Lecture 13

Fluid Dynamics III.

Incompressible viscous fluid flow

$$\frac{\partial}{\partial t}\left(\operatorname{curl}\vec{v}\right) = \operatorname{curl}\left(\vec{v}\times\operatorname{curl}\vec{v}\right) + \nu\,\Delta\left(\operatorname{curl}\vec{v}\right)$$

Navier-Stokes equation

$\operatorname{div}\vec{v} = 0$ incompressible

$$\frac{\partial}{\partial t}\left(\operatorname{curl}\vec{v}\right) + (\vec{v}\cdot\operatorname{grad})\operatorname{curl}\vec{v} - (\operatorname{curl}\vec{v}\cdot\operatorname{grad})\vec{v}$$
$$= \nu\,\Delta\operatorname{curl}\vec{v}$$

When the velocity distribution is known, the pressure distribution in the fluid can be found by taking

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$$\operatorname{curl}(\vec{f}\times\vec{g}) = \vec{f}\cdot\operatorname{div}\vec{g} - \vec{g}\operatorname{div}\vec{f} + (\vec{g}\cdot\operatorname{grad})\vec{f}$$
$$- (\vec{f}\cdot\operatorname{grad})\vec{g}$$

$\vec{f} = \vec{v}$, $\vec{g} = \operatorname{curl}\vec{v}$

$$\operatorname{curl}(\vec{v}\times\operatorname{curl}\vec{v}) = \vec{v}\cdot\underbrace{\operatorname{div}\operatorname{curl}\vec{v}}_{=0} - \operatorname{curl}\vec{v}\cdot\underbrace{\operatorname{div}\vec{v}}_{=0}$$
$$+ (\operatorname{curl}\vec{v}\cdot\operatorname{grad})\vec{v} - (\vec{v}\cdot\operatorname{grad})\operatorname{curl}\vec{v}$$

The divergence of

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \operatorname{grad}) \vec{v} = -\frac{1}{\rho} \operatorname{grad} p + \frac{\eta}{\rho} \Delta \vec{v}$$

$$\operatorname{div} \frac{\partial \vec{v}}{\partial t} = \frac{\partial}{\partial t} \operatorname{div} \vec{v} = 0$$

useful vector analysis:

$$\operatorname{div}(\phi \cdot \vec{f}) = \phi \cdot \operatorname{div} \vec{f} + \vec{f} \cdot \operatorname{grad} \phi$$

$$\operatorname{grad}(\phi + \psi) = \operatorname{grad} \phi + \operatorname{grad} \psi$$

$$\operatorname{grad}(\vec{f} \cdot \vec{g}) = (\vec{f} \cdot \operatorname{grad}) \vec{g} + (\vec{g} \cdot \operatorname{grad}) \vec{f} + \\ + \vec{f} \times \operatorname{curl} \vec{g} + \vec{g} \times \operatorname{curl} \vec{f}$$

$$\operatorname{grad}(\phi \cdot \psi) = \phi \operatorname{grad} \psi + \psi \operatorname{grad} \phi$$

$$\operatorname{div}(\vec{f} \times \vec{g}) = \vec{g} \cdot \operatorname{curl} \vec{f} - \vec{f} \cdot \operatorname{curl} \vec{g}$$

$$\operatorname{curl}(\phi \vec{f}) = \phi \operatorname{curl} \vec{f} + \operatorname{grad} \phi \times \vec{f}$$

$$\operatorname{curl}(\operatorname{curl} \vec{f}) = \operatorname{grad}(\operatorname{div} \vec{f}) - \nabla^2 \vec{f}$$

$$\operatorname{curl}(\operatorname{grad} \phi) = 0$$

$$\operatorname{div}(\operatorname{curl} \vec{f}) = 0$$

$$\nabla^2(\phi \cdot \psi) = \phi \nabla^2 \psi + 2 \operatorname{grad} \phi \cdot \operatorname{grad} \psi + \psi \nabla^2 \phi$$

$$\operatorname{div} \Delta \vec{v} = \Delta \cdot \operatorname{div} \vec{v} = 0$$

ρ is constant

$$\operatorname{div}\left((\vec{v} \cdot \operatorname{grad}) \vec{v}\right) \rightarrow \frac{\partial v_i}{\partial x_k} \cdot \frac{\partial v_k}{\partial x_i} = \frac{\partial^2 v_i v_k}{\partial x_k \partial x_i}$$

$$\Delta p = -\rho \frac{\partial v_i}{\partial x_k} \cdot \frac{\partial v_k}{\partial x_i} = -\rho \frac{\partial^2 v_i v_k}{\partial x_k \partial x_i}$$

Poisson - type equation

Equation of stream function $\psi(x,y)$ in two-dimensional flow of incompressible viscous fluid:

$$\frac{\partial}{\partial t} \Delta \psi - \frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial y} + \frac{\partial \psi}{\partial y} \cdot \frac{\partial \Delta \psi}{\partial x} - \nu \Delta \Delta \psi = 0$$

where $v_x = \frac{\partial \psi}{\partial y}$ $\quad v_y = -\frac{\partial \psi}{\partial x}$

$\vec{v} = 0$ at fixed solid surface

boundary condition

There are always molecular forces between viscous fluid and surface of solid body : layer of fluid adjacent to surface is brought to rest, adheres to surface

For moving surface $\vec{v}$ is equal to the velocity of surface

What is the force acting on solid surface?
Force is the momentum flux:

$$\Pi_{ik}\, df_k = (\rho v_i v_k - \sigma_{ik})\, df_k$$

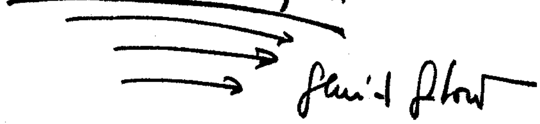
momentum flux through surface element $d\vec{f}$

$$df_k = n_k\, df$$

(we are looking at each surface element in its rest frame)

$$P_i = -\sigma_{ik} \cdot n_k = \underbrace{p n_i}_{\text{fluid pressure}} - \underbrace{\sigma'_{ik} n_k}_{\text{friction force}} \quad \text{force}$$

$\vec{n}$ is inward normal to solid body



Note on 2-dimensional flow

incompressible $\quad \operatorname{div} \vec{v} = 0$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

satisfied if $\quad v_x = \dfrac{\partial \psi}{\partial y}$

$$v_y = -\frac{\partial \psi}{\partial x}$$

$\psi(x,y)$ stream function

Equation of continuity automatic

$$\frac{\partial}{\partial t}\left(\operatorname{curl} \vec{v}\right) = \operatorname{curl}\left(\vec{v} \times \operatorname{curl} \vec{v}\right) \quad \text{ideal fluid}$$

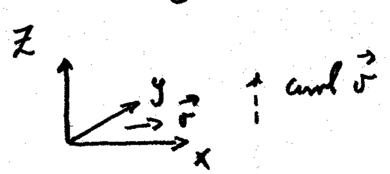
Euler Eq.

$$\frac{\partial}{\partial t}\left(\operatorname{curl} \vec{v}\right) = \operatorname{curl}\left(\vec{v} \times \operatorname{curl} \vec{v}\right) + \nu \, \Delta \left(\operatorname{curl} \vec{v}\right)$$

Navier - Stokes eq.

curl $\vec{v}$ "along z-direction" for 2D flow (or think about curl $\vec{v}$ as a scalar)

$$\operatorname{curl} \vec{v} = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = \Delta \psi$$



We can use the identity

$$\text{curl}\,(\vec{v} \times \text{curl}\,\vec{v}) = (\text{curl}\,\vec{v} \cdot \text{grad})\,\vec{v} - (\vec{v} \cdot \text{grad})\,\text{curl}\,\vec{v}$$

for incompressible liquid

$$(\text{curl}\,\vec{v} \cdot \text{grad})\,\vec{v} = 0$$

$$(\vec{v} \cdot \text{grad})\,\text{curl}\,\vec{v} = v_x \cdot \frac{\partial}{\partial x} \cdot \Delta\psi + v_y \frac{\partial}{\partial y} \Delta\psi$$

$$= \frac{\partial\psi}{\partial y} \frac{\partial}{\partial x} \Delta\psi - \frac{\partial\psi}{\partial x} \frac{\partial}{\partial y} \Delta\psi$$

Putting it together

$$\frac{\partial}{\partial t} \Delta\psi - \frac{\partial\psi}{\partial x} \frac{\partial}{\partial y} \Delta\psi + \frac{\partial\psi}{\partial y} \frac{\partial}{\partial x} \Delta\psi - \nu \Delta\Delta\psi = 0$$

If we know the stream function we can immediately determine the form of the streamlines for steady flow

$$\frac{dx}{v_x} = \frac{dy}{v_y} \quad \text{or} \quad v_y\,dx - v_x\,dy = 0$$

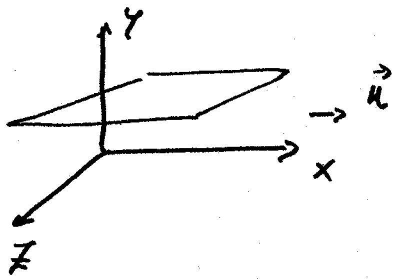
differential eq. of streamlines in two-dimensional flow

$$\frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy = d\psi = 0$$

$\psi(x, y) = \text{constant}$ family of curves: streamlines

Incompressible viscous flow in pipe

First consider fluid enclosed between two parallel planes moving with a constant relative velocity $\vec{u}$



one of the plates is xz-plane
x-axis along $\vec{u}$

quantities depend on y only
fluid velocity is in x-direction

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \text{grad})\vec{v} = -\frac{1}{\rho}\,\text{grad}\, p + \frac{\eta}{\rho}\Delta \vec{v}$$

Navier – Stokes

x - component of equation for steady state:

$$\frac{d^2 v}{dy^2} = 0 \quad \rightarrow \quad v = a \cdot y + b$$

y - component: $\quad \frac{dp}{dy} = 0 \quad \rightarrow \quad p = \text{constant}$

$y = 0 \quad v = 0$

$y = h \quad v = u$

h is distance between plates

$$v = \frac{y \cdot u}{h}$$

fluid velocity is linear

$$\bar{v} = \frac{1}{h}\int_0^h v\,dy = \frac{1}{2}u \qquad \text{mean fluid velocity}$$

$$P_i = p n_i - \sigma'_{ik} n_k = -\sigma_{ik} n_k$$

force acting on surface

normal component of force on either plate is just p

tangential friction force on the plane at $y=0$ is

$$\sigma_{xy} = \eta \frac{dv}{dy} = \eta \frac{u}{h}$$

force on plane at $y=h$ is $-\eta \frac{u}{h}$

Next example: steady flow between two fixed parallel planes in the presence of a pressure gradient.
Same set-up for coordinate system

x-component of NS eq.:

$$\frac{\partial^2 v}{\partial y^2} = \frac{1}{\eta}\frac{\partial p}{\partial x}$$

depends on y — depends on x

y component: $\frac{\partial p}{\partial y} = 0$ $\quad$ p is independent of y

$$\frac{dP}{dx} = \text{constant}$$

$$v = \frac{1}{2\eta} \frac{dP}{dx} \cdot y^2 + ay + b$$

$v = 0$ at $y = 0$

$v = 0$ at $y = h$

$$v = -\frac{1}{2\eta} \frac{dP}{dx} y (y - h)$$

parabolic variation across the fluid
maximum in the middle

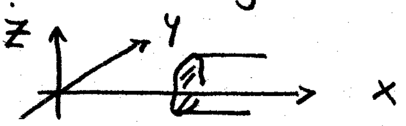
$$\overline{v} = -\frac{h^2}{12\eta} \frac{dP}{dx}$$ average over depth

$$\sigma_{xy} = \eta \left. \frac{\partial v}{\partial y} \right|_{y=0} = -\frac{1}{2} h \frac{dP}{dx}$$

frictional force acting on fixed plane

<u>Pipe with arbitrary cross section</u>

same along length of pipe



v is along x-axis $\quad v(y,z)$

y component of NS eq: $\quad \dfrac{\partial p}{\partial y} = 0$

z component : $\quad \dfrac{\partial p}{\partial z} = 0$

pressure is constant over cross-section of pipe

x-component :

$$\underbrace{\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}}_{\text{on } y,z \text{ only}} = \underbrace{\frac{1}{\eta}\frac{dp}{dx}}_{\text{on } x\text{-only}}$$

$$\Downarrow$$

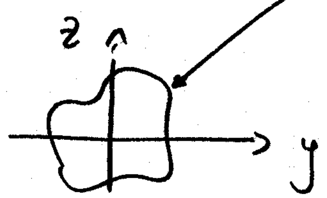
$$\frac{dp}{dx} = \text{constant}$$

$-\dfrac{\Delta p}{l}$ pressure gradient

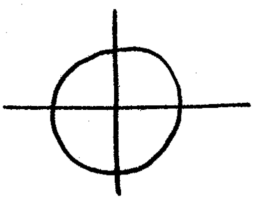
l pipe length

Δp pressure difference between ends

$\Delta v = \text{constant} \qquad v = 0$ on boundary



solution for circle:



$v(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\frac{\Delta P}{\eta \cdot l}$$

$$v = -\frac{\Delta P}{4\eta l}\cdot r^2 + a\ln r + b$$

$a = 0$ to keep v finite at $r = 0$

$v = 0 \quad r = R$

$$v = \frac{\Delta P}{4\cdot\eta l}\left(R^2 - r^2\right)$$

parabolic profile

$$Q = 2\pi\rho\int_0^R r v\,dr = \frac{\pi\cdot\Delta P}{8\nu\cdot l}R^4$$

mass Q of fluid passing per unit time through cross section of pipe

$\rho\, 2\pi r v\, dr$ in annular element $2\pi r\, dr$