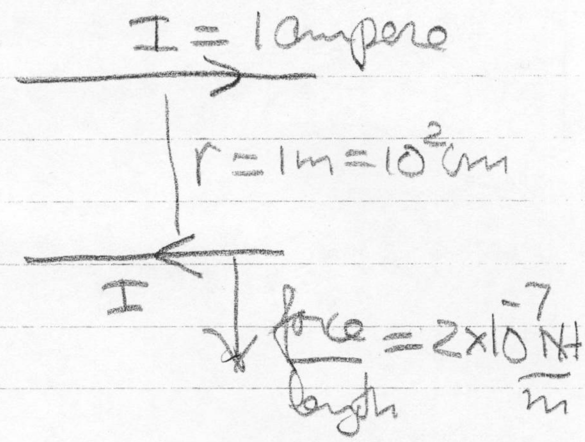
Solutions to homework problems

(1)



In Gaussian units,

$$\frac{\text{force}}{\text{length}} = \frac{2I^2}{c^2 r}$$

$$2\times10^{-7}\,\frac{(10^5\,\text{dyne})}{10^2\,\text{cm}} = \frac{2}{9\times10^{20}\,\frac{\text{cm}^2}{\text{sec}^2}}\,\frac{(1\,\text{amp})^2}{10^2\,\text{cm}}$$

$$\therefore\ 9\cdot10^{-7+5-2+22}\,\underbrace{\frac{\text{dyne cm}^2}{\text{sec}^2}}_{(\text{statamp})^2} = (1\,\text{amp})^2$$

$$3\times10^9\,\text{statamp} = 1\,\text{amp}$$

$\longrightarrow$ 3×10^9 statcoul = 1 coul

$\text{volt} = \frac{\text{Joule}}{\text{Coul.}} = \frac{10^7\,\text{erg}}{3\times 10^9\,\text{statCoul}} = \frac{1}{3}10^{-2}\,\text{statvolt}$

$\text{henry} = \frac{\text{Volt sec}}{\text{amp}} = \frac{1}{3}\frac{10^{-2}\,\text{statvolt sec}}{3\cdot 10^9\,\text{statamp}}$

$\text{henry} = \frac{1}{9}10^{-11}\,\frac{\text{statvolt sec}}{\underbrace{\text{statamp}}_{\text{sec}^2/\text{cm}}}$

for $q = 1\,\text{coul}$, $B = 1\,\text{tesla}$

$V = 1\,\frac{\text{m}}{\text{sec}} \qquad F = qVB = 1\,\text{Nt}$

$F = q\,V/c\,B$

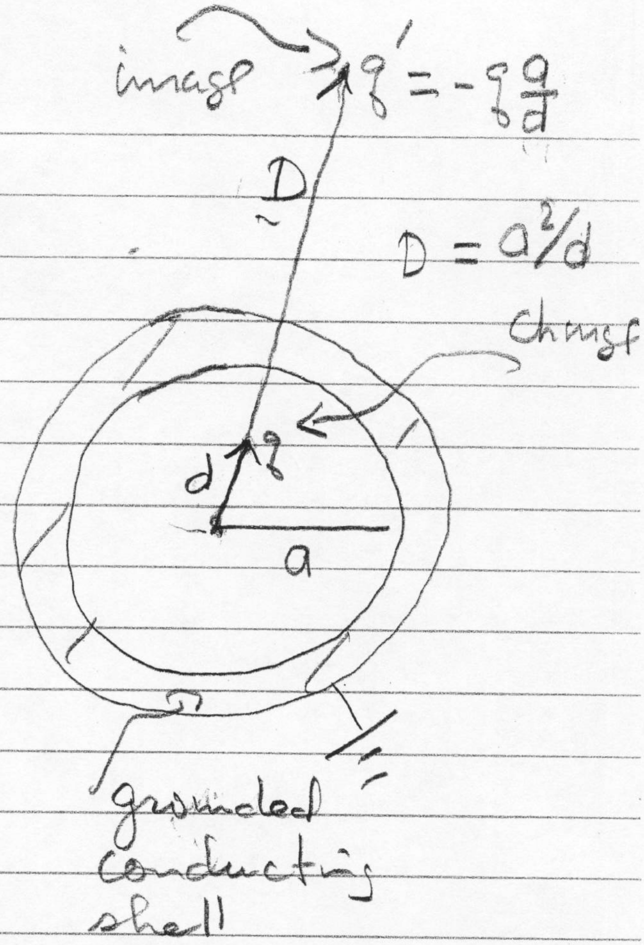
$10^5\,\text{dyne} = \frac{3\times 10^9\,\text{statcoul}}{3\times 10^{10}\,\frac{\text{cm}}{\text{sec}}}\;10^2\frac{\text{cm}}{\text{sec}}\,(1\,\text{tesla})$

$10^4\,\underbrace{\frac{\text{dyne}}{\text{statcoul}}}_{\text{gauss}} = 1\,\text{tesla}$

$$\text{Farad} = \frac{\text{Coul}}{\text{Volt}} = \frac{3\times10^{9}\ \text{statcoul}}{\frac{1}{3}\,10^{-2}\ \text{statvolt}}$$

$$\text{"} = 9\times10^{11}\ \text{cm}$$

2. Jackson 2,2



(a) for $r < a$

$$\Phi(r) = \left\{ \frac{q}{|\underset{\sim}{r} - \underset{\sim}{d}|} - \frac{q\,a/d}{|\underset{\sim}{r} - \underset{\sim}{D}|} \right\}$$

$$\underset{\sim}{D} = \frac{a^2}{d^2}\underset{\sim}{d}$$

(b) induced surface charge

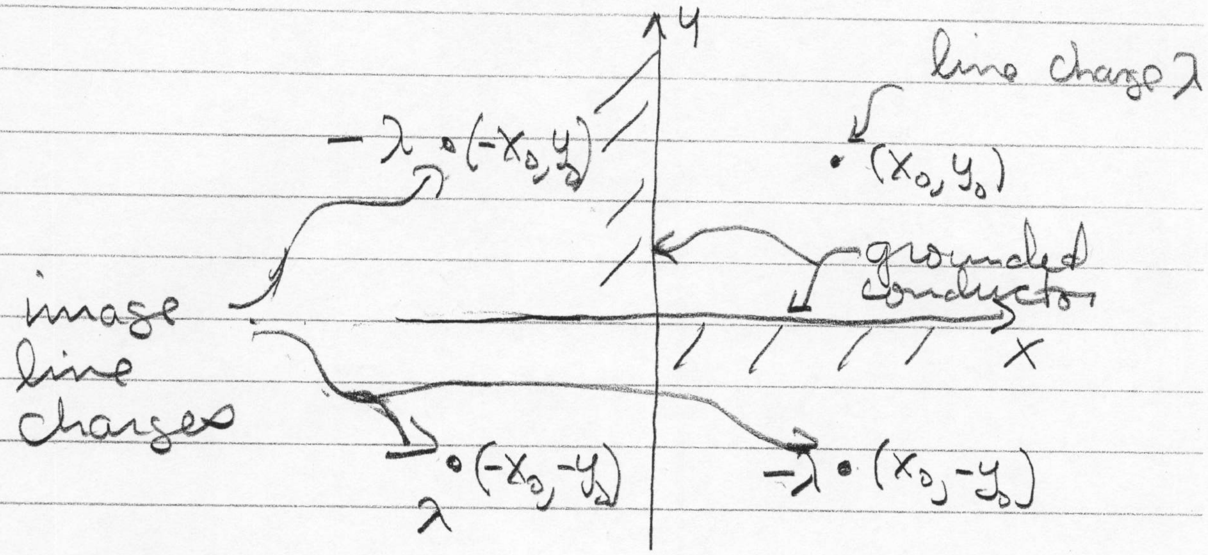
$$\sigma = -\frac{1}{4\pi} E_r(r=a) = +\frac{1}{4\pi}\frac{\partial \Phi}{\partial r}\bigg|_{r=a}$$

(c) $$F_r = \frac{-q\,q'}{|d-D|^2} = \frac{q^2 a/d}{|d - a^2/d|^2}$$

(d) $\Phi \rightarrow \Phi + V$, but $\underset{\sim}{E}$ and σ unchanged.

No changes if charge Q is added.
Q goes to outside

3. Jackson 2.3



for $x \geq 0,\ y \geq 0$ (1st quadrant)

(a) $$\varphi = -\lambda \ln\left[(x-x_0)^2+(y-y_0)^2\right] + \lambda \ln\left[(x-x_0)^2+(y+y_0)^2\right]$$
$$-\lambda \ln\left[(x+x_0)^2+(y+y_0)^2\right] + \lambda \ln\left[(x+x_0)^2+(y-y_0)^2\right]$$

$$\varphi = -\lambda \ln\left[\frac{\left[(x-x_0)^2+(y-y_0)^2\right]\left[(x+x_0)^2+(y+y_0)^2\right]}{\left[(x-x_0)^2+(y+y_0)^2\right]\left[(x+x_0)^2+(y-y_0)^2\right]}\right]$$

note that $\varphi(x=0, y) = \varphi(x, y=0) = 0$

$$\therefore \left.\frac{\partial \varphi}{\partial y}\right|_{x=0} = \left.\frac{\partial \varphi}{\partial x}\right|_{y=0} = 0$$

b. $$\sigma = -\frac{1}{4\pi}\frac{\partial \varphi}{\partial y}\bigg|_{y=0} = +\frac{\lambda}{4\pi}\left\{ \frac{-2y_0}{(x-x_0)^2+y_0^2}\right.$$

$$\left. -\frac{2y_0}{(x-x_0)^2+y_0^2} + \frac{2y_0}{(x+x_0)^2+y_0^2} + \frac{2y_0}{(x+x_0)^2+y_0^2}\right\}$$

$$\frac{\sigma}{\lambda} = \frac{\cancel{4}y_0}{\cancel{4}\pi}\left\{\frac{-1}{(x-x_0)^2+y_0^2} + \frac{1}{(x+x_0)^2+y_0^2}\right\}$$

$$\frac{\sigma}{\lambda}\frac{\pi x_0^2}{y_0} = \underbrace{\left\{\frac{-1}{\left(\frac{x}{x_0}-1\right)^2+\left(\frac{y_0}{x_0}\right)^2} + \frac{1}{\left(\frac{x}{x_0}+1\right)^2+\left(\frac{y_0}{x_0}\right)^2}\right\}}_{}$$

only one parameter $\left(\frac{y_0}{x_0}\right)$ $\Big\} \longrightarrow g\left(\frac{x}{x_0}, \frac{y_0}{x_0}\right)$

note that $g \leq 0$ for $\frac{x}{x_0} > 0$

$g = 0$ for $\frac{x}{x_0} = 0, \infty$

c.

$$Q_x = \int_0^\infty dx\, \sigma_x = \frac{\lambda y_0}{\pi}\left\{ -\int_0^\infty \frac{dx}{(x-x_0)^2+y_0^2} + \int_0^\infty \frac{dx}{(x+x_0)^2+y_0^2} \right\}$$

$$= \frac{\lambda y_0}{\pi}\left\{ -\int_{-x_0}^\infty \frac{dx}{x^2+y_0^2} + \int_{x_0}^\infty \frac{dx}{x^2+y_0^2} \right\}$$

$$= -\frac{\lambda y_0}{\pi}\int_{-x_0}^{x_0} \frac{dx}{x^2+y_0^2} = -\frac{2\lambda \cancel{y_0}}{\pi}\frac{1}{\cancel{y_0}}\tan^{-1}\frac{x_0}{y_0}$$

d. For $\rho = \sqrt{x^2+y^2} \gg \rho_0 = \sqrt{x_0^2+y_0^2}$

expect quadrupole term in Taylor expansion for potential

let $\lambda_1 = \lambda \quad (x_1, y_1) = (x_0, y_0)$

$\lambda_2 = -\lambda \quad (x_2, y_2) = (-x_0, y_0)$

$$\lambda_3 = \lambda \qquad (x_3, y_3) = (-x_0, -y_0)$$

$$\lambda_4 = -\lambda \qquad (x_4, y_4) = (x_0, -y_0)$$

$$\Phi = -\sum_{i=1}^{4} \lambda_i \ln(\underset{\sim}{\rho} - \underset{\sim}{\rho_i})^2 \simeq -\underbrace{\left(\sum_{i=1}^{4} \lambda_i\right)}_{0} \ln \rho^2$$

Taylor expand using $|\rho| >> |\rho_i|$

$$-\underbrace{\left(\sum_{i=1}^{4} \lambda_i \underset{\sim}{\rho_i}\right)}_{0} \cdot \underset{\sim}{\nabla_\perp} \ln \rho^2$$

$$-\frac{1}{2}\underbrace{\left(\sum_{i=1}^{4} x_i^2 \lambda_i\right)}_{0} \frac{\partial^2}{\partial x^2} \ln \rho^2$$

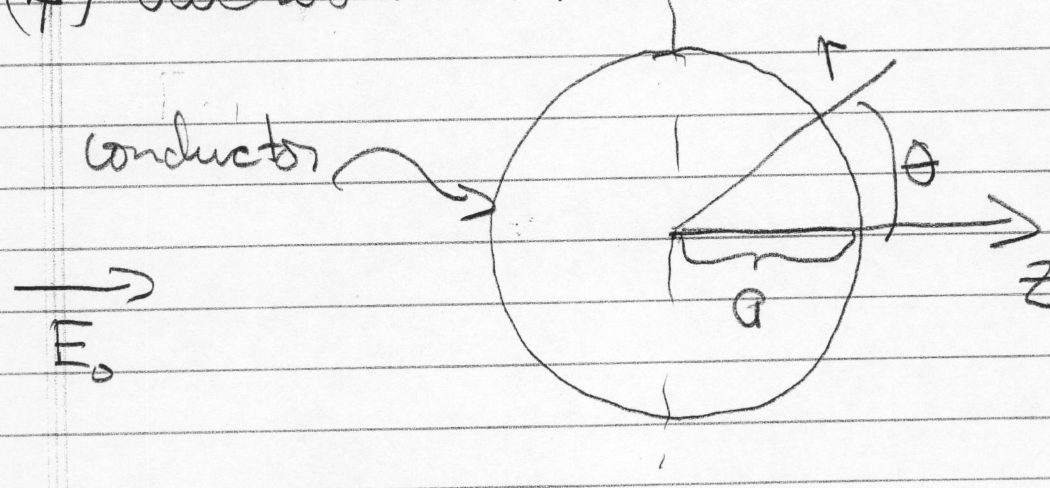
$$-\frac{1}{2}\underbrace{\left(\sum_{i=1}^{4} y_i^2 \lambda_i\right)}_{0} \frac{\partial^2}{\partial y^2} \ln \rho^2$$

$$-\frac{1}{2}\underbrace{\left(\sum_{i=1}^{4} 2 x_i y_i \lambda_i\right)}_{8 x_0 y_0 \lambda} \frac{\partial^2}{\partial x \partial y} \ln \rho^2$$

$$\phi \simeq -4x_0 y_0 \lambda \underbrace{\frac{\partial^2}{\partial x \partial y} \ln(x^2+y^2)}_{\frac{-4xy}{(x^2+y^2)^2}}$$

$$\phi \simeq \frac{+16\lambda x_0 y_0 x y}{\rho^4}$$

(4) Jackson 2.9



(a) outside conductor

$$\phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

$$F_z = \int_0^{\pi/2} 2\pi a^2 \sin\theta \, d\theta \, \underbrace{\frac{1}{8\pi}\left(\frac{\partial\phi}{\partial r}\right)^2}_{\substack{\text{tension} \\ \text{in field} \\ \text{lines at} \\ \text{surface}}} \cos\theta$$

$$F_z = \int_0^{\pi/2} \frac{2\pi a^2}{8\pi} \sin\theta \, d\theta \overset{\cos\theta}{\downarrow} \left[-E_0 \cos\theta - \frac{2E_0 a^3 \cos\theta}{a^3}\right]^2$$

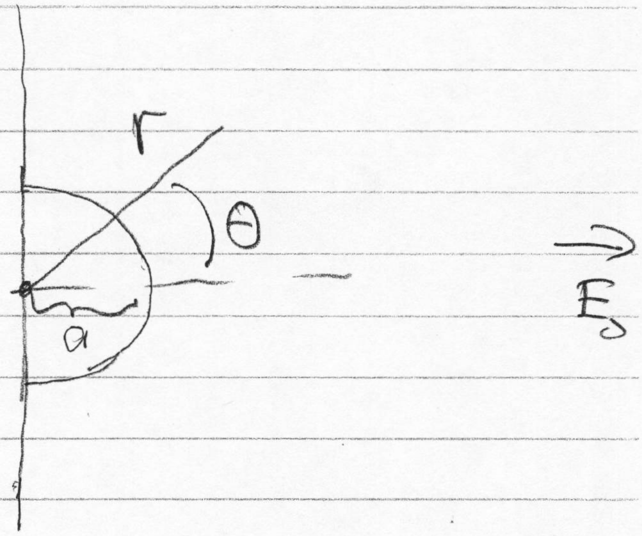
$$F_z = \frac{a^2}{4} 9E_0^2 \underbrace{\int_0^{\pi/2} \underbrace{\sin\theta \, d\theta}_{-d(\cos\theta)} \cos^3\theta}_{\frac{1}{4}} = \frac{9}{16} a^2 E_0^2$$

(b) $$\phi(r,\theta) = -E_0 r \cos\theta + \frac{E_0 a^3 \cos\theta}{r^2} + Q/r$$

$$\left.\frac{\partial\phi}{\partial r}\right|_{r=a} = -3E_0 \cos\theta - \frac{Q}{a^2}, \text{ etc.}$$

(15) Jackson 2.10

(a)



Outside the hemispherical boss, the potential is given by

$$\Phi(r,\theta) = -E_0 r\cos\theta + \frac{E_0 a^3}{r^2}\cos\theta$$

on the boss

$$\sigma = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial r} = \frac{3E_0\cos\theta}{4\pi}$$

on the plane

$$\sigma = -\frac{1}{4\pi}\frac{\partial\Phi}{\partial z}\bigg|_{\theta=\pi/2} = -\frac{1}{4\pi}\left(-\frac{\partial\Phi}{r\partial\theta}\right)_{\theta=\pi/2}$$

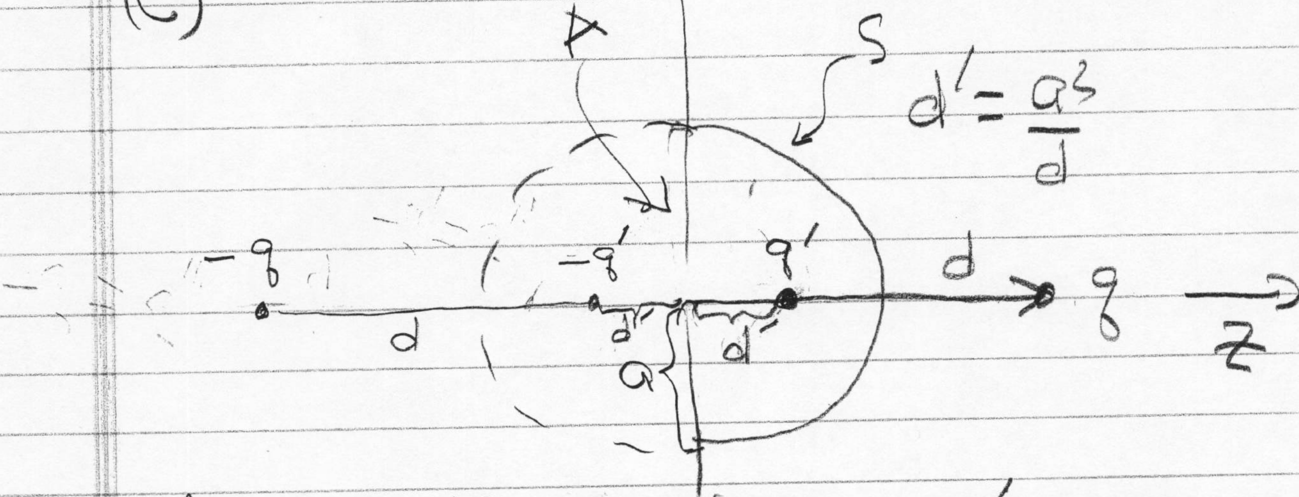
$$\sigma = \frac{1}{4\pi r}\left[E_0 r\sin\theta - \frac{E_0 a^3}{r^2}\sin\theta\right]\Bigg|_{\theta=\pi/2}$$

$$\sigma = \frac{E_0}{4\pi}\left(1 - \frac{a^3}{r^3}\right) \quad \text{for } r > a$$

(b) $$Q = \int_0^{\pi/2} 2\pi a^2 \sin\theta\, d\theta\, \frac{3E_0\cos\theta}{4\pi}$$

$$= \frac{3}{2}E_0 a^2 \cdot \frac{1}{2} = \frac{3}{4}E_0 a^2$$

(c)



There are three images: $q' = -q\frac{a}{d}$, $-q'$, and $-q$ (see figure)

The charge and three images make the potential zero on the plane and on the boss.

Charge on boss $= q_b = \int_S \frac{\underset{\sim}{E} \cdot d\underset{\sim}{S}}{4\pi}$

S is the hemispherical surface

Let A be the base of the hemisphere (a circular disc).

By Gauss' law

$$-q\frac{a}{d} = q' = \int_S \frac{\underset{\sim}{E} \cdot d\underset{\sim}{S}}{4\pi} + \int_A \frac{\underset{\sim}{E} \cdot d\underset{\sim}{S}}{4\pi}$$

$$\therefore \quad q_b = -q\frac{a}{d} - \int_A \frac{\underset{\sim}{E} \cdot d\underset{\sim}{S}}{4\pi}$$

$$q_b = -q\frac{a}{d} - \int_0^a \frac{2\pi r\,dr}{4\pi} \left.\frac{\partial \varphi}{\partial z}\right|_{z=0}$$

$$\frac{\partial \varphi}{\partial z} = \frac{+2q\,d}{(d^2+r^2)^{3/2}} + \frac{2q'd'}{(d'^2+r^2)^{3/2}}$$

$$q_b = -q\frac{a}{d} - \frac{1}{\cancel{2}}\int_0^a r\,dr\left[\frac{\cancel{2}qd}{(d^2+r^2)^{3/2}} + \frac{\cancel{2}q'd'}{(d'^2+r^2)^{3/2}}\right]$$

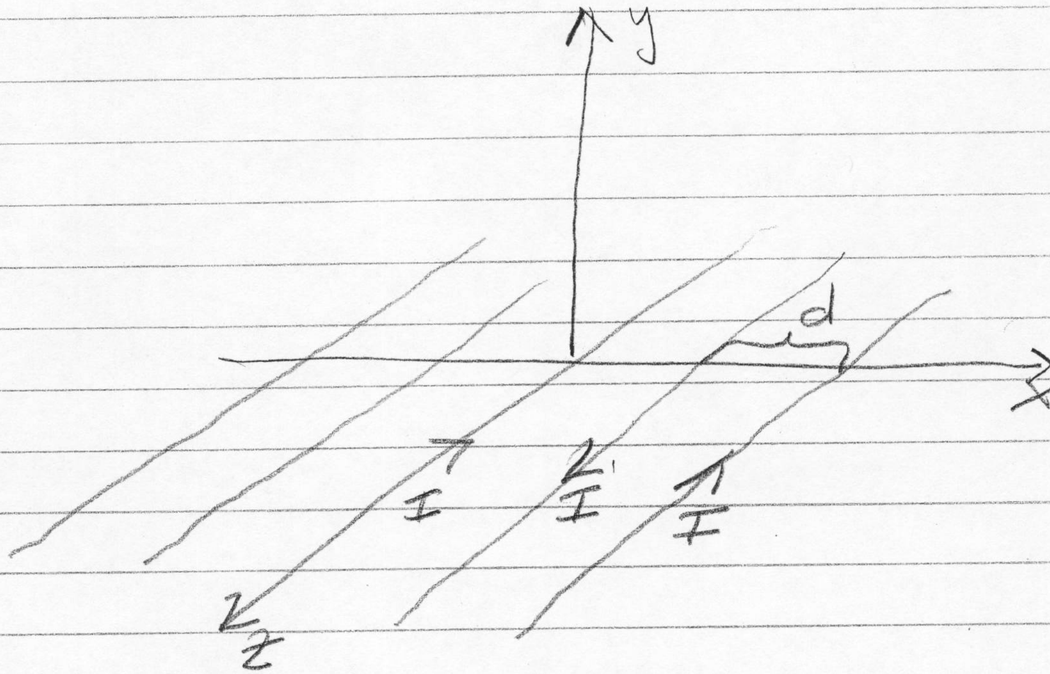
$$q_b = -q\frac{a}{d} - \left[\frac{qd}{(d^2+r^2)^{1/2}} \quad \frac{q'd'}{(d^2+r^2)^{1/2}}\right]_a^0$$

$$q_b = -q\frac{a}{d} - \left[q + q' - \frac{2qd}{\sqrt{d^2+a^2}} - \frac{2q'd'}{\sqrt{d'^2+a^2}}\right]$$

$$q_b = -q\frac{a}{d} - q\left(1 - \frac{d}{\sqrt{d^2+a^2}}\right) + \frac{qa}{d}\left(1 - \frac{a^2/d}{\sqrt{\frac{a^4}{d^2}+a^2}}\right)$$

$$q_b = -q \qquad + \frac{q}{\sqrt{d^2+a^2}}\underbrace{\left(+d - \frac{a^2}{d}\right)}_{\frac{d^2-a^2}{d}}$$

(6)



for $y > 0$ $\quad \underset{\sim}{J} = 0 \quad \therefore \underset{\sim}{\nabla} \times \underset{\sim}{B} = 0$

$$\therefore \underset{\sim}{B} = -\underset{\sim}{\nabla}\varphi$$

$$0 = \underset{\sim}{\nabla} \cdot \underset{\sim}{B} = -\nabla^2\varphi$$

$$\varphi \to 0 \text{ as } y \to \infty$$

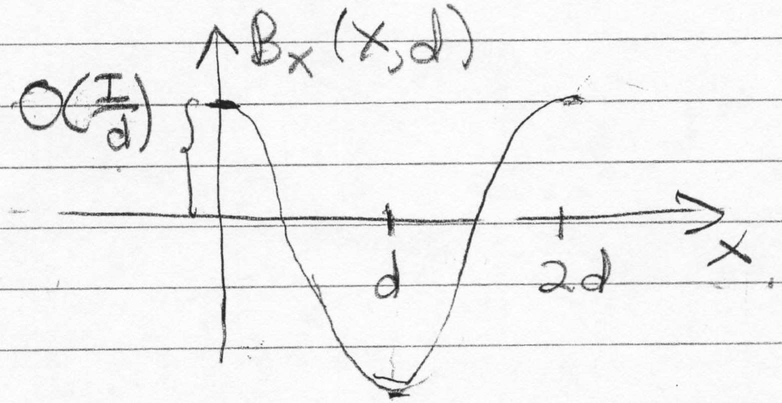
$$\varphi(x+2d, y) = \varphi(x, y)$$

$$\therefore \varphi(x,y) = \sum_n \left[A_n \cos\left(\frac{n\pi x}{d}\right) + B_n \sin\left(\frac{n\pi x}{d}\right)\right] e^{-\frac{n\pi}{d}y}$$

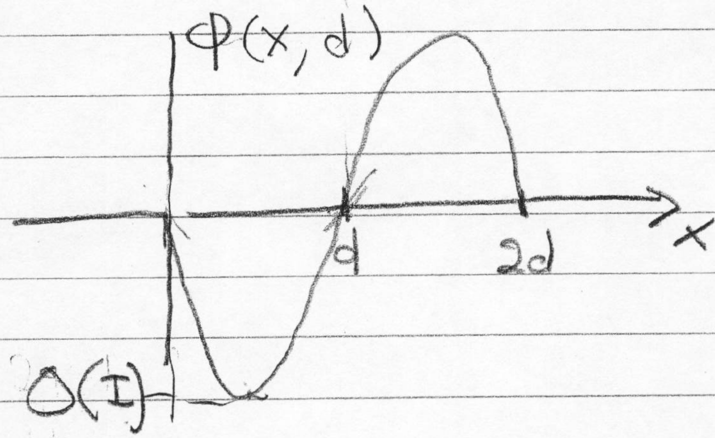
$$\int_0^{2d} \varphi(x, y=d) \cos\frac{n'\pi x}{d}\,dx = A_{n'}\, 2d \cdot \frac{1}{\cancel{2}}\, e^{-\frac{\pi n' \cancel{d}}{\cancel{d}}}$$

$$\int_0^{2d} \varphi(x, y=d) \sin\frac{n'\pi x}{d}\,dx = B_{n'}\, d\, e^{-\pi n'}$$

expect $B_x(x, d) = -\frac{\partial \varphi(x,d)}{\partial x}$ to be of form



Hence



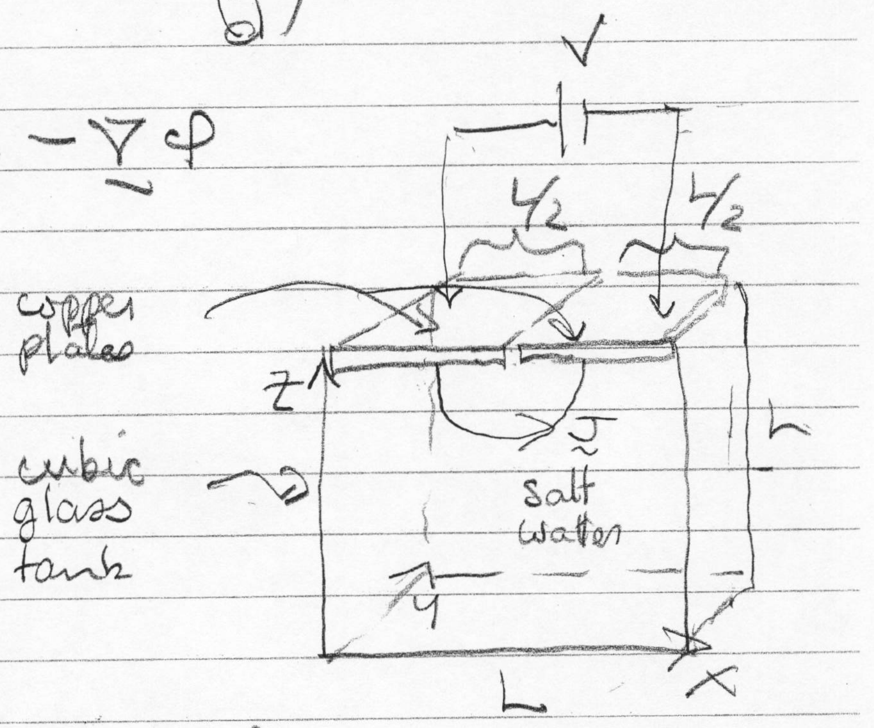
$\therefore$ $A_{n'} = 0$, $B_0 = 0$, $B_1 \simeq -I$

for large y/d

$$\varphi(x,y) \simeq - I \sin\left(\frac{\pi}{d}\right) e^{-\frac{\pi y}{d}}$$

$$\underset{\sim}{B}(x,y) = -\underset{\sim}{\nabla}\varphi$$

(7)



in the salt water

$$\underset{\sim}{E} = -\underset{\sim}{\nabla}\varphi \quad , \quad \underset{\sim}{J} = -\sigma \underset{\sim}{\nabla}\varphi$$

$$0 = \frac{\partial \rho}{\partial t} = -\underset{\sim}{\nabla}\cdot \underset{\sim}{J} = \sigma \nabla^2 \varphi$$

$$0 = J_n = \sigma \frac{\partial \varphi}{\partial n} \quad \text{on glass walls}$$

$$\varphi(x,y,L) = \left\{ \begin{array}{ll} V & 0 < x < L/2 \\ 0 & \frac{L}{2} < x < L \end{array} \right\} \text{on copper plates}$$

$$\phi(x,y,z) = \sum_n A_n \cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi z}{L}\right)$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = \left.\frac{\partial\phi}{\partial x}\right)_{x=L} = \left.\frac{\partial\phi}{\partial y}\right|_{y=0} = \left.\frac{\partial\phi}{\partial y}\right|_{y=L} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0} = 0$$

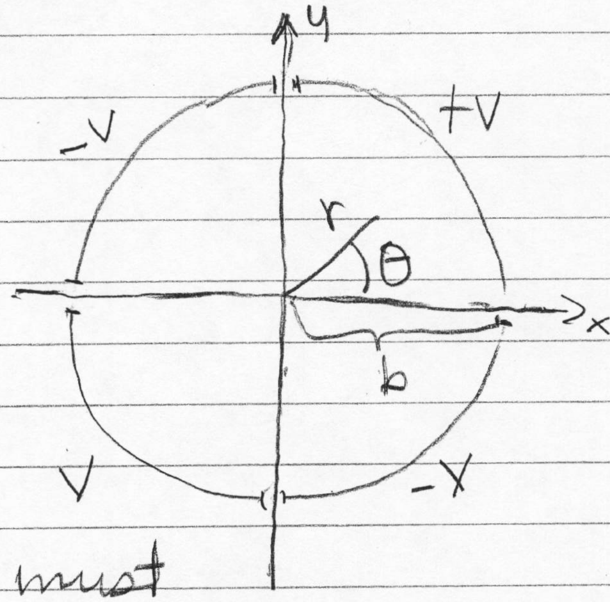
$$\int_0^L \phi(x,y,z=L)\cos\frac{n'\pi}{L}x\,dx = A_{n'}\frac{L}{2}\cosh(\pi n')$$

$$V\int_0^{L/2}\cos\left(\frac{n'\pi x}{L}\right)dx = A_{n'}\frac{L}{2}\cosh(\pi n')$$

$$V\underbrace{\frac{L}{n'\pi}\sin\frac{n'\pi x}{L}\Big|_0^{L/2}}_{\sin\frac{n'\pi}{2}} = \text{"}$$

$$A_{n'} = \begin{cases} 0 & \text{for } n' \text{ even} \\ \dfrac{-V\,2\,(-)^{(n'+1)/2}}{n'\pi\cosh(\pi n')} & \text{for } n' \text{ odd} \end{cases}$$

(8) J. 2.14 (a)



Inside cylinder, must discard solns varying like

$$\ln r \; , \; \frac{1}{r^l}$$

Since $\Phi(r,\theta)$ is odd in θ, need retain only $\sin l\theta$ terms

$$\Phi(r,\theta) = \sum_{l=1}^{\infty} A_l \, r^l \sin(l\theta)$$

$$\int_0^{2\pi} d\theta \, \Phi(b,\theta) \sin l\theta = A_l \, b^l \, 2\pi \frac{1}{2}$$

$$\pi b^l A_l = 2V \int_0^{\pi/2} d\theta \sin l\theta - 2V \int_{\pi/2}^{\pi} d\theta \sin l\theta$$

$$= \frac{2V}{l}\left[\left(1 - \cos\frac{l\pi}{2}\right) + \cos l\pi - \cos\frac{l\pi}{2}\right]$$

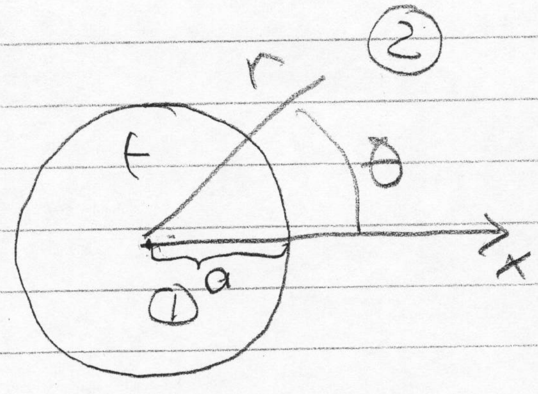
$$= \frac{2V}{l}\begin{cases} 0 & l \neq 2n+2 \\ 4 & l = 2n+2 \end{cases}$$

where $n = 0, 1, 2 \cdots$

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \frac{\pi 8V}{(2n+2)}\left(\frac{r}{b}\right)^{2n+2} \sin(2n+2)$$

(4)

$\vec{E_0 \hat{x}}$



$$\varphi = \begin{cases} \varphi_1(r,\theta) & \text{for } r<a \\ \varphi_2(r,\theta) & \text{for } r>a \end{cases}$$

$$\varphi_1(a,\theta) = \varphi_2(a,\theta)$$

$$\epsilon \left.\frac{\partial \varphi_1}{\partial r}\right|_{r=a} = \left.\frac{\partial \varphi_2}{\partial r}\right|_{r=a}$$

$$\varphi_2(r,\theta) \rightarrow -E_0 r\cos\theta \quad \text{for } r \to \infty$$

try

$$\varphi_1(r,\theta) = A r\cos\theta$$

$$\varphi_2(r,\theta) = -E_0 r\cos\theta + \frac{B}{r}\cos\theta$$

$$A\cancel{a} = -E_0\cancel{a} + \frac{B}{a^2}$$

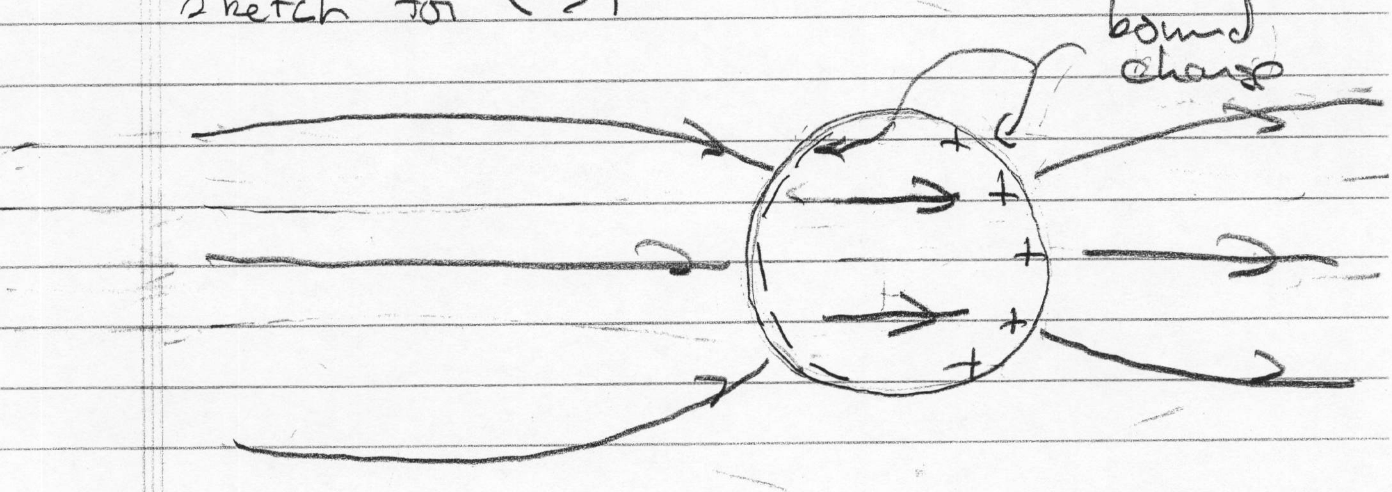
$$\epsilon A = -E_0 - \frac{B}{a^2}$$

$$(\epsilon+1)A = -2E_0 \quad , \quad B = a^2E_0\left(\frac{-2}{1+\epsilon}+1\right)$$

$$= a^2E_0\left(\frac{\epsilon-1}{\epsilon+1}\right)$$

$$\therefore \varphi(r,\theta) = \begin{cases} \dfrac{-2E_0 r\cos\theta}{\epsilon+1} & r<a \\[2ex] -E_0 r\cos\theta + \dfrac{E_0 a^2}{r}\left(\dfrac{\epsilon-1}{\epsilon+1}\right)\cos\theta & r>a \end{cases}$$

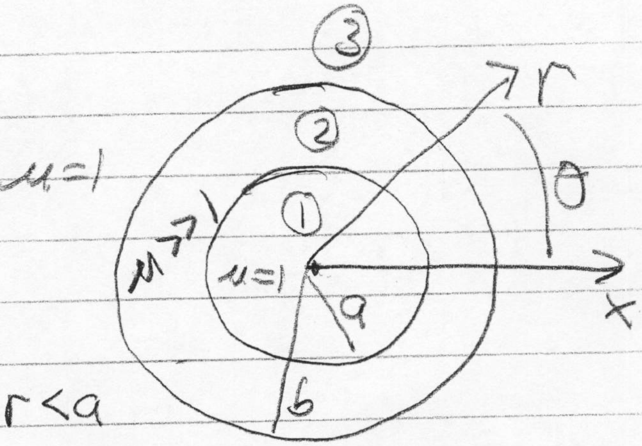
sketch for $\epsilon > 1$



(10) $\nabla \times \underset{\sim}{H} = 0$

$\underset{\sim}{H} = -\underset{\sim}{\nabla}\phi$

$\underset{\sim}{B} = -\mu \underset{\sim}{\nabla}\phi$



$$\phi(r,\theta) = \begin{cases} \phi_1 & \text{for } r<a \\ \phi_2 & \text{for } a<r<b \\ \phi_3 & \text{for } b<r \end{cases}$$

$$\nabla^2\phi_1 = \nabla^2\phi_2 = \nabla^2\phi_3 = 0$$

b.c.

$\phi_1(a,\theta) = \phi_2(a,\theta)$, $\phi_2(b,\theta) = \phi_3(b,\theta)$

$\underset{\sim}{B}\cdot\hat{n}$ continuous across interface

$\left.\frac{\partial\phi_1}{\partial r}\right|_a = \mu \left.\frac{\partial\phi_2}{\partial r}\right|_a$, $\mu \left.\frac{\partial\phi_2}{\partial r}\right|_b = \left.\frac{\partial\phi_3}{\partial r}\right|_b$

as $r \to \infty$ $\phi_3(r,\theta) = -B_0 r\cos\theta$

$$\phi_1(r,\theta) = A r \cos\theta$$

$$\phi_2(r,\theta) = C r \cos\theta + \frac{D}{r}\cos\theta$$

$$\phi_3 = -B_0 r \cos\theta + \frac{E}{r}\cos\theta$$

$$A\cancel{a} = C\cancel{a} + \frac{D}{a^2}, \quad C + \frac{D}{b^2} = -B_0 + \frac{E}{b^2}$$

$$A = \mu\left(C - \frac{D}{a^2}\right), \quad \mu\left(C - \frac{D}{b^2}\right) = -B_0 - \frac{E}{b^2}$$

$$0 = C(\mu-1) - \frac{D}{a^2}(\mu+1), \quad C(\mu+1) + \frac{D}{b^2}(1-\mu) = -2B_0$$

$$D\left\{\frac{(\mu+1)^2}{(\mu-1)a^2} + \frac{(1-\mu)}{b^2}\right\} = -2B_0$$

$$C = \frac{-2B_0(\mu+1)}{\left\{\frac{(\mu+1)^2}{(\mu-1)} + \frac{a^2}{b^2}(1-\mu)\right\}(\mu-1)} = \frac{-2B_0(\mu+1)}{\left[(\mu+1)^2 - \frac{a^2}{b^2}(1-\mu)^2\right]}$$

$$A = \mu\left[C - \frac{D}{a^2}\right] = \mu\left\{\frac{-2B_0(\mu+1)}{\left[(\mu+1)^2 - \frac{a^2}{b^2}(1-\mu)^2\right]}\right.$$

$$\left.\frac{2B_0(1-\mu)}{\left[(\mu+1)^2 - \frac{a^2}{b^2}(1-\mu)^2\right]}\right\}$$

$$A = \frac{-2B_0 \, 2\mu}{\left[(\mu+1)^2 - \frac{a^2}{b^2}(\mu-1)^2\right]}$$

check limits

(1) $\mu = 1$

$$\left.\begin{array}{l} A = -B_0 \\ C = -B_0 \\ D = 0 \\ E = 0 \end{array}\right\} \text{correct}$$

(2) $\mu \to \infty \quad a < b$

$A \to 0$

magnetic shielding