Fluid Flows

law of similarity

Consider steady flow (incompressible)

$\nu = \frac{\eta}{\rho}$ kinematic viscosity
this is the only characteristic fluid parameter
in Navier-Stokes equation

We have to solve for $\vec{v}$, $\frac{p}{\rho}$

l characteristic linear size of body
u main stream velocity (constant past solid body)

ν, u, l flow characterized by these parameters

$$\left.\begin{aligned} \nu &\sim \frac{cm^2}{sec} \\ l &\sim cm \\ u &\sim \frac{cm}{sec} \end{aligned}\right\} \text{dimensions}$$

$$Re = \frac{u \cdot l}{\nu} = \frac{\rho \cdot u \cdot l}{\eta}$$

Reynolds number
only characteristic dimensionless quantity

$\frac{\vec{r}}{l}$, $\frac{\vec{v}}{u}$ scaled physical quantities

$$\vec{v} = u \cdot f\left(\frac{\vec{r}}{l}, Re\right)$$

In two different flows of the same type (for example, flow past spheres with different radii by fluids with different viscosities), the velocities $\vec{v}/u$ are the same functions of the ratio $\frac{\vec{r}}{l}$ if the Reynolds number is the same for each flow.

Flows which can be obtained from one another by simply changing the unit of measurement of coordinates and velocities are said to be similar. Thus flows of the same type with the same Reynolds number are similar. This is called <u>the law of similarity</u>.

$$p = \rho u^2 f\left(\frac{\vec{r}}{l}, Re\right)$$ for pressure

$\frac{p}{\rho u^2}$ dimensionless

Similar considerations can be applied to quantities which characterize the flow but are not functions of the coordinates

$$F = \rho u^2 \cdot l^2 f(R_e) \qquad \text{drag force}$$

since $\rho u^2 \cdot l^2$ has force dimension

In non-steady flow τ some characteristic time (like oscillation period)

ν, u, l, τ for non-steady flow

$$S = \frac{u\tau}{l} \qquad \text{Strouhal number, dimensionless}$$

$$S = f(R_e) \qquad \text{for spontaneous oscillations}$$

Flow with small Reynolds numbers

$$\underbrace{(\vec{v} \cdot \text{grad})\vec{v}}_{\sim \frac{u^2}{l}} = -\frac{1}{\rho}\,\text{grad}\,p + \underbrace{\frac{\eta}{\rho}\Delta\vec{v}}_{\sim \eta \frac{u}{\rho l^2}}$$

ratio of $$\frac{(\vec{v}\cdot \text{grad})\vec{v}}{\frac{\eta}{\rho}\Delta \vec{v}} = \frac{u^2}{l}\cdot\frac{\rho l^2}{\eta u} = \frac{\rho\, u\, l}{\eta} = Re$$

For small Re :

$$\eta \Delta \vec{v} - \text{grad}\, p = 0$$

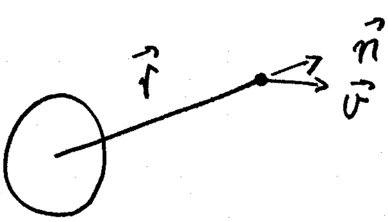
$$\text{div}\, \vec{v} = 0 \quad \text{continuity eq.}$$

sphere

far past sphere

velocity field :

$$\vec{V} = -\frac{3}{4} R \frac{\vec{u} + \vec{n}(\vec{u}\cdot\vec{n})}{r} - \frac{1}{4} R^3 \frac{\vec{u} - 3\vec{n}(\vec{u}\cdot\vec{n})}{r^3}$$
$$+ \vec{u}$$



pressure : $$p = p_0 - \frac{3}{2}\eta \frac{\vec{u}\cdot\vec{n}}{r^2} R$$

pressure at infinity

Drag force on sphere $\vec{F}$ parallel to $\vec{u}$

$\vec{P}$ force acting on unit surface area:

$$P_i = -\sigma_{ik} n_k = p \cdot n_i - \sigma'_{ik} n_k$$

$$\sigma'_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)$$

$$F = \oint \left(-p \cos\theta + \sigma'_{rr} \cos\theta - \sigma'_{r\theta} \sin\theta \right) df$$

surface of sphere

$$\sigma'_{rr} = 2\eta \frac{\partial v_r}{\partial r} \qquad \sigma'_{r\theta} = \eta \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)$$

$$v_r = u \cos\theta \left[1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right]$$

$$v_\theta = -u \cdot \sin\theta \left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3} \right]$$

$\vec{u}$ is along z direction

$$\longrightarrow \quad \sigma'_{rr} = 0, \quad \sigma'_{r\theta} = -\frac{3\eta}{2R} u \sin\theta$$

$$p = p_0 - \frac{3\eta}{2R} u \cos\theta$$

$$F = \frac{3\eta u}{2R} \oint df$$

$$F = 6\pi \eta R \cdot u$$ Stokes' formula

drag on sphere moving slowly in fluid

$$F = \rho \cdot u^2 \cdot l^2 \cdot f(Re)$$

$$l = 2R, \quad f(Re) \longrightarrow f(Re) = \frac{3\pi}{2Re}$$

<u>Oseen's improvement</u>

flow past sphere is not valid at large distances

$(\vec{v} \cdot \text{grad}) \vec{v}$ estimate $\vec{v} \simeq \vec{u}$

$\frac{uR}{r^2}$ velocity derivative

$(\vec{v} \cdot \text{grad}) \vec{v} \sim \frac{u^2 R}{r^2}$ was neglected

retained term was order $\eta R \frac{u}{\rho r^3}$

$\eta \frac{Ru}{\rho r^3} \gg u^2 R / r^2$ only satisfied if $r \ll \frac{\nu}{u}$

$$(\vec{v}\cdot\mathrm{grad})\,\vec{v} \;\rightarrow\; (\vec{u}\cdot\mathrm{grad})\,\vec{v}$$

$$(\vec{u}\cdot\mathrm{grad})\,\vec{v} = -\frac{1}{\rho}\,\mathrm{grad}\,p + \nu\,\Delta\vec{v}$$

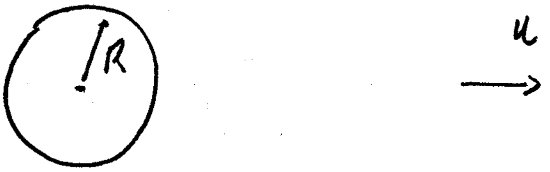
Oseen's approximation to N-S eq.

for $Re = u\dfrac{R}{\nu}$ def. :

$$F = \underbrace{6\pi\eta u R}_{\sim\frac{1}{Re}}\Big(1 + \underbrace{3\frac{uR}{8\nu}}_{O(Re)}\Big)$$

correction term

Infinite cylinder



only Oseen's approximation has solution

$$F = \frac{4\pi\eta u}{\frac{1}{2} - C - \log\left(\frac{uR}{4\nu}\right)} = \frac{4\pi\eta u}{\log\left(3.703\frac{\nu}{uR}\right)}$$ drag per unit length

$C = 0.5772\ldots$ Euler's constant

drag coefficients:

$$C_D = \frac{F_{sphere}}{\frac{1}{2}\rho u^2 \pi \cdot R^2} = \frac{24}{Re}\left(1 + \frac{3}{16} Re\right)$$

sphere

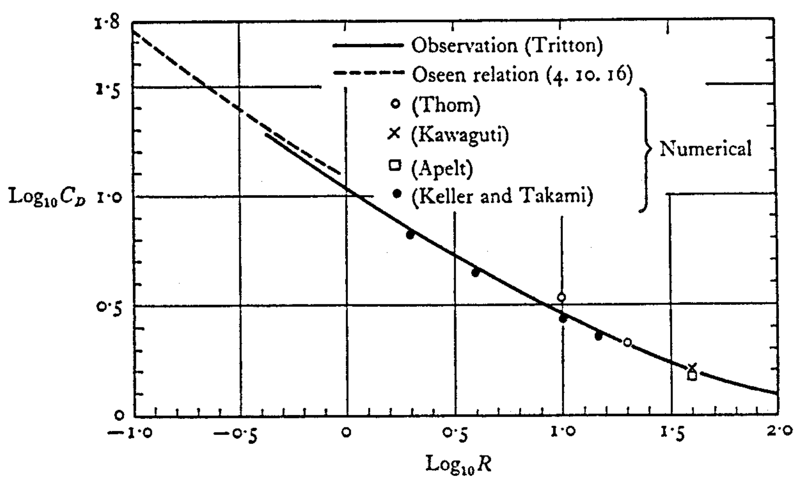
$$Re = \frac{2R \cdot u\rho}{\eta}$$

$$C_D = \frac{F_{cyl}}{\frac{1}{2}\rho u^2 \cdot 2R} = \frac{8\pi}{Re \log\left(\frac{7.407}{Re}\right)}$$

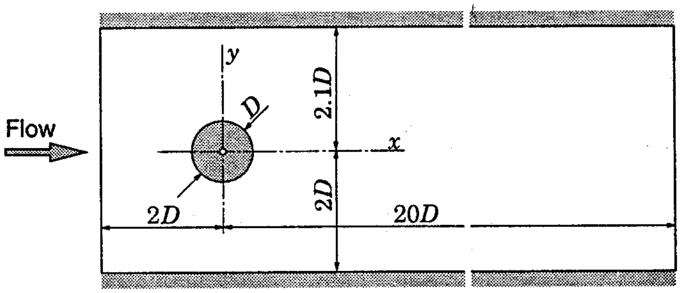
cylinder

$$Re = 2R\rho u/\eta$$

Experiment on cylinder:



Computational example:



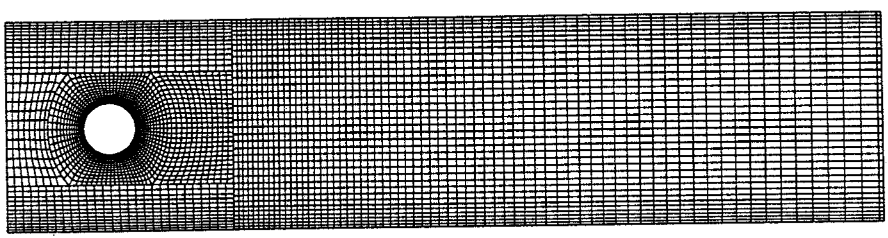
at inlet: $$u_x = \frac{6u}{H^2}\left[(y-y_B)H - (y-y_B)^2\right], \quad u_y = 0$$

parabolic profile

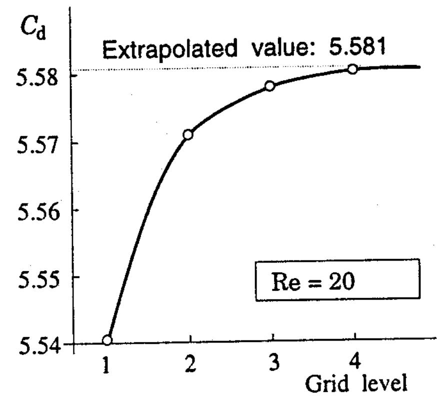
$$H = 4.1D \qquad y_B = -2D$$

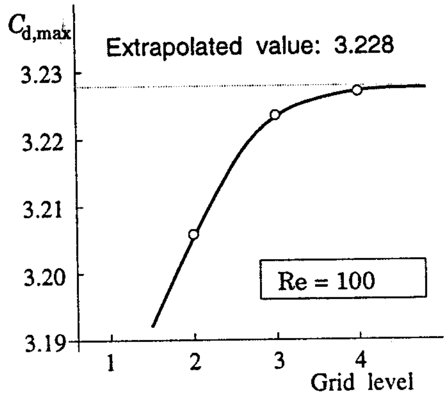
flow is slightly asymmetric

block structured grid:



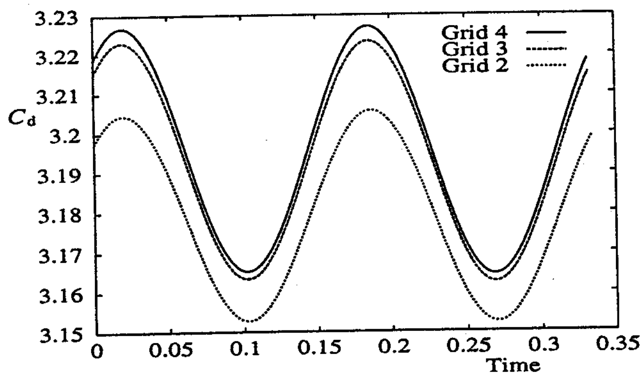
Drag coefficients at different Reynolds numbers:





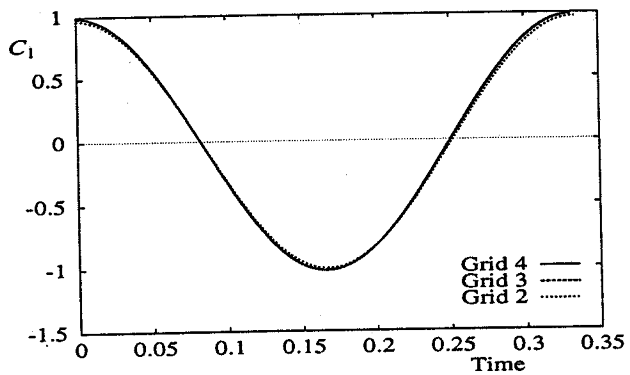
time averaged

Re = 100



drag

three time-level implicit scheme



lift

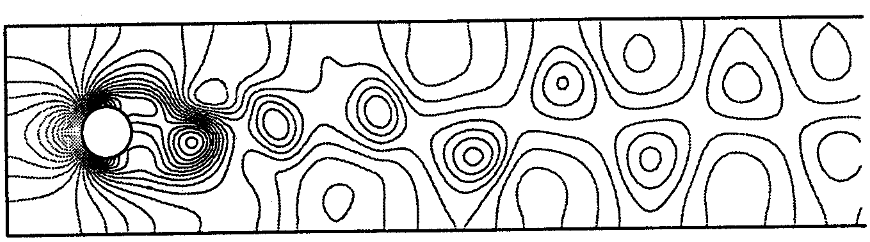
Starting impulsively from rest, the flow goes through a development stage at $Re = 100$ and eventually becomes periodic. Due to vortex shedding, both the drag and lift force oscillate.
663 time steps per oscillation period of lift force

Drag has twice the frequency as the lift (drag force has one maximum and one minimum during the growth and shedding of each vortex while the sign of the lift force depends on the location of the vortex (above or below the cylinder)

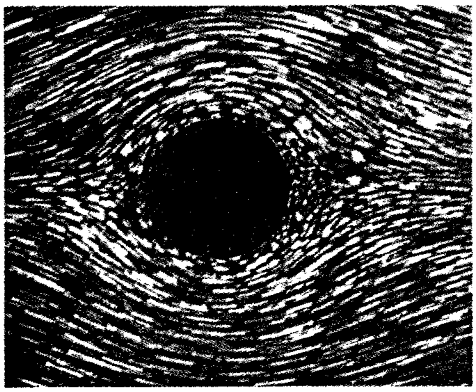
$$S = \frac{D}{uT} = 0.3018 \qquad (0.18 - 0.2 \text{ in infinite stream})$$

confinement in channel speeds up vortex shedding

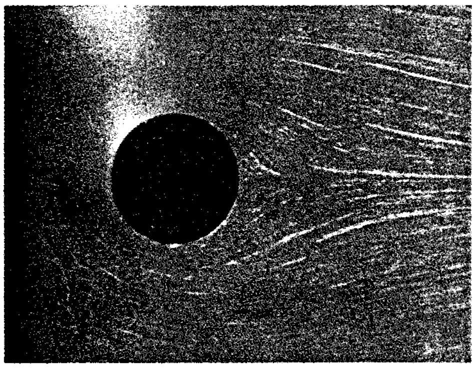
isobars:



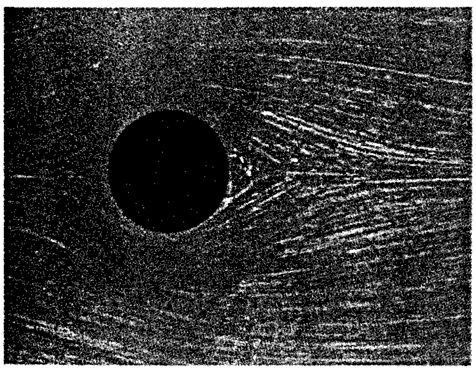
Experiments on flows around cylinder:



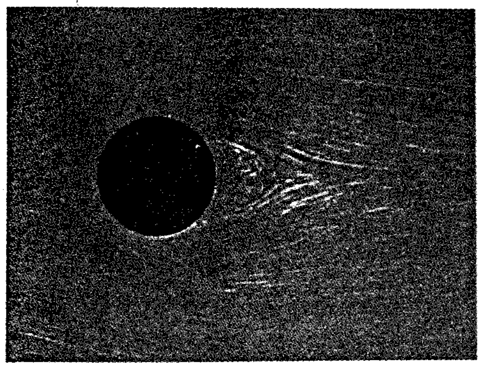
$R = 0{\cdot}25$



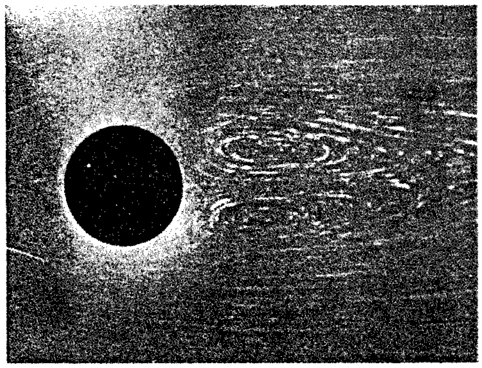
$R = 3{\cdot}64$



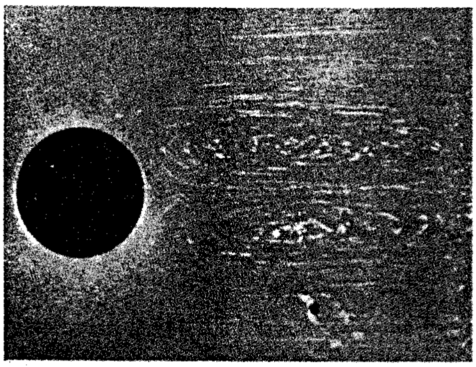
$R = 9{\cdot}10$



$R = 13{\cdot}05$



$R = 39{\cdot}0$

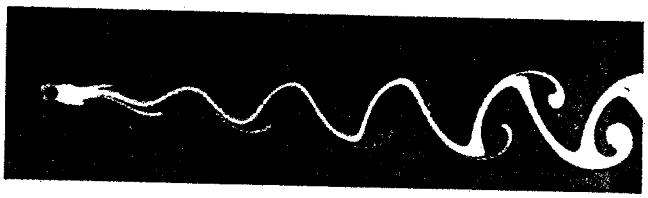


$R = 57{\cdot}7$

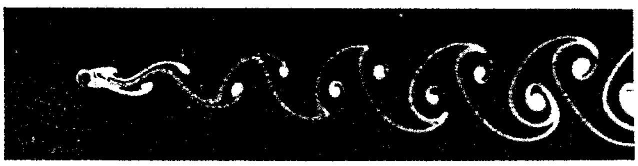
Streak lines in the wake behind a circular cylinder in a stream of oil:

$R = 32$

$R = 55$

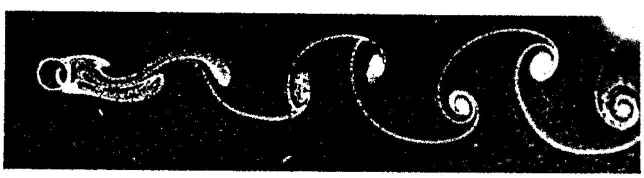


$R = 65$



$R = 73$

$R = 102$



$R = 161$