PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #5

(1) You know that at most one fermion may occupy any given single-particle state. A *parafermion* is a particle for which the maximum occupancy of any given single-particle state is k, where k is an integer greater than zero. (For $k = 1$, parafermions are regular everyday fermions; for $k = \infty$, parafermions are regular everyday bosons.) Consider a system with one single-particle level whose energy is ε, *i.e.* the Hamiltonian is simply $\mathcal{H} = \varepsilon\, n$, where n is the particle number.

(a) Compute the partition function $\Xi(\mu, T)$ in the grand canonical ensemble for parafermions.

(b) Compute the occupation function $n(\mu, T)$. What is n when $\mu = -\infty$? When $\mu = \varepsilon$? When $\mu = +\infty$? Does this make sense? Show that $n(\mu, T)$ reduces to the Fermi and Bose distributions in the appropriate limits.

(c) Sketch $n(\mu, T)$ as a function of μ for both $T = 0$ and $T > 0$.

(d) Can a gas of ideal parafermions condense in the sense of Bose condensation?

(2) Consider a system of N spin-$\frac{1}{2}$ particles occupying a volume V at temperature T. Opposite spin fermions may bind in a singlet state to form a boson:

$$\mathrm{f}\!\uparrow \; + \; \mathrm{f}\!\downarrow \quad \rightleftharpoons \quad \mathrm{b}$$

with a binding energy $-\Delta < 0$. Assume that all the particles are nonrelativistic; the fermion mass is m and the boson mass is $2m$. Assume further that spin-flip processes exist, so that the $\uparrow$ and $\downarrow$ fermion species have identical chemical potential μ_{f}.

(a) What is the equilibrium value of the boson chemical potential, μ_{b}? *Hint* : the answer is $\mu_{\mathrm{b}} = 2\mu_{\mathrm{f}}$.

(b) Let the total mass density be ρ. Derive the equation of state $\rho = \rho(\mu_{\mathrm{f}}, T)$, assuming the bosons have not condensed. You may wish to abbreviate

$$\zeta_p(z) \equiv \sum_{n=1}^{\infty} \frac{z^n}{n^p} \ .$$

(c) At what value of μ_{f} do the bosons condense?

(d) Derive an equation for the Bose condensation temperature T_{c}. Solve for T_{c} in the limits $\varepsilon_0 \ll \Delta$ and $\varepsilon_0 \gg \Delta$, respectively, where

$$\varepsilon_0 \equiv \frac{\pi\hbar^2}{m} \left(\frac{\rho/2m}{\zeta(\frac{3}{2})} \right)^{2/3} .$$

(e) What is the equation for the condensate fraction $\rho_0(T, \rho)/\rho$ when $T < T_{\mathrm{c}}$?

(3) A three-dimensional system of spin-0 bosonic particles obeys the dispersion relation

$$\varepsilon(\boldsymbol{k}) = \Delta + \frac{\hbar^2 \boldsymbol{k}^2}{2m} \ .$$

The quantity Δ is the formation energy and m the mass of each particle. These particles are not conserved – they may be created and destroyed at the boundaries of their environment. (A possible example: vacancies in a crystalline lattice.) The Hamiltonian for these particles is

$$\mathcal{H} = \sum_{\boldsymbol{k}} \varepsilon(\boldsymbol{k})\, \hat{n}_{\boldsymbol{k}} + \frac{U}{2V}\, \hat{N}^2 \ ,$$

where $\hat{n}_{\boldsymbol{k}}$ is the number operator for particles with wavevector $\boldsymbol{k}$, $\hat{N} = \sum_{\boldsymbol{k}} \hat{n}_{\boldsymbol{k}}$ is the total number of particles, V is the volume of the system, and U is an interaction potential.

(a) Treat the interaction term within mean field theory. That is, define $\hat{N} = \langle \hat{N} \rangle + \delta\hat{N}$, where $\langle \hat{N} \rangle$ is the thermodynamic average of $\hat{N}$, and derive the mean field self-consistency equation for the number density $\rho = \langle \hat{N} \rangle / V$ by neglecting terms quadratic in the fluctuations $\delta\hat{N}$. Show that the mean field Hamiltonian is

$$\mathcal{H}_{\rm MF} = -\tfrac{1}{2} V U \rho^2 + \sum_{\boldsymbol{k}} \Big[\varepsilon(\boldsymbol{k}) + U\rho \Big]\, \hat{n}_{\boldsymbol{k}} \ ,$$

(b) Derive the criterion for Bose condensation. Show that this requires $\Delta < 0$. For $\Delta = -|\Delta_0|$, find an equation relating $T_{\rm c}$, U, and Δ_0.

(4) The $n^{\rm th}$ moment of the normalized Gaussian distribution $P(x) = (2\pi)^{-1/2} \exp\big(-\tfrac{1}{2}x^2\big)$ is defined by

$$\langle x^n \rangle = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \! dx\, x^n \, \exp\big(-\tfrac{1}{2}x^2\big)$$

Clearly $\langle x^n \rangle = 0$ if n is a nonnegative odd integer. Next consider the *generating function*

$$Z(j) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \! dx \, \exp\big(-\tfrac{1}{2}x^2\big) \exp(jx) = \exp\big(\tfrac{1}{2}j^2\big) \ .$$

(a) Show that

$$\langle x^n \rangle = \frac{d^n Z}{dj^n}\bigg|_{j=0}$$

and provide an explicit result for $\langle x^{2k} \rangle$ where $k \in \mathbb{N}$.

(b) Now consider the following integral:

$$F(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \! dx \, \exp\Big(-\tfrac{1}{2}x^2 - \tfrac{1}{4!}\lambda\, x^4\Big) \ .$$

This has no analytic solution but we may express the result as a power series in the parameter λ by Taylor expanding $\exp\left(-\frac{\lambda}{4!}\,x^4\right)$ and then using the result of part (a) for the moments $\langle x^{4k}\rangle$. Find the coefficients in the perturbation expansion,

$$F(\lambda) = \sum_{k=0}^{\infty} C_k\,\lambda^k \ .$$

(c) Define the *remainder after N terms* as

$$R_N(\lambda) = F(\lambda) - \sum_{k=0}^{N} C_k\,\lambda^k \ .$$

Compute $R_N(\lambda)$ by evaluating numerically the integral for $F(\lambda)$ (using `Mathematica` or some other numerical package) and subtracting the finite sum. Then define the ratio $S_N(\lambda) = R_N(\lambda)/F(\lambda)$, which is the relative error from the N term approximation and plot the absolute relative error $\left|S_N(\lambda)\right|$ *versus* N for several values of λ. (I suggest you plot the error on a log scale.) What do you find?? Try a few values of λ including $\lambda = 0.01$, $\lambda = 0.05$, $\lambda = 0.2$, $\lambda = 0.5$, $\lambda = 1$, $\lambda = 2$.