PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #7

(1) Consider the ferromagnetic XY model, with

$$\hat{H} = -\sum_{i<j} J_{ij}\,\cos(\phi_i - \phi_j) - \mathsf{H}\sum_i \cos\phi_i \ .$$

Defining $z_i \equiv \exp(i\phi_i)$, write $z_i = \langle z_i \rangle + \delta z_i$ with

$$\langle z_i \rangle = m\, e^{i\alpha} \ .$$

(a) Assuming $\mathsf{H} > 0$, what should you take for α?

(b) Making this choice for α, find the mean field free energy using the 'neglect of fluctuations' method. *Hint* : Note that $\cos(\phi_i - \phi_j) = \mathsf{Re}\,(z_i z_j^*)$.

(c) Find the self-consistency equation for m.

(d) Find T_c.

(e) Find the mean field critical behavior for $m(T,\mathsf{H}=0)$, $m(T=T_c,\mathsf{H})$, $C_V(T,\mathsf{H}=0)$, and $\chi(T,\mathsf{H}=0)$, and identify the critical exponents α, β, γ, and δ.

(2) Consider a nearest neighbor two-state Ising *antiferromagnet* on a triangular lattice. The Hamiltonian is

$$\hat{H} = J\sum_{\langle ij \rangle} \sigma_i \sigma_j - \mathsf{H}\sum_i \sigma_i \ ,$$

with $J > 0$.

(a) Show graphically that the triangular lattice is *tripartite*, *i.e.* that it may be decomposed into three component sublattices A, B, and C such that every neighbor of A is either B or C, *etc.*

(b) Use a variational density matrix which is a product over single site factors, where

$$\begin{aligned} \rho(\sigma_i) &= \frac{1+m}{2}\,\delta_{\sigma_i,+1} + \frac{1-m}{2}\,\delta_{\sigma_i,-1} \qquad \text{if } i \in \text{A or } i \in \text{B} \\ &= \frac{1+m_\text{C}}{2}\,\delta_{\sigma_i,+1} + \frac{1-m_\text{C}}{2}\,\delta_{\sigma_i,-1} \qquad \text{if } i \in \text{C} \ . \end{aligned}$$

Compute the variational free energy $F(m, m_\text{C}, T, \mathsf{H}, N)$.

(c) Find the mean field equations.

(d) Find the mean field phase diagram.

(e) While your mean field analysis predicts the existence of an ordered phase, it turns out that $T_c = 0$ for this model because it is so highly frustrated. The ground state is highly degenerate. Show that for any ground state, no triangle can be completely ferromagnetically aligned. What is the ground state energy? Find a lower bound for the ground state entropy per spin.

(3) A system is described by the Hamiltonian

$$\hat{H} = -J \sum_{\langle ij \rangle} \mathcal{I}(\mu_i, \mu_j) - \mathsf{H} \sum_i \delta_{\mu_i,\mathrm{A}} \ , \tag{1}$$

where on each site i there are four possible choices for μ_i: $\mu_i \in \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$. The interaction matrix $\mathcal{I}(\mu, \mu')$ is given in the following table:

$\mathcal{I}$	A	B	C	D
A	$+1$	-1	-1	0
B	-1	$+1$	0	-1
C	-1	0	$+1$	-1
D	0	-1	-1	$+1$

(a) Write a trial density matrix

$$\varrho(\mu_1, \ldots, \mu_N) = \prod_{i=1}^{N} \varrho_1(\mu_i)$$
$$\varrho_1(\mu) = x\, \delta_{\mu,\mathrm{A}} + y(\delta_{\mu,\mathrm{B}} + \delta_{\mu,\mathrm{C}} + \delta_{\mu,\mathrm{D}}) \ .$$

What is the relationship between x and y? Henceforth use this relationship to eliminate y in terms of x.

(b) What is the variational energy per site, $E(x, T, \mathsf{H})/N$?

(c) What is the variational entropy per site, $S(x, T, \mathsf{H})/N$?

(d) What is the mean field equation for x?

(e) What value x^* does x take when the system is disordered?

(f) Write $x = x^* + \frac{3}{4}\varepsilon$ and expand the free energy to fourth order in ε. (The factor $\frac{3}{4}$ should generate manageable coefficients in the Taylor series expansion.)

(g) Sketch ε as a function of T for $\mathsf{H} = 0$ and find T_c. Is the transition first order or second order?