PHYSICS 210A : STATISTICAL PHYSICS
HW ASSIGNMENT #4 SOLUTIONS

(1) Consider a three-dimensional ultrarelativistic gas, with dispersion $\varepsilon = \hbar c|\boldsymbol{k}|$. Find the viral expansion of the equation of state $p = p(n,T)$ to order n^3 for both bosons and fermions.

Solution : We have

$$\beta p = \mp \mathsf{g}\int\!\frac{d^3\!k}{(2\pi)^3}\ \ln\big(1 \mp z\, e^{-\beta\varepsilon(\boldsymbol{k})}\big)$$
$$z = \mathsf{g}\int\!\frac{d^3\!k}{(2\pi)^3}\ \frac{1}{z^{-1}\,e^{\beta\varepsilon(\boldsymbol{k})}\mp 1}\ ,$$

where g is the degeneracy of each $\boldsymbol{k}$ mode. WIth $\varepsilon(\boldsymbol{k}) = \hbar c k$, we change variables to $t = \beta\hbar c k$ and find

$$\beta p = \frac{\mathsf{g}}{6\pi^2}\bigg(\frac{k_{\rm B}T}{\hbar c}\bigg)^3\int\limits_{-\infty}^{\infty}\!\!dt\ \frac{t^3}{z^{-1}\,e^t \mp 1} = \frac{\mathsf{g}}{\pi^2}\bigg(\frac{k_{\rm B}T}{\hbar c}\bigg)^3\ \sum_{j=1}^{\infty}(\pm 1)^{j-1}\,\frac{z^j}{j^4}$$
$$n = \frac{\mathsf{g}}{2\pi^2}\bigg(\frac{k_{\rm B}T}{\hbar c}\bigg)^3\int\limits_{-\infty}^{\infty}\!\!dt\ \frac{t^2}{z^{-1}\,e^t \mp 1} = \frac{\mathsf{g}}{\pi^2}\bigg(\frac{k_{\rm B}T}{\hbar c}\bigg)^3\ \sum_{j=1}^{\infty}(\pm 1)^{j-1}\,\frac{z^j}{j^3}\ ,$$

where we have integrated by parts in the first of these equations. Now it's time to ask `Mathematica`:

```
In[1] =  y = InverseSeries[x + x^2/2^3 + x^3/3^3 + x^4/4^3 + x^5/5^3 + O[x]^6]
```

$$\texttt{Out[1] =}\quad x - \frac{x^2}{8} - \frac{5\,x^3}{864} - \frac{31\,x^4}{13\,824} - \frac{56\,039\,x^5}{62\,208\,000} + O[x]^6$$

```
In[2] =  w = y + y^2/2^4 + y^3/3^4 + y^4/4^4 + y^5/5^4
```

$$\texttt{Out[2] =}\quad x - \frac{x^2}{16} - \frac{47\,x^3}{5184} - \frac{25\,x^4}{9216} - \frac{2\,014\,561\,x^5}{1\,866\,240\,000} + O[x]^6$$

So with the definition

$$\lambda_T = \pi^{2/3}\,\mathsf{g}^{-1/3}\,\frac{\hbar c}{k_{\rm B}T}\ ,$$

we have

$$p = nk_{\rm B}T\big(1 + B_2\,n + B_3\,n^2 + \ldots\big)\ ,$$

where

$$B_2 = \mp\tfrac{1}{16}\,\lambda_T^3 \quad , \quad B_3 = -\tfrac{47}{5184}\,\lambda_T^6 \quad , \quad B_4 = \mp\tfrac{25}{9216}\,\lambda_T^9 \quad , \quad B_4 = -\tfrac{2014561}{1866240000}\,\lambda_T^{12}\ .$$

(2) Suppose photons had a dispersion $\varepsilon = J\boldsymbol{k}^2$. All other things being equal (surface temperature of the sun, earth-sun distance, earth and solar radii, *etc.*), what would be the surface temperature of the earth? *Hint: Derive the corresponding version of Stefan's law.*

Solution : This material has been added to the notes; see §4.4.4. Assume a dispersion of the form $\varepsilon(k)$ for the (nonconserved) bosons. Then the energy current incident on a differential area dA of surface normal to $\hat{\boldsymbol{z}}$ is

$$dP = dA \cdot \int \frac{d^3k}{(2\pi)^3}\, \Theta(\cos\theta) \cdot \varepsilon(k) \cdot \frac{1}{\hbar}\frac{\partial\varepsilon(k)}{\partial k_z} \cdot \frac{1}{e^{\varepsilon(k)/k_\mathrm{B}T} - 1} \ .$$

Note that

$$\frac{\partial\varepsilon(k)}{\partial k_z} = \frac{k_z}{k}\frac{\partial\varepsilon}{\partial k} = \cos\theta\, \varepsilon'(k) \ .$$

Now let us assume a power law dispersion $\varepsilon(k) = Ak^\alpha$. Changing variables to $t = Ak^\alpha/k_\mathrm{B}T$, we find

$$\frac{dP}{dA} = \sigma\, T^{2+\frac{2}{\alpha}} \ ,$$

where

$$\sigma = \zeta\big(2 + \tfrac{2}{\alpha}\big)\, \Gamma\big(2 + \tfrac{2}{\alpha}\big) \cdot \frac{\mathsf{g}\, k_\mathrm{B}^{2+\frac{2}{\alpha}}\, A^{-\frac{2}{\alpha}}}{8\pi^2\hbar} \ .$$

One can check that for $\mathsf{g} = 2$, $A = \hbar c$, and $\alpha = 1$ that this result reduces to Stefan's Law.

Equating the power incident on the earth to that radiated by the earth, we obtain

$$T_\mathrm{e} = \left(\frac{R_\odot}{2\, a_\mathrm{e}}\right)^{\frac{\alpha}{\alpha+1}} T_\odot \ .$$

Plugging in the appropriate constants and setting $\alpha = 2$, we obtain $T_\mathrm{e} = 101.3\,\mathrm{K}$. Brrr!

(3) Almost all elements freeze into solids well before they can undergo Bose condensation. Setting the Lindemann temperature equal to the Bose condensation temperature, show that this implies a specific ratio of $k_\mathrm{B}\Theta_\mathrm{D}$ to $\hbar^2/Ma^2$, where M is the atomic mass and a is the lattice spacing. Evaluate this ratio for the noble gases He, Ne, Ar, Kr, and Xe. (You will have to look up some numbers.)

Solution : The Lindemann melting temperature T_M and the Bose condensation temperature T_c for monatomic solids are given by

$$T_\mathrm{M} = x^2 \cdot \frac{M k_\mathrm{B} \Theta_\mathrm{D}^2\, a^2}{9\hbar^2} \quad , \quad T_\mathrm{c} = \frac{2\pi\hbar^2}{M k_\mathrm{B}} \left(\frac{n}{\zeta(3/2)}\right)^{2/3} ,$$

where a is the lattice constant, M the atomic mass, and Θ_D the Debye temperature. For a simple cubic lattice, the number density is $n = a^{-3}$. Helium solidifies into a hexagonal close packed (HCP) structure, while Neon, Argon, Krypton, and Xenon solidify into a face-centered cubic (FCC) structure. The unit cell volume for both HCP and FCC is $a^3/\sqrt{2}$, where a is the lattice spacing, so $n = \sqrt{2}\, a^{-3}$ for the rare gas solids. Thus, we find

$$\frac{T_\mathrm{M}}{T_\mathrm{c}} = \frac{x}{\alpha} \cdot \left(\frac{k_\mathrm{B}\Theta_\mathrm{D}}{\hbar^2/Ma^2}\right)^2 \ .$$

where

$$\alpha = 18\pi \left(\frac{\sqrt{2}}{\zeta(3/2)} \right)^{2/3} \approx 40 \ .$$

If we set $x = 0.1$ we find $\frac{x}{\alpha} \approx \frac{1}{400}$. Now we need some data for $\Theta_{\rm D}$ and a. The most convenient table of data I've found is from H. Glyde's article on solid helium in the *Encyclopedia of Physics*. The table entry for ^{4}He is for the BCC structure at a pressure $p = 25$ bar. For a BCC structure the unit cell volume is $4a^3/3\sqrt{3}$. Define the ratio $R \equiv k_{\rm B}\Theta_{\rm D}/(\hbar^2/Ma^2)$.

As one can see from the table and from the above equation for $T_{\rm M}/T_{\rm c}$. the R values are such that the melting temperature is predicted to be several orders of magnitude higher than the ideal Bose condensation temperature in every case except ^{4}He, where the ratio is on the order of unity (and is less than unity if the actual melting temperature is used). The reason that ^{4}He under high pressure is a solid rather than a Bose condensate at low temperatures is because the ^{4}He atoms are not free particles.

crystal	a (Å)	M (amu)	$\Theta_{\rm D}$ (K)	$T_{\rm M}^{\rm actual}$ (K)	$T_{\rm c}$	$\hbar^2/Ma^2k_{\rm B}$ (K)	R
^{4}He	3.57	4.00	25	1.6	3.9	0.985	25
Ne	4.46	20.2	66	24.6	0.50	0.125	530
Ar	5.31	39.9	84	83.8	0.18	0.0446	1900
Kr	5.65	83.8	64	161.4	0.076	0.0188	3400
Xe	6.13	131	55	202.0	0.041	0.0102	20000

Table 1: Lattice constants for Ne, Ar, Kr, and Xe from F. W. de Wette and R. M. J. Cotterill, *Solid State Comm.* **6**, 227 (1968). Debye temperatures and melting temperatures from H. Glyde, *Solid Helium* in *Encyclopedia of Physics*. ^{4}He data are for $p = 25$ bar, in the bcc phase (from Glyde).

(4) A nonrelativistic Bose gas consists of particles of spin $S = 1$. Each boson has mass m and magnetic moment μ_0. A gas of these particles is placed in an external field H.

(a) What is the relationship of the Bose condensation temperature $T_{\rm c}(H)$ to $T_{\rm c}(\mathsf{H} = 0)$ when $\mu_0\mathsf{H} \gg k_{\rm B}T$?

(b) Find the magnetization M for $T < T_{\rm c}$ when $\mu_0\mathsf{H} \gg k_{\rm B}T$. Calculate through order $\exp(-\mu_0\mathsf{H}/k_{\rm B}T)$.

Solution : The number density of bosons is given by

$$n(T,z) = \lambda_T^{-3} \left\{ \zeta_{3/2}\big(z\, e^{\mu_0\mathsf{H}/k_{\rm B}T}\big) + \zeta_{3/2}\big(z\big) + \zeta_{3/2}\big(z\, e^{-\mu_0\mathsf{H}/k_{\rm B}T}\big) \right\} .$$

The argument of $\zeta_z(z)$ cannot exceed unity, thus Bose condensation occurs for $z = \exp(-\mu_0\mathsf{H}/k_{\rm B}T)$ (assuming $\mathsf{H} > 0$). Thus, the condition for Bose condensation is given by

$$n\lambda_{T_{\rm c}}^3 = \zeta(3/2) + \zeta_{3/2}\big(e^{-\mu_0\mathsf{H}/k_{\rm B}T_{\rm c}}\big) + \zeta_{3/2}\big(e^{-2\mu_0\mathsf{H}/k_{\rm B}T_{\rm c}}\big) \ .$$

This is a transcendental equation for $T = T_c(n, \mathsf{H})$. In the limit $\mu_0 \mathsf{H} \gg k_B T_c$, the second two terms become negligible, since

$$\zeta_s(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^s} \ .$$

Thus,

$$T_c(\mathsf{H} \to \infty) = \frac{2\pi\hbar^2}{m} \left(\frac{n}{\zeta(3/2)} \right)^{2/3} \ .$$

When $\mathsf{H} = 0$, we have Thus,

$$T_c(\mathsf{H} \to 0) = \frac{2\pi\hbar^2}{m} \left(\frac{n}{3\,\zeta(3/2)} \right)^{2/3} \ .$$

Thus,

$$\frac{T_c(\mathsf{H} \to \infty)}{T_c(\mathsf{H} \to 0)} = 3^{2/3} = 2.08008\ldots$$

The magnetization density is

$$M = \mu_0 \, \lambda_T^{-3} \left\{ \zeta_{3/2}\big(z \, e^{\mu_0 \mathsf{H}/k_B T}\big) - \zeta_{3/2}\big(z \, e^{-\mu_0 \mathsf{H}/k_B T}\big) \right\} \ .$$

For $T < T_c$, we have $z = \exp(-\mu_0 \mathsf{H}/k_B T)$ and therefore

$$\begin{aligned} M &= \mu_0 \, \lambda_T^{-3} \left\{ \zeta(3/2) - \sum_{j=1}^{\infty} j^{-3/2} \, e^{-2j\mu_0 \mathsf{H}/k_B T} \right\} \\ &= n\mu_0 \left\{ 1 - \frac{e^{-2\mu_0 \mathsf{H}/k_B T}}{\zeta(3/2)} + \mathcal{O}\big(e^{-4\mu_0 \mathsf{H}/k_B T}\big) \right\} \end{aligned}$$