\delta D = \int\limits_{x_a}^{x_b}\!dx\,\bigg\{ -\frac{d}{dx}\bigg[\tau(x)\,\frac{d\psi(x)}{dx}\bigg] + v(x)\,\psi(x) - \omega^2\mu(x)\,\psi(x)\bigg\}\,\delta\psi(x)\\ + \bigg[\tau(x)\bigg(\psi'(x) + \frac{\alpha}{\beta}\,\psi(x)\bigg)\,\delta\psi(x)\bigg]_{x=x_a}^{x=x_b} \ ,\end{aligned} \tag{14.108}$$

and the last term vanishes as a result of the boundary conditions.

For all four classes of boundary conditions we may write

$$\omega^2\big[\psi(x)\big] = \frac{\int\limits_{x_a}^{x_b}\!dx\,\psi(x)\,\overbrace{\Big[-\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x)\Big]}^{K}\,\psi(x)}{\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi^2(x)} \tag{14.109}$$

If we expand $\psi(x)$ in the basis of eigenfunctions of the Sturm-Liouville operator K,

$$\psi(x) = \sum_{n=1}^{\infty} \mathcal{C}_n\,\psi_n(x) \ , \tag{14.110}$$

we obtain

$$\omega^2\big[\psi(x)\big] = \omega^2(\mathcal{C}_1,\ldots,\mathcal{C}_\infty) = \frac{\sum_{j=1}^{\infty}|\mathcal{C}_j|^2\,\omega_j^2}{\sum_{k=1}^{\infty}|\mathcal{C}_k|^2} \ . \tag{14.111}$$

If $\omega_1^2 \le \omega_2^2 \le \ldots$, then we see that $\omega^2 \ge \omega_1^2$, so an arbitrary function $\psi(x)$ will always yield an upper bound to the lowest eigenvalue.

As an example, consider a violin string ($v = 0$) with a mass m affixed in the center. We write $\mu(x) = \mu + m\,\delta(x - \frac{1}{2}L)$, hence

$$\omega^2\big[\psi(x)\big] = \frac{\tau\int\limits_0^L dx\,\psi'^2(x)}{m\,\psi^2(\frac{1}{2}L) + \mu\int\limits_0^L dx\,\psi^2(x)} \tag{14.112}$$

Now consider a trial function

$$\psi(x) = \begin{cases} A\,x^\alpha & \text{if} \quad 0 \le x \le \frac{1}{2}L \\ \\ A\,(L-x)^\alpha & \text{if} \quad \frac{1}{2}L \le x \le L \ . \end{cases} \tag{14.113}$$

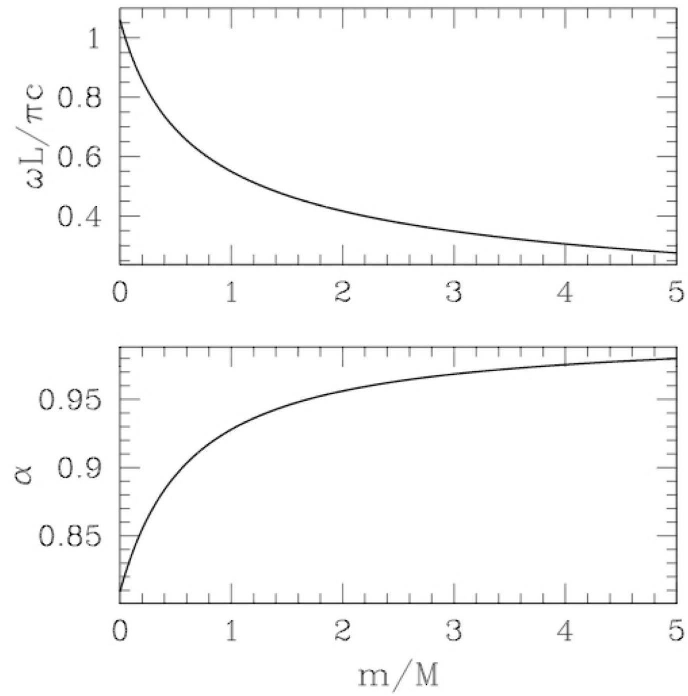

Figure 14.8: One-parameter variational solution for a string with a mass m affixed at $x = \frac{1}{2}L$.

The functional $\omega^2\big[\psi(x)\big]$ now becomes an ordinary function of the trial parameter α, with

$$\omega^2(\alpha) = \frac{2\tau\int_0^{L/2} dx\,\alpha^2\,x^{2\alpha-2}}{m\left(\frac{1}{2}L\right)^{2\alpha} + 2\mu\int\limits_0^{L/2} dx\,x^{2\alpha}} = \left(\frac{2c}{L}\right)^2 \cdot \frac{\alpha^2(2\alpha+1)}{(2\alpha-1)\big[1+(2\alpha+1)\frac{m}{M}\big]}\ , \qquad (14.114)$$

where $M = \mu L$ is the mass of the string alone. We minimize $\omega^2(\alpha)$ to obtain the optimal solution of this form:

$$\frac{d}{d\alpha}\,\omega^2(\alpha) = 0 \qquad \Longrightarrow \qquad 4\alpha^2 - 2\alpha - 1 + (2\alpha+1)^2\,(\alpha-1)\,\frac{m}{M} = 0\ . \qquad (14.115)$$

For $m/M \to 0$, we obtain $\alpha = \frac{1}{4}\big(1+\sqrt{5}\big) \approx 0.809$. The variational estimate for the eigenvalue is then 6.00% larger than the exact answer $\omega_1^0 = \pi c/L$. In the opposite limit, $m/M \to \infty$, the inertia of the string may be neglected. The normal mode is then piecewise linear, in the shape of an isosceles triangle with base L and height y. The equation of motion is then $m\ddot{y} = -2\tau \cdot (y/\frac{1}{2}L)$, assuming $|y/L| \ll 1$. Thus, $\omega_1 = (2c/L)\sqrt{M/m}$. This is reproduced exactly by the variational solution, for which $\alpha \to 1$ as $m/M \to \infty$.

14.5 Continua in Higher Dimensions

In higher dimensions, we generalize the operator K as follows:

$$K = -\frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\boldsymbol{x})\,\frac{\partial}{\partial x^\beta} + v(\boldsymbol{x}) \ . \tag{14.116}$$

The eigenvalue equation is again

$$K\psi(\boldsymbol{x}) = \omega^2\,\mu(\boldsymbol{x})\,\psi(\boldsymbol{x}) \ , \tag{14.117}$$

and the Green's function again satisfies

$$\Big[K - \omega^2\,\mu(\boldsymbol{x})\Big]\,G_\omega(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \ , \tag{14.118}$$

and has the eigenfunction expansion,

$$G_\omega(\boldsymbol{x},\boldsymbol{x}') = \sum_{n=1}^\infty \frac{\psi_n(\boldsymbol{x})\,\psi_n(\boldsymbol{x}')}{\omega_n^2-\omega^2} \ . \tag{14.119}$$

The eigenfunctions form a complete and orthonormal basis:

$$\mu(\boldsymbol{x})\sum_{n=1}^\infty \psi_n(\boldsymbol{x})\,\psi_n(\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \tag{14.120}$$

$$\int\limits_\Omega d\boldsymbol{x}\,\mu(\boldsymbol{x})\,\psi_m(\boldsymbol{x})\,\psi_n(\boldsymbol{x}) = \delta_{mn} \ , \tag{14.121}$$

where Ω is the region of space in which the continuous medium exists. For purposes of simplicity, we consider here fixed boundary conditions $u(\boldsymbol{x},t)\big|_{\partial\Omega} = 0$, where $\partial\Omega$ is the boundary of Ω. The general solution to the wave equation

$$\bigg[\mu(\boldsymbol{x})\,\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\boldsymbol{x})\,\frac{\partial}{\partial x^\beta} + v(\boldsymbol{x})\bigg]\,u(\boldsymbol{x},t) = 0 \tag{14.122}$$

is

$$u(\boldsymbol{x},t) = \sum_{n=1}^\infty \mathcal{C}_n\,\psi_n(\boldsymbol{x})\,\cos(\omega_n\,t + \delta_n) \ . \tag{14.123}$$

The variational approach generalizes as well. We define

$$\mathcal{N}\big[\psi(\boldsymbol{x})\big] = \int\limits_\Omega d\boldsymbol{x}\,\bigg[\tau_{\alpha\beta}\,\frac{\partial\psi}{\partial x^\alpha}\,\frac{\partial\psi}{\partial x^\beta} + v\,\psi^2\bigg] \tag{14.124}$$

$$\mathcal{D}\big[\psi(\boldsymbol{x})\big] = \int\limits_\Omega d\boldsymbol{x}\,\mu\,\psi^2 \ , \tag{14.125}$$

and

$$\omega^2\big[\psi(\boldsymbol{x})\big] = \frac{\mathcal{N}\big[\psi(\boldsymbol{x})\big]}{\mathcal{D}\big[\psi(\boldsymbol{x})\big]} \ . \tag{14.126}$$

Setting the variation $\delta\omega^2 = 0$ recovers the eigenvalue equation $K\psi = \omega^2\mu\,\psi$.

14.5.1 Membranes

Consider a surface where the height z is a function of the lateral coordinates x and y:

$$z = u(x,y) \ . \tag{14.127}$$

The equation of the surface is then

$$F(x,y,z) = z - u(x,y) = 0 \ . \tag{14.128}$$

Let the differential element of surface area be dS. The projection of this element onto the (x,y) plane is

$$\begin{aligned} dA &= dx\,dy \\ &= \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}\, dS \ . \end{aligned} \tag{14.129}$$

The unit normal $\hat{\boldsymbol{n}}$ is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla} F}{\big|\boldsymbol{\nabla} F\big|} = \frac{\hat{\boldsymbol{z}} - \boldsymbol{\nabla} u}{\sqrt{1 + (\boldsymbol{\nabla} u)^2}} \ . \tag{14.130}$$

Thus,

$$dS = \frac{dx\,dy}{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}} = \sqrt{1 + (\boldsymbol{\nabla} u)^2}\, dx\,dy \ . \tag{14.131}$$

The potential energy for a deformed surface can take many forms. In the case we shall consider here, we consider only the effect of surface tension σ, and we write the potential energy functional as

$$\begin{aligned} U\big[u(x,y,t)\big] &= \sigma \!\int\! dS \\ &= U_0 + \tfrac{1}{2}\!\int\! dA\, (\boldsymbol{\nabla} u)^2 + \ldots \ . \end{aligned} \tag{14.132}$$

The kinetic energy functional is

$$T\big[u(x,y,t)\big] = \tfrac{1}{2}\!\int\! dA\, \mu(\boldsymbol{x})\, (\partial_t u)^2 \ . \tag{14.133}$$

Thus, the action is

$$S\big[u(\boldsymbol{x},t)\big] = \int\! d^2\!x\, \mathcal{L}(u, \boldsymbol{\nabla} u, \partial_t u, \boldsymbol{x}) \ , \tag{14.134}$$

where the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu(\boldsymbol{x})\, (\partial_t u)^2 - \tfrac{1}{2}\sigma(\boldsymbol{x})\, (\boldsymbol{\nabla} u)^2 \ , \tag{14.135}$$

where here we have allowed both $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ to depend on the spatial coordinates. The equations of motion are

$$0 = \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\, \partial_t u} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla} u} - \frac{\partial \mathcal{L}}{\partial u} \tag{14.136}$$

$$= \mu(\boldsymbol{x})\, \frac{\partial^2 u}{\partial t^2} - \boldsymbol{\nabla} \cdot \Big\{ \sigma(\boldsymbol{x})\, \boldsymbol{\nabla} u \Big\} \ . \tag{14.137}$$

14.5.2 Helmholtz equation

When μ and σ are each constant, we obtain the Helmholtz equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\boldsymbol{x},t) = 0 \ , \tag{14.138}$$

with $c = \sqrt{\sigma/\mu}$. The d'Alembert solution still works – waves of arbitrary shape can propagate *in a fixed direction* $\hat{\boldsymbol{k}}$:

$$u(\boldsymbol{x},t) = f(\hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct) \ . \tag{14.139}$$

This is called a *plane wave* because the three dimensional generalization of this wave has wavefronts which are planes. In our case, it might better be called a *line wave*, but people will look at you funny if you say that, so we'll stick with *plane wave*. Note that the locus of points of constant f satisfies

$$\phi(\boldsymbol{x},t) = \hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct = \text{constant} \ , \tag{14.140}$$

and setting $d\phi = 0$ gives

$$\hat{\boldsymbol{k}}\cdot\frac{d\boldsymbol{x}}{dt} = c \ , \tag{14.141}$$

which means that the velocity along $\hat{\boldsymbol{k}}$ is c. The component of $\boldsymbol{x}$ perpendicular to $\hat{\boldsymbol{k}}$ is arbitrary, hence the regions of constant ϕ correspond to lines which are orthogonal to $\hat{\boldsymbol{k}}$.

Owing to the linearity of the wave equation, we can construct arbitrary superpositions of plane waves. The most general solution is written

$$u(\boldsymbol{x},t) = \int\!\frac{d^2\!k}{(2\pi)^2}\,\Big[A(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-ckt)} + B(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+ckt)}\Big] \ . \tag{14.142}$$

The first term in the bracket on the RHS corresponds to a plane wave moving in the $+\hat{\boldsymbol{k}}$ direction, and the second term to a plane wave moving in the $-\hat{\boldsymbol{k}}$ direction.

14.5.3 Rectangles

Consider a rectangular membrane where $x \in [0,a]$ and $y \in [0,b]$, and subject to the boundary conditions $u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0$. We try a solution of the form

$$u(x,y,t) = X(x)\,Y(y)\,T(t) \ . \tag{14.143}$$

This technique is known as *separation of variables*. Dividing the Helmholtz equation by u then gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \ . \tag{14.144}$$

The first term on the LHS depends only on x. The second term on the LHS depends only on y. The RHS depends only on t. Therefore, each of these terms must individually be constant. We write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad , \quad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \ , \tag{14.145}$$

with

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2} \ . \tag{14.146}$$

Thus, $\omega = \pm c|\boldsymbol{k}|$. The most general solution is then

$$X(x) = A\,\cos(k_x x) + B\sin(k_x x) \tag{14.147}$$
$$Y(y) = C\,\cos(k_y y) + D\sin(k_y y) \tag{14.148}$$
$$T(t) = E\,\cos(\omega t) + B\sin(\omega t) \ . \tag{14.149}$$

The boundary conditions now demand

$$A = 0 \quad , \quad C = 0 \quad , \quad \sin(k_x a) = 0 \quad , \quad \sin(k_y b) = 0 \ . \tag{14.150}$$

Thus, the most general solution subject to the boundary conditions is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathcal{A}_{mn} \sin\!\left(\frac{m\pi x}{a}\right) \sin\!\left(\frac{n\pi y}{b}\right) \cos\big(\omega_{mn} t + \delta_{mn}\big) \ , \tag{14.151}$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2} \ . \tag{14.152}$$

14.5.4 Circles

For a circular membrane, such as a drumhead, it is convenient to work in two-dimensional polar coordinates (r,φ). The Laplacian is then

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} \ . \tag{14.153}$$

We seek a solution to the Helmholtz equation which satisfies the boundary conditions $u(r = a, \varphi, t) = 0$. Once again, we invoke the separation of variables method, writing

$$u(r,\varphi,t) = R(r)\,\Phi(\varphi)\,T(t) \ , \tag{14.154}$$

resulting in

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial R}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \ . \tag{14.155}$$

The azimuthal and temporal functions are

$$\Phi(\varphi) = e^{im\varphi} \quad , \quad T(t) = \cos(\omega t + \delta) \ , \tag{14.156}$$

where m is an integer in order that the function $u(r, \varphi, t)$ be single-valued. The radial equation is then

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\right) R = 0 \ . \tag{14.157}$$

This is Bessel's equation, with solution

$$R(r) = A\, J_m\!\left(\frac{\omega r}{c}\right) + B\, N_m\!\left(\frac{\omega r}{c}\right) \ , \tag{14.158}$$

where $J_m(z)$ and $N_m(z)$ are the Bessel and Neumann functions of order m, respectively. Since the Neumann functions diverge at $r = 0$, we must exclude them, setting $B = 0$ for each m.

We now invoke the boundary condition $u(r = a, \varphi, t) = 0$. This requires

$$J_m\!\left(\frac{\omega a}{c}\right) = 0 \qquad \Longrightarrow \qquad \omega = \omega_{m\ell} = x_{m\ell}\, \frac{c}{a} \ , \tag{14.159}$$

where $J_m(x_{m\ell}) = 0$, *i.e.* $x_{m\ell}$ is the ℓ^{th} zero of $J_m(x)$. The mose general solution is therefore

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} \sum_{\ell=1}^{\infty} \mathcal{A}_{m\ell}\, J_m\big(x_{m\ell}\, r/a\big)\, \cos\big(m\varphi + \beta_{m\ell}\big)\, \cos(\omega_{m\ell}\, t + \delta_{m\ell}\big) \ . \tag{14.160}$$

14.5.5 Sound in fluids

Let $\varrho(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$ be the density and velocity fields in a fluid. Mass conservation requires

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho\, \boldsymbol{v}) = 0 \ . \tag{14.161}$$

This is the continuity equation for mass.

Focus now on a small packet of fluid of infinitesimal volume dV. The total force on this fluid element is $d\boldsymbol{F} = \big(-\boldsymbol{\nabla} p + \varrho\, \boldsymbol{g}\big)\, dV$. By Newton's Second Law,

$$d\boldsymbol{F} = \big(\varrho\, dV\big)\, \frac{d\boldsymbol{v}}{dt} \tag{14.162}$$

Note that the chain rule gives

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v} \ . \tag{14.163}$$

Thus, dividing eqn, 14.162 by dV, we obtain

$$\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \varrho\, \boldsymbol{g} \ . \tag{14.164}$$

This is the inviscid (*i.e.* zero viscosity) form of the Navier-Stokes equation.

Locally the fluid can also be described in terms of thermodynamic variables $p(\boldsymbol{x},t)$ (pressure) and $T(\boldsymbol{x},t)$ (temperature). For a one-component fluid there is necessarily an equation of state of the form $p = p(\varrho, T)$. Thus, we may write

$$dp = \frac{\partial p}{\partial \varrho}\bigg|_T d\varrho + \frac{\partial p}{\partial T}\bigg|_\varrho dT \ . \qquad (14.165)$$

We now make the following approximations. First, we assume that the fluid is close to equilibrium at $\boldsymbol{v} = 0$, meaning we write $p = \bar{p} + \delta p$ and $\varrho = \bar{\varrho} + \delta\varrho$, and assume that δp, $\delta\varrho$, and $\boldsymbol{v}$ are small. The smallness of $\boldsymbol{v}$ means we can neglect the nonlinear term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ in eqn. 14.164. Second, we neglect gravity (more on this later). The continuity equation then takes the form

$$\frac{\partial\,\delta\varrho}{\partial t} + \bar{\varrho}\,\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \ , \qquad (14.166)$$

and the Navier-Stokes equation becomes

$$\bar{\varrho}\,\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\delta p \ . \qquad (14.167)$$

Taking the time derivative of the former, and then invoking the latter of these equations yields

$$\frac{\partial^2\,\delta\varrho}{\partial t^2} = \nabla^2 p = \bigg(\frac{\partial p}{\partial \varrho}\bigg)\nabla^2\,\delta\varrho \equiv c^2\,\nabla^2\delta\varrho \ . \qquad (14.168)$$

The speed of wave propagation, *i.e.* the speed of sound, is given by

$$c = \sqrt{\frac{\partial p}{\partial \varrho}} \ . \qquad (14.169)$$

Finally, we must make an assumption regarding the conditions under which the derivative $\partial p/\partial\varrho$ is computed. If the fluid is an excellent conductor of heat, then the temperature will equilibrate quickly and it is a good approximation to take the derivative at fixed temperature. The resulting value of c is called the *isothermal* sound speed c_T. If, on the other hand, the fluid is a poor conductor of heat, as is the case for air, then it is more appropriate to take the derivative at constant entropy, yielding the *adiabatic* sound speed. Thus,

$$c_T = \sqrt{\bigg(\frac{\partial p}{\partial \varrho}\bigg)_T} \quad , \quad c_S = \sqrt{\bigg(\frac{\partial p}{\partial \varrho}\bigg)_S} \ . \qquad (14.170)$$

In an ideal gas, $c_S/c_T = \sqrt{\gamma}$, where $\gamma = c_p/c_V$ is the ratio of the specific heat at constant pressure to that at constant volume. For a (mostly) diatomic gas like air (comprised of N_2 and O_2 and just a little Ar), $\gamma = \frac{7}{5}$. Note that one can write $c^2 = 1/\varrho\kappa$, where

$$\kappa = \frac{1}{\varrho}\bigg(\frac{\partial \varrho}{\partial p}\bigg) \qquad (14.171)$$

is the *compressibility*, which is the inverse of the *bulk modulus*. Again, one must specify whether one is talking about κ_T or κ_S. For reference in air at $T = 293\,\mathrm{K}$, using $M =$

28.8 g/mol, one obtains $c_T = 290.8\,\mathrm{m/s}$ and $c_S = 344.0\,\mathrm{m/s}$. In H_2O at 293 K, $c = 1482\,\mathrm{m/s}$. In Al at 273 K, $c = 6420\,\mathrm{m/s}$.

If we retain gravity, the wave equation becomes

$$\frac{\partial^2 \delta\varrho}{\partial t^2} = c^2\,\nabla^2 \delta\varrho - \boldsymbol{g}\cdot\boldsymbol{\nabla}\delta\varrho\ . \tag{14.172}$$

The dispersion relation is then

$$\omega(\boldsymbol{k}) = \sqrt{c^2 k^2 + i\boldsymbol{g}\cdot\boldsymbol{k}}\ . \tag{14.173}$$

We are permitted to ignore the effects of gravity so long as $c^2k^2 \gg gk$. In terms of the wavelength $\lambda = 2\pi/k$, this requires

$$\lambda \ll \frac{2\pi c^2}{g} = 75.9\,\mathrm{km}\ (\text{at}\ T = 293\,\mathrm{K})\ . \tag{14.174}$$

14.6 Dispersion

The one-dimensional Helmholtz equation $\partial_x^2\, y = c^{-2}\,\partial_t^2\, y$ is solved by a plane wave

$$y(x,t) = A\,e^{ikx}\,e^{-i\omega t}\ , \tag{14.175}$$

provided $\omega = \pm ck$. We say that there are *two branches* to the *dispersion relation* $\omega(k)$ for this equation. In general, we may add solutions, due to the linearity of the Helmholtz equation. The most general solution is then

$$\begin{aligned} y(x,t) &= \int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,\Big[\hat f(k)\,e^{ik(x-ct)} + \hat g(k)\,e^{ik(x+ct)}\Big] \\ &= f(x-ct) + g(x+ct)\ , \end{aligned} \tag{14.176}$$

which is consistent with d'Alembert's solution.

Consider now the free particle Schrödinger equation in one space dimension,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\frac{\partial^2\psi}{\partial x^2}\ . \tag{14.177}$$

The function $\psi(x,t)$ is the quantum mechanical wavefunction for a particle of mass m moving freely along a one-dimensional line. The *probability density* for finding the particle at position x at time t is

$$\rho(x,t) = \big|\psi(x,t)\big|^2\ . \tag{14.178}$$

Conservation of probability therefore requires

$$\int\limits_{-\infty}^{\infty}\!dx\,\big|\psi(x,t)\big|^2 = 1\ . \tag{14.179}$$

This condition must hold at all times t.

As is the case with the Helmholtz equation, the Schrödinger equation is solved by a plane wave of the form

$$\psi(x,t) = A\, e^{ikx}\, e^{-i\omega t} \ , \tag{14.180}$$

where the dispersion relation now only has one branch, and is given by

$$\omega(k) = \frac{\hbar k^2}{2m} \ . \tag{14.181}$$

The most general solution is then

$$\psi(x,t) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{\psi}(k)\, e^{ikx}\, e^{-i\hbar k^2 t/2m} \ . \tag{14.182}$$

Let's suppose we start at time $t=0$ with a Gaussian wavepacket,

$$\psi(x,0) = \big(\pi \ell_0^2\big)^{-1/4}\, e^{-x^2/2\ell_0^2}\, e^{ik_0 x} \ . \tag{14.183}$$

To find the amplitude $\hat{\psi}(k)$, we perform the Fourier transform:

$$\begin{aligned} \hat{\psi}(k) &= \int\limits_{-\infty}^{\infty} dx\, \psi(x,0)\, e^{-ikx} \\ &= \sqrt{2}\, \big(\pi \ell_0^2\big)^{-1/4}\, e^{-(k-k_0)^2 \ell_0^2/2} \ . \end{aligned} \tag{14.184}$$

We now compute $\psi(x,t)$ valid for all times t:

$$\psi(x,t) = \sqrt{2}\, \big(\pi \ell_0^2\big)^{-1/4} \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-(k-k_0)^2 \ell_0^2/2}\, e^{ikx}\, e^{-i\hbar k^2 t/2m} \tag{14.185}$$

$$\begin{aligned} &= \big(\pi \ell_0^2\big)^{-1/4}\, \big(1 + it/\tau\big)^{-1/2} \exp\left[-\frac{\big(x - \hbar k_0 t/m\big)^2}{2\,\ell_0^2\,\big(1 + t^2/\tau^2\big)} \right] \\ &\qquad \times \exp\left[\frac{i\big(2k_0\,\ell_0^2\, x + x^2\, t/\tau - k_0^2\, \ell_0^4\, t/\tau\big)}{2\,\ell_0^2\,\big(1 + t^2/\tau^2\big)} \right] \ , \end{aligned} \tag{14.186}$$

where $\tau \equiv m\ell_0^2/\hbar$. The probability density is then the normalized Gaussian

$$\rho(x,t) = \frac{1}{\sqrt{\pi\, \ell^2(t)}}\, e^{-(x - v_0 t)^2/\ell^2(t)} \ , \tag{14.187}$$

where $v_0 = \hbar k_0/m$ and

$$\ell(t) = \ell_0\, \sqrt{1 + t^2/\tau^2} \ . \tag{14.188}$$

Note that $\ell(t)$ gives the width of the wavepacket, and that this width increases as a function of time, with $\ell(t \gg \tau) \simeq \ell_0\, t/\tau$.

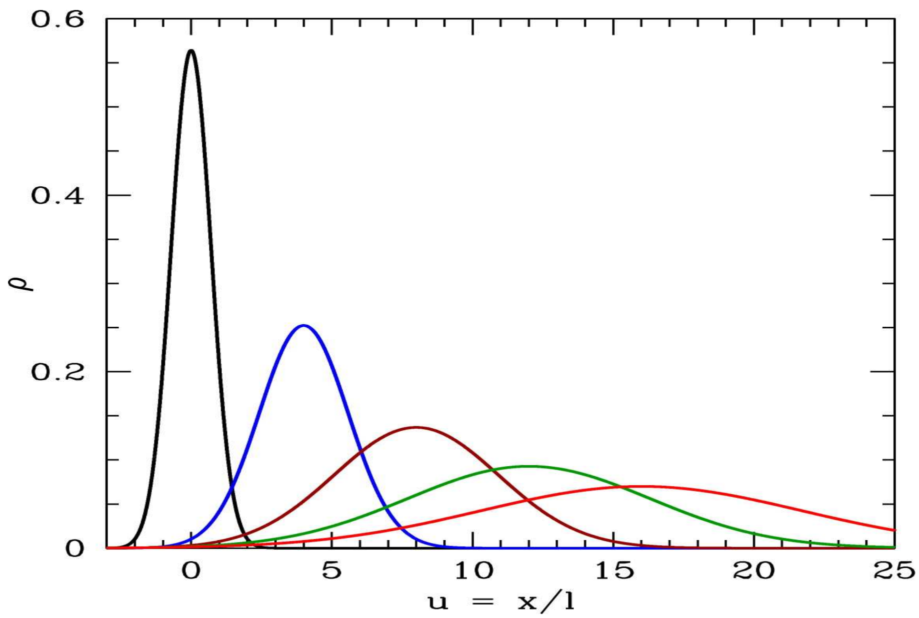

Figure 14.9: Wavepacket spreading for $k_0\,\ell_0 = 2$ with $t/\tau = 0$, 2, 4, 6, and 8.

Unlike the case of the Helmholtz equation, the solution to the Schrödinger equation does not retain its shape as it moves. This phenomenon is known as the *spreading of the wavepacket*. In fig. 14.9, we show the motion and spreading of the wavepacket.

For a given plane wave $e^{ikx}\,e^{-i\omega(k)t}$, the wavefronts move at the *phase velocity*

$$v_{\rm p}(k) = \frac{\omega(k)}{k} \ . \tag{14.189}$$

The center of the wavepacket, however, travels at the *group velocity*

$$v_{\rm g}(k) = \frac{d\omega}{dk}\bigg|_{k_0} \ , \tag{14.190}$$

where $k = k_0$ is the maximum of $\big|\hat\psi(k)\big|^2$.

14.7 Appendix I : Three Strings

Problem: Three identical strings are connected to a ring of mass m as shown in fig. 14.10. The linear mass density of each string is σ and each string is under identical tension τ. In equilibrium, all strings are coplanar. All motion on the string is in the $\hat{\boldsymbol{z}}$-direction, which is perpendicular to the equilibrium plane. The ring slides frictionlessly along a vertical pole.

It is convenient to describe each string as a half line $[-\infty, 0]$. We can choose coordinates x_1, x_2, and x_3 for the three strings, respectively. For each string, the ring lies at $x_i = 0$.

A pulse is sent down the first string. After a time, the pulse arrives at the ring. Transmitted waves are sent down the other two strings, and a reflected wave down the first string. The solution to the wave equation in the strings can be written as follows. In string #1, we have

$$z = f(ct - x_1) + g(ct + x_1) \ . \tag{14.191}$$

In the other two strings, we may write $z = h_{\rm A}(ct + x_2)$ and $z = h_{\rm B}(ct + x_3)$, as indicated in the figure.

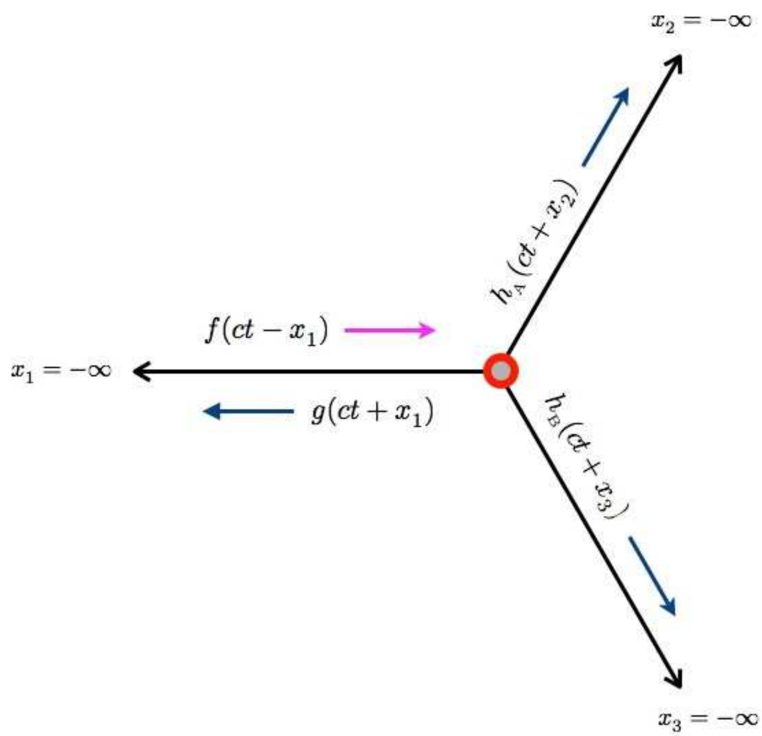

Figure 14.10: Three identical strings arranged symmetrically in a plane, attached to a common end. All motion is in the direction perpendicular to this plane. The red ring, whose mass is m, slides frictionlessly in this direction along a pole.

(a) Write the wave equation in string #1. Define all constants.

(b) Write the equation of motion for the ring.

(c) Solve for the reflected wave $g(\xi)$ in terms of the incident wave $f(\xi)$. You may write this relation in terms of the Fourier transforms $\hat{f}(k)$ and $\hat{g}(k)$.

(d) Suppose a very long wavelength pulse of maximum amplitude A is incident on the ring. What is the maximum amplitude of the reflected pulse? What do we mean by "very long wavelength"?

Solution:

(a) The wave equation is

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 z}{\partial t^2}\ , \tag{14.192}$$

where x is the coordinate along the string, and $c = \sqrt{\tau/\sigma}$ is the speed of wave propagation.

(b) Let Z be the vertical coordinate of the ring. Newton's second law says $m\ddot{Z} = F$, where the force on the ring is the sum of the vertical components of the tension in the three strings at $x = 0$:

$$F = -\tau\Big[-f'(ct) + g'(ct) + h'_{\rm A}(ct) + h'_{\rm B}(ct)\Big]\ , \tag{14.193}$$

where prime denotes differentiation with respect to argument.

(c) To solve for the reflected wave, we must eliminate the unknown functions $h_{\rm A,B}$ and then obtain g in terms of f. This is much easier than it might at first seem. We start by demanding continuity at the ring. This means

$$Z(t) = f(ct) + g(ct) = h_{\rm A}(ct) = h_{\rm B}(ct) \tag{14.194}$$

for all t. We can immediately eliminate $h_{\rm A,B}$:

$$h_{\rm A}(\xi) = h_{\rm B}(\xi) = f(\xi) + g(\xi)\ , \tag{14.195}$$

for all ξ. Newton's second law from part (b) may now be written as

$$mc^2\big[f''(\xi) + g''(\xi)\big] = -\tau\big[f'(\xi) + 3g'(\xi)\big]\ . \tag{14.196}$$

This linear ODE becomes a simple linear algebraic equation for the Fourier transforms,

$$f(\xi) = \int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,\hat{f}(k)\,e^{ik\xi}\ , \tag{14.197}$$

etc. We readily obtain

$$\hat{g}(k) = -\bigg(\frac{k - iQ}{k - 3iQ}\bigg)\,\hat{f}(k)\ , \tag{14.198}$$

where $Q \equiv \tau/mc^2$ has dimensions of inverse length. Since $h_{\rm A,B} = f + g$, we have

$$\hat{h}_{\rm A}(k) = \hat{h}_{\rm B}(k) = -\bigg(\frac{2iQ}{k - 3iQ}\bigg)\,\hat{f}(k)\ . \tag{14.199}$$

(d) For a very long wavelength pulse, composed of plane waves for which $|k| \ll Q$, we have $\hat{g}(k) \approx -\frac{1}{3}\,\hat{f}(k)$. Thus, the reflected pulse is inverted, and is reduced by a factor $\frac{1}{3}$ in amplitude. Note that for a very *short* wavelength pulse, for which $k \gg Q$, we have perfect reflection with inversion, and no transmission. This is due to the inertia of the ring.

It is straightforward to generalize this problem to one with n strings. The transmission into each of the $(n-1)$ channels is of course identical (by symmetry). One then finds the reflection and transmission amplitudes

$$r(k) = -\bigg(\frac{k - i(n-2)Q}{k - inQ}\bigg)\qquad,\qquad t(k) = -\bigg(\frac{2iQ}{k - inQ}\bigg)\ . \tag{14.200}$$

Conservation of energy means that the sum of the squares of the reflection amplitude and all the $(n-1)$ transmission amplitudes must be unity:

$$\big|r(k)\big|^2 + (n-1)\,\big|t(k)\big|^2 = 1 \ . \tag{14.201}$$

14.8 Appendix II : General Field Theoretic Formulation

Continuous systems possess an infinite number of degrees of freedom. They are described by a set of fields $\phi_a(\boldsymbol{x},t)$ which depend on space and time. These fields may represent local displacement, pressure, velocity, *etc.* The equations of motion of the fields are again determined by extremizing the action, which, in turn, is an integral of the *Lagrangian density* over all space and time. Extremization yields a set of (generally coupled) *partial* differential equations.

14.8.1 Euler-Lagrange equations for classical field theories

Suppose $\phi_a(\boldsymbol{x})$ depends on n independent variables, $\{x^1, x^2, \ldots, x^n\}$. Consider the functional

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_{\Omega} d\boldsymbol{x}\,\mathcal{L}(\phi_a\,\partial_\mu\phi_a, \boldsymbol{x}) \ , \tag{14.202}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^n. Then the first variation of S is

$$\begin{aligned}\delta S &= \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\,\delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} d\Sigma\, n^\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a + \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right\}\delta\phi_a \ ,\end{aligned} \tag{14.203}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right)\right]_{\boldsymbol{x}} \ , \tag{14.204}$$

where the subscript means we are to evaluate the term in brackets at $\boldsymbol{x}$. In a mechanical system, one of the n independent variables (usually x^0), is the time t. However, we may be interested in a time-independent context in which we wish to extremize the energy functional, for example. In any case, setting the first variation of S to zero yields the Euler-Lagrange equations,

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right) = 0 \tag{14.205}$$

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} - J_\mu\, A^\mu \ . \tag{14.206}$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu\, F^{\mu\nu} = 4\pi J^\nu \ , \tag{14.207}$$

which are Maxwell's equations.

14.8.2 Conserved currents in field theory

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{14.208}$$

where $\tilde{q}_\sigma = \tilde{q}_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x},t) \ , \tag{14.209}$$

where $\{\phi_a(\boldsymbol{x},t)\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $dQ/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{14.210}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu\, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} \ . \tag{14.211}$$

We call $Q = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu\, J^\mu$ over the spatial region Ω gives

$$\frac{dQ}{dt} = \int\limits_\Omega d^3\!x\; \partial_0\, J^0 = -\int\limits_\Omega d^3\!x\; \boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\; \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \ , \tag{14.212}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[2]

$$\mathcal{L}(\psi,,\psi^*,\partial_\mu\psi,\partial_\mu\psi^*) = \tfrac{1}{2}K\,(\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \ . \tag{14.213}$$

[2]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$.

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\,e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^* \ , \tag{14.214}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} \cdot (i\psi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^*\right)} \cdot (-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big) \ . \end{aligned} \tag{14.215}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

14.8.3 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g\,\big(|\psi|^2 - n_0\big)^2 \ . \tag{14.216}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x},t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned} \delta S[\psi^*,\psi] &= S[\psi^* + \delta\psi^*, \psi + \delta\psi] \\ &= \int\!dt\!\int\!d^d\!x\,\bigg\{ i\hbar\,\psi^*\,\frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^*\cdot\boldsymbol{\nabla}\psi \\ &\qquad\qquad - 2g\,\big(|\psi|^2 - n_0\big)\,(\psi^*\delta\psi + \psi\delta\psi^*)\bigg\} \\ &= \int\!dt\!\int\!d^d\!x\,\bigg\{ \bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \bigg]\,\delta\psi \\ &\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\,\big(|\psi|^2 - n_0\big)\,\psi \bigg]\,\delta\psi^* \bigg\} \ , \end{aligned} \tag{14.217}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*,\psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + 2g\,\big(|\psi|^2 - n_0\big)\,\psi \tag{14.218}$$

as well as its complex conjugate,

$$-i\hbar\,\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi^* + 2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \ . \tag{14.219}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\frac{\delta S}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\right) \tag{14.220}$$

$$\frac{\delta S}{\delta\psi^*} = \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\right) , \tag{14.221}$$

with $x^\mu = (t,\boldsymbol{x})$[3] Plugging in

$$\frac{\partial\mathcal{L}}{\partial\psi} = -2g\left(|\psi|^2 - n_0\right)\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi} = i\hbar\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{14.222}$$

and

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\,\psi - 2g\left(|\psi|^2 - n_0\right)\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi^*} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi , \tag{14.223}$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$\psi(\boldsymbol{x},t) \to \tilde\psi(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \tilde\psi^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) . \tag{14.224}$$

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\,\frac{\partial\tilde\psi}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\,\frac{\partial\tilde\psi^*}{\partial\zeta}\bigg|_{\zeta=0}$$

$$J^0 = -\hbar\,|\psi|^2 \tag{14.225}$$

$$\boldsymbol{J} = -\frac{\hbar^2}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) . \tag{14.226}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 , \tag{14.227}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) . \tag{14.228}$$

are the particle density and the particle current, respectively.

[3]In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

14.9 Appendix III : Green's Functions

Suppose we add a forcing term,

$$\mu(x)\,\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] + v(x)\,y = \text{Re}\Big[\mu(x)\,f(x)\,e^{-i\omega t}\Big] \ . \tag{14.229}$$

We write the solution as

$$y(x,t) = \text{Re}\Big[y(x)\,e^{-i\omega t}\Big] \ , \tag{14.230}$$

where

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dy(x)}{dx}\bigg] + v(x)\,y(x) - \omega^2\,\mu(x)\,y(x) = \mu(x)\,f(x) \ , \tag{14.231}$$

or

$$\Big[K - \omega^2\mu(x)\Big]\,y(x) = \mu(x)\,f(x) \ , \tag{14.232}$$

where K is a differential operator,

$$K \equiv -\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x) \ . \tag{14.233}$$

Note that the eigenfunctions of K are the $\{\psi_n(x)\}$:

$$K\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x) \ . \tag{14.234}$$

The formal solution to equation 14.232 is then

$$y(x) = \Big[\,K - \omega^2\mu\,\Big]^{-1}_{x,x'}\,\mu(x')\,f(x') \tag{14.235}$$

$$= \int\limits_{x_a}^{x_b} dx'\,\mu(x')\,G_\omega(x,x')\,f(x'). \tag{14.236}$$

What do we mean by the term in brackets? If we define the *Green's function*

$$G_\omega(x,x') \equiv \Big[\,K - \omega^2\mu\,\Big]^{-1}_{x,x'} \ , \tag{14.237}$$

what this means is

$$\Big[\,K - \omega^2\mu(x)\Big]\,G_\omega(x,x') = \delta(x-x') \ . \tag{14.238}$$

Note that the Green's function may be expanded in terms of the (real) eigenfunctions, as

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2} \ , \tag{14.239}$$

which follows from completeness of the eigenfunctions:

$$\mu(x)\sum_{n=1}^{\infty}\psi_n(x)\,\psi_n(x') = \delta(x-x') \ . \tag{14.240}$$

The expansion in eqn. 14.239 is formally exact, but difficult to implement, since it requires summing over an infinite set of eigenfunctions. It is more practical to construct the Green's function from solutions to the homogeneous Sturm Liouville equation, as follows. When $x \neq x'$, we have that $(K - \omega^2\mu)\, G_\omega(x,x') = 0$, which is a homogeneous ODE of degree two. Consider first the interval $x \in [x_a, x']$. A second order homogeneous ODE has two solutions, and further invoking the boundary condition at $x = x_a$, there is a unique solution, up to a multiplicative constant. Call this solution $y_1(x)$. Next, consider the interval $x \in [x', x_b]$. Once again, there is a unique solution to the homogeneous Sturm-Liouville equation, up to a multiplicative constant, which satisfies the boundary condition at $x = x_b$. Call this solution $y_2(x)$. We then can write

$$G_\omega(x,x') = \begin{cases} A(x')\, y_1(x) & \text{if } x_a \leq x < x' \\ \\ B(x')\, y_2(x) & \text{if } x' < x \leq x_b \ . \end{cases} \tag{14.241}$$

Here, $A(x')$ and $B(x')$ are undetermined functions. We now invoke the inhomogeneous Sturm-Liouville equation,

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dG_\omega(x,x')}{dx}\bigg] + v(x)\, G_\omega(x,x') - \omega^2\mu(x)\, G_\omega(x,x') = \delta(x-x') \ . \tag{14.242}$$

We integrate this from $x = x' - \epsilon$ to $x = x' + \epsilon$, where ϵ is a positive infinitesimal. This yields

$$\tau(x')\Big[A(x')\, y_1'(x') - B(x')\, y_2'(x')\Big] = 1 \ . \tag{14.243}$$

Continuity of $G_\omega(x,x')$ itself demands

$$A(x')\, y_1(x') = B(x')\, y_2(x') \ . \tag{14.244}$$

Solving these two equations for $A(x')$ and $B(x')$, we obtain

$$A(x') = -\frac{y_2(x')}{\tau(x')\, \mathcal{W}_{y_1,y_2}(x')} \qquad , \qquad B(x') = -\frac{y_1(x')}{\tau(x')\, \mathcal{W}_{y_1,y_2}(x')} \ , \tag{14.245}$$

where $\mathcal{W}_{y_1,y_2}(x)$ is the *Wronskian*

$$\begin{aligned} \mathcal{W}_{y_1,y_2}(x) &= \det\begin{pmatrix} y_1(x) & y_2(x) \\ \\ y_1'(x) & y_2'(x) \end{pmatrix} \\ &= y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) \ . \end{aligned} \tag{14.246}$$

Now it is easy to show that $\mathcal{W}_{y_1,y_2}(x)\, \tau(x) = \mathcal{W}\,\tau$ is a constant. This follows from the fact that

$$\begin{aligned} 0 &= y_2\, K\, y_1 - y_2\, K\, y_1 \\ &= \frac{d}{dx}\bigg\{\tau(x)\Big[y_1\, y_2' - y_2\, y_1'\Big]\bigg\} \ . \end{aligned} \tag{14.247}$$

Thus, we have

$$G_\omega(x,x') = \begin{cases} -y_1(x)\, y_2(x')/\mathcal{W} & \text{if } x_a \le x < x' \\ \\ -y_1(x')\, y_2(x)/\mathcal{W} & \text{if } x' < x \le x_b \ , \end{cases} \tag{14.248}$$

or, in compact form,

$$G_\omega(x,x') = -\frac{y_1(x_<)\, y_2(x_>)}{\mathcal{W}\,\tau} \ , \tag{14.249}$$

where $x_< = \min(x,x')$ and $x_> = \max(x,x')$.

As an example, consider a uniform string (*i.e.* μ and τ constant, $v=0$) with fixed endpoints at $x_a = 0$ and $x_b = L$. The normalized eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{\mu L}}\, \sin\left(\frac{n\pi x}{L}\right) \ , \tag{14.250}$$

and the eigenvalues are $\omega_n = n\pi c/L$. The Green's function is

$$G_\omega(x,x') = \frac{2}{\mu L} \sum_{n=1}^\infty \frac{\sin(n\pi x/L)\, \sin(n\pi x'/L)}{(n\pi c/L)^2 - \omega^2} \ . \tag{14.251}$$

Now construct the homogeneous solutions:

$$(K - \omega^2 \mu)\, y_1 = 0 \quad , \quad y_1(0) = 0 \qquad \Longrightarrow \qquad y_1(x) = \sin\left(\frac{\omega x}{c}\right) \tag{14.252}$$

$$(K - \omega^2 \mu)\, y_2 = 0 \quad , \quad y_2(L) = 0 \qquad \Longrightarrow \qquad y_2(x) = \sin\left(\frac{\omega (L-x)}{c}\right) \ . \tag{14.253}$$

The Wronskian is

$$\mathcal{W} = y_1\, y_2' - y_2\, y_1' = -\frac{\omega}{c}\, \sin\left(\frac{\omega L}{c}\right) \ . \tag{14.254}$$

Therefore, the Green's function is

$$G_\omega(x,x') = \frac{\sin\big(\omega x_</c\big)\, \sin\big(\omega(L - x_>)/c\big)}{(\omega\tau/c)\, \sin(\omega L/c)} \ . \tag{14.255}$$

14.9.1 Perturbation theory

Suppose we have solved for the Green's function for the linear operator K_0 and mass density $\mu_0(x)$. *I.e.* we have

$$\big(K_0 - \omega^2 \mu_0(x)\big)\, G^0_\omega(x,x') = \delta(x - x') \ . \tag{14.256}$$

We now imagine perturbing $\tau_0 \to \tau_0 + \lambda\tau_1$, $v_0 \to v_0 + \lambda v_2$, $\mu_0 \to \mu_0 + \lambda\mu_1$. What is the new Green's function $G_\omega(x,x')$? We must solve

$$\big(L_0 + \lambda L_1\big)\, G_\omega(x,x') = \delta(x - x') \ , \tag{14.257}$$

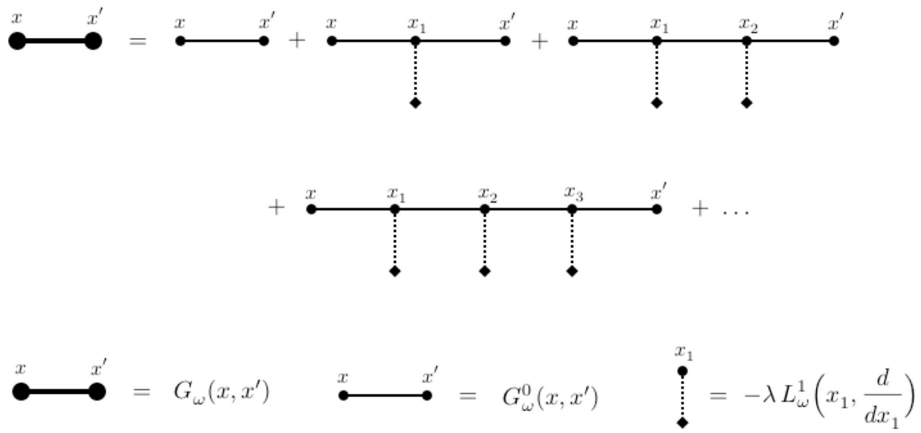

Figure 14.11: Diagrammatic representation of the perturbation expansion in eqn. 14.260..

where

$$
\begin{aligned}
L^0_\omega &\equiv K_0 - \omega^2\,\mu_0 && (14.258)\\
L^1_\omega &\equiv K_1 - \omega^2\,\mu_1 \ . && (14.259)
\end{aligned}
$$

Dropping the ω subscript for simplicity, the full Green's function is then given by

$$
\begin{aligned}
G_\omega &= \Big[L^0_\omega + \lambda L^1_\omega\Big]^{-1}\\
&= \Big[\,\big(G^0_\omega\big)^{-1} + \lambda L^1_\omega\,\Big]^{-1}\\
&= \Big[\,1 + \lambda\, G^0_\omega\, L^1_\omega\,\Big]^{-1}\, G^0_\omega\\
&= G^0_\omega - \lambda\, G^0_\omega\, L^1_\omega\, G^0_\omega + \lambda^2\, G^0_\omega\, L^1_\omega\, G^0_\omega\, L^1_\omega\, G^0_\omega + \ldots \ . && (14.260)
\end{aligned}
$$

The 'matrix multiplication' is of course a convolution, *i.e.*

$$
G_\omega(x,x') = G^0_\omega(x,x') - \lambda \int\limits_{x_a}^{x_b}\! dx_1\, G^0_\omega(x,x_1)\, L^1_\omega\big(x_1,\tfrac{d}{dx_1}\big)\, G^0_\omega(x_1,x') + \ldots \ . \qquad (14.261)
$$

Each term in the perturbation expansion of eqn. 14.260 may be represented by a diagram, as depicted in Fig. 14.11.

As an example, consider a string with $x_a = 0$ and $x_b = L$ with a mass point m affixed at the point $x = d$. Thus, $\mu_1(x) = m\,\delta(x-d)$, and $L^1_\omega = -m\omega^2\,\delta(x-d)$, with $\lambda = 1$. The perturbation expansion gives

$$
\begin{aligned}
G_\omega(x,x') &= G^0_\omega(x,x') + m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x') + m^2\omega^4\, G^0_\omega(x,d)\, G^0_\omega(d,d)\, G^0_\omega(d,x') + \ldots\\
&= G^0_\omega(x,x') + \frac{m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x')}{1 - m\omega^2\, G^0_\omega(d,d)} \ . && (14.262)
\end{aligned}
$$

Note that the eigenfunction expansion,

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2} \ , \tag{14.263}$$

says that the exact eigenfrequencies are poles of $G_\omega(x,x')$, and furthermore the residue at each pole is

$$\operatorname*{Res}_{\omega=\omega_n} G_\omega(x,x') = -\frac{1}{2\omega_n}\,\psi_n(x)\,\psi_n(x') \ . \tag{14.264}$$

According to eqn. 14.262, the poles of $G_\omega(x,x')$ are located at solutions to[4]

$$m\omega^2\, G^0_\omega(d,d) = 1 \ . \tag{14.265}$$

For simplicity let us set $d = \frac{1}{2}L$, so the mass point is in the middle of the string. Then according to eqn. 14.255,

$$\begin{aligned} G^0_\omega\big(\tfrac{1}{2}L,\tfrac{1}{2}L\big) &= \frac{\sin^2(\omega L/2c)}{(\omega\tau/c)\,\sin(\omega L/c)} \\ &= \frac{c}{2\omega\tau}\,\tan\!\left(\frac{\omega L}{2c}\right) \ . \end{aligned} \tag{14.266}$$

The eigenvalue equation is therefore

$$\tan\!\left(\frac{\omega L}{2c}\right) = \frac{2\tau}{m\omega c} \ , \tag{14.267}$$

which can be manipulated to yield

$$\frac{m}{M}\,\lambda = \operatorname{ctn}\lambda \ , \tag{14.268}$$

where $\lambda = \omega L/2c$ and $M = \mu L$ is the total mass of the string. When $m = 0$, the LHS vanishes, and the roots lie at $\lambda = (n + \frac{1}{2})\pi$, which gives $\omega = \omega_{2n+1}$. Why don't we see the poles at the even mode eigenfrequencies ω_{2n}? The answer is that these poles are present in the Green's function. They do not cancel for $d = \frac{1}{2}L$ because the perturbation does not couple to the even modes, which all have $\psi_{2n}(\frac{1}{2}L) = 0$. The case of general d may be instructive in this regard. One finds the eigenvalue equation

$$\frac{\sin(2\lambda)}{2\lambda\,\sin\big(2\epsilon\lambda\big)\,\sin\big(2(1-\epsilon)\lambda\big)} = \frac{m}{M} \ , \tag{14.269}$$

where $\epsilon = d/L$. Now setting $m = 0$ we recover $2\lambda = n\pi$, which says $\omega = \omega_n$, and all the modes are recovered.

[4]Note in particular that there is no longer any divergence at the location of the original poles of $G^0_\omega(x,x')$. These poles are cancelled.

14.9.2 Perturbation theory for eigenvalues and eigenfunctions

We wish to solve

$$\big(K_0 + \lambda K_1\big)\,\psi = \omega^2\big(\mu_0 + \lambda\mu_1\big)\,\psi\ , \tag{14.270}$$

which is equivalent to

$$L^0_\omega\,\psi = -\lambda L^1_\omega\,\psi\ . \tag{14.271}$$

Multiplying by $\big(L^0_\omega\big)^{-1} = G^0_\omega$ on the left, we have

$$\psi(x) = -\lambda\int\limits_{x_a}^{x_b}\!dx'\,G_\omega(x,x')\,L^1_\omega\,\psi(x') \tag{14.272}$$

$$= \lambda\sum_{m=1}^\infty \frac{\psi_m(x)}{\omega^2-\omega^2_m}\int\limits_{x_a}^{x_b}\!dx'\,\psi_m(x')\,L^1_\omega\,\psi(x')\ . \tag{14.273}$$

We are free to choose any normalization we like for $\psi(x)$. We choose

$$\big\langle\,\psi\,\big|\,\psi_n\,\big\rangle = \int\limits_{x_a}^{x_b}\!dx\;\mu_0(x)\,\psi_n(x)\,\psi(x) = 1\ , \tag{14.274}$$

which entails

$$\omega^2 - \omega^2_n = \lambda\int\limits_{x_a}^{x_b}\!dx\,\psi_n(x)\,L^1_\omega\,\psi(x) \tag{14.275}$$

as well as

$$\psi(x) = \psi_n(x) + \lambda\sum_{\substack{k\\(k\neq n)}} \frac{\psi_k(x)}{\omega^2-\omega^2_k}\int\limits_{x_a}^{x_b}\!dx'\,\psi_k(x')\,L^1_\omega\,\psi(x')\ . \tag{14.276}$$

By expanding ψ and ω^2 in powers of λ, we can develop an order by order perturbation series.

To lowest order, we have

$$\omega^2 = \omega^2_n + \lambda\int\limits_{x_a}^{x_b}\!dx\,\psi_n(x)\,L^1_{\omega_n}\psi_n(x)\ . \tag{14.277}$$

For the case $L^1_\omega = -m\,\omega^2\,\delta(x-d)$, we have

$$\begin{aligned}\frac{\delta\omega_n}{\omega_n} &= -\tfrac{1}{2}m\big[\psi_n(d)\big]^2\\ &= -\frac{m}{M}\,\sin^2\!\Big(\frac{n\pi d}{L}\Big)\ .\end{aligned} \tag{14.278}$$

For $d=\frac{1}{2}L$, only the odd n modes are affected, as the even n modes have a node at $x=\frac{1}{2}L$.

Carried out to second order, one obtains for the eigenvalues,

$$\begin{aligned}\omega^2 = \omega_n^2 &+ \lambda \int_{x_a}^{x_b} dx\, \psi_n(x)\, L^1_{\omega_n}\, \psi_n(x) \\ &+ \lambda^2 \sum_{\substack{k \\ (k \neq n)}} \frac{\left| \int_{x_a}^{x_b} dx\, \psi_k(x)\, L^1_{\omega_n}\, \psi_n(x) \right|^2}{\omega_n^2 - \omega_k^2} + \mathcal{O}(\lambda^3) \\ &- \lambda^2 \int_{x_a}^{x_b} dx\, \psi_n(x)\, L^1_{\omega_n}\, \psi_n(x) \cdot \int_{x_a}^{x_b} dx'\, \mu_1(x') \left[\psi_n(x')\right]^2 + \mathcal{O}(\lambda^3)\ .\end{aligned} \tag{14.279}$$