Physics 4E   Final Solutions   Spring 2010

Problem 1

Fn well: $E_n = \frac{\hbar^2\pi^2 n^2}{2m_e L^2}$. Lowest energy transition is $n=1 \to n=2$

$$\Rightarrow \Delta E = \frac{\hbar^2\pi^2}{2m_e L^2}(2^2-1^2) = \frac{3\hbar^2\pi^2}{2m_e L^2} = \frac{hc}{\lambda} \Rightarrow \cancel{x} L^2 = \frac{3\hbar^2\pi^2}{2m_e}\frac{\lambda}{hc} \Rightarrow$$

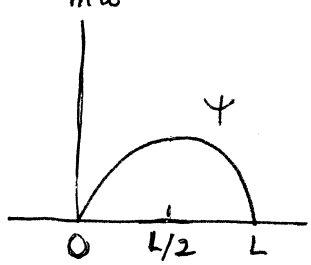
$$\Rightarrow L^2 = \frac{3\hbar^2\pi^2\lambda}{2m_e hc} = \frac{3}{2}\pi^2 \frac{7.62 \cdot 1000}{12{,}400} \text{Å}^2 = 9.098\,\text{Å}^2 \Rightarrow \boxed{L = 3.02\,\text{Å}} \text{ (a)}$$

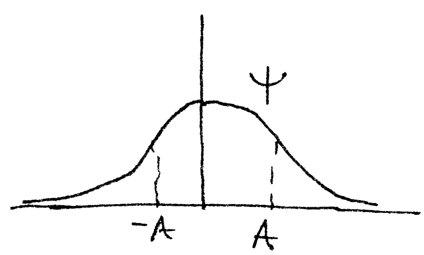
(b) $E_n = \hbar\omega(n+\frac{1}{2}) \Rightarrow \Delta E = \hbar\omega = \frac{hc}{\lambda} = \frac{12{,}400}{1000}\text{eV} = 12.4\text{eV}$

$\boxed{\hbar\omega = 12.4\text{eV}}$ Classical amplitude: $\frac{1}{2}m\omega^2A^2 = \frac{\hbar\omega}{2} \Rightarrow$

$$\Rightarrow A^2 = \frac{\hbar\omega}{m\omega^2} = \frac{\hbar}{m\omega} = \frac{\hbar^2}{m\hbar\omega} = \frac{7.62}{12.4}\text{Å}^2 = 0.615\,\text{Å}^2 \Rightarrow \boxed{A = 0.78\,\text{Å}}$$

(c)





Fn oscillator: $\psi_0(x) = Ce^{-\frac{m\omega}{2\hbar}x^2} = Ce^{-x^2/2A^2}$; $\int dx\,\psi_0^2 = 1 \Rightarrow$

$$\Rightarrow C^2\sqrt{\pi A^2} = 1 \Rightarrow C = \frac{1}{\pi^{1/4}A^{1/2}} \Rightarrow \boxed{\psi_0(x) = \frac{1}{\pi^{1/4}A^{1/2}}e^{-x^2/2A^2}}$$

or $\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-m\omega x^2/2\hbar}}$ $\boxed{\psi_0(0) = \frac{1}{\pi^{1/4}A^{1/2}} = 0.850\,\text{Å}^{-1}}$

Fn square well, $\psi_0(x) = \sqrt{\frac{2}{L}}\sin\frac{\pi x}{L} \Rightarrow \boxed{\psi_0\left(\frac{L}{2}\right) = \sqrt{\frac{2}{L}} = 0.814\,\text{Å}^{-1}}$

So the probability is higher for the electron to be at the center in the harmonic oscillator case.

Problem 2

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} \quad ; \quad \psi(x) = C x e^{-x^2/2x_0^2}$$

$$\langle x\rangle = \int_{-\infty}^{\infty} dx \cdot x \, |\psi(x)|^2 = 0$$

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2 |\psi(x)|^2 = \frac{\int_{-\infty}^{\infty} dx\, x^4 e^{-x^2/x_0^2}}{\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2/x_0^2}}$$

Use $\int_{-\infty}^{\infty} dx\, x^2 e^{-\lambda x^2} = \frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}}$, $\int_{-\infty}^{\infty} dx\, x^4 e^{-\lambda x^2} = \frac{3}{4}\sqrt{\frac{\pi}{\lambda^5}}$

$$\Rightarrow \langle x^2\rangle = \frac{\frac{3}{4}\sqrt{\pi}\, x_0^5}{\frac{1}{2}\sqrt{\pi}\, x_0^3} = \frac{3}{2} x_0^2 \Rightarrow \boxed{\Delta x = \sqrt{\frac{3}{2}}\, x_0 = 1.22 x_0}$$

(b) $P(x) = |\psi(x)|^2 = C^2 x^2 e^{-x^2/x_0^2}$; maximize $P(x) \Rightarrow$

$$P'(x) = 0 = \cancel{2} x\, e^{-x^2/x_0^2} - \frac{\cancel{2} x^3}{x_0^2} e^{-x^2/x_0^2} = 0 \Rightarrow$$

$$\Rightarrow \boxed{x = x_0}$$

$$\frac{P(x = 0.02 x_0)}{P(x = 0.01 x_0)} \sim \frac{0.02^2}{0.01^2} = 4 \quad \boxed{\text{four times more likely}}$$

$\left(e^{-x^2/x_0^2} \simeq 1 \text{ for } x = 0.01 x_0 \text{ or } x = 0.02 x_0 \right)$

Problem 3

$$\psi(x) = C\,(1+\cos(x/a)) \quad ; \quad V(x) = V_0 \frac{1-\cos(x/a)}{1+\cos(x/a)}$$

$$-\frac{\hbar^2}{2m_e}\frac{d^2\psi}{dx^2} + V(x)\,\psi = E\psi$$

$$\frac{d\psi}{dx} = -\frac{C}{a}\sin\frac{x}{a}, \quad \frac{d^2\psi}{dx^2} = -\frac{C}{a^2}\cos\frac{x}{a} \Rightarrow$$

$$\frac{\hbar^2}{2m_e a^2}\cos\frac{x}{a} + V_0\frac{1-\cos x/a}{\cancel{1+\cos x/a}}\cdot\cancel{(1+\cos x/a)} = E(1+\cos x/a) \Rightarrow$$

$$\cos\frac{x}{a}\left(\frac{\hbar^2}{2m_e a^2} - V_0 - E\right) + V_0 = E$$; for this to be valid for

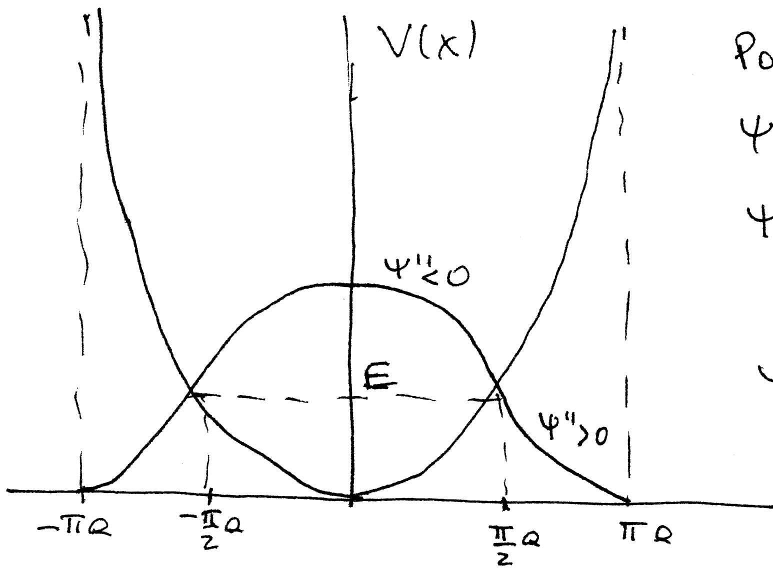
all x, we need: $V_0 = E$; $\frac{\hbar^2}{2m_e a^2} - V_0 - E = 0 = \frac{\hbar^2}{2m_e a^2} - 2V_0 \Rightarrow$

$$\Rightarrow \boxed{V_0 = \frac{\hbar^2}{4m_e a^2} = E = \frac{1}{4}\cdot\frac{7.62\,\text{eV}\,\text{Å}^2}{1\,\text{Å}^2} = 1.905\,\text{eV}}$$

(b) Regime where $V(x) < E \Rightarrow V_0\frac{1-\cos x/a}{1+\cos x/a} < V_0 \Rightarrow 1-\cos x/a < 1+\cos x/a$

$\Rightarrow \cos x/a > 0 \Rightarrow \frac{|x|}{a} < \frac{\pi}{2} \Rightarrow \boxed{-\frac{\pi a}{2} < x < \frac{\pi a}{2}}$

(c)



Potential diverges at $x = \pm\pi a$

$\psi(x=\pi a) = \psi(x=-\pi a) = 0$

$\psi'' < 0$ for $-\frac{\pi}{2}a < x < \frac{\pi}{2}a$

= regime where $E > V(x)$

$\psi'' > 0$ for $\frac{\pi}{2}a < |x| < \pi a$

Problem 4

For a harmonic oscillator, $\langle E \rangle = \dfrac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}$, $k_B = \dfrac{1}{11{,}600}\dfrac{eV}{K}$

For N three-dimensional harmonic oscillators,

$$\langle E \rangle_N = 3N \frac{\hbar\omega}{e^{\hbar\omega/k_BT}-1}$$

For very high T, $e^{\hbar\omega/k_BT} \simeq 1 + \dfrac{\hbar\omega}{k_BT} \Rightarrow \langle E \rangle_N = 3Nk_BT \Rightarrow C_v = 3Nk_B$

(a) At T = 100 K

$$\frac{\hbar\omega}{k_BT} = \frac{0.02\,eV \times 11{,}600\,K}{100\,K \times eV} = 2.32$$

$$\langle E \rangle = \frac{\hbar\omega}{e^{2.32}-1} = \frac{\hbar\omega}{k_BT}\frac{1}{e^{2.32}-1}k_BT = 0.253\,k_BT$$

$$\Rightarrow \boxed{\langle E \rangle_N = 3N \times 0.253\,k_BT = 0.759\,Nk_BT}$$

(b) at T = 1000 K

$$\frac{\hbar\omega}{k_BT} = 0.232$$

$$\langle E \rangle = \frac{0.232}{e^{0.232}-1}k_BT = 0.888\,k_BT$$

$$\boxed{\langle E \rangle_N = 2.665\,Nk_BT}$$

(c) at T = 10,000 K, $\dfrac{\hbar\omega}{k_BT} = 0.0232$

$$\langle E \rangle = \frac{0.0232}{e^{0.0232}-1}k_BT = 0.988$$

$$\boxed{\langle E \rangle_N = 2.965\,k_BT}$$

Problem 5

$$\psi(r,\theta,\phi) = C r e^{-r/a_0} \cos\theta$$

(a) exponential dependence is e^{-Zr/na_0}

Since it's $\propto r \Rightarrow \boxed{n=2}$; since it's $e^{-r/a_0} \Rightarrow \boxed{Z=2}$

No ϕ angular dependence $\Rightarrow \boxed{l=1,\ m=0}$

(b) Most probable r:

$$P(r) = r^2 R^2(r) = C r^4 e^{-2r/a_0}$$

$$P'(r_m) = 0 \Rightarrow 4 r_m^3 - \frac{2 r_m^4}{a_0} = 0 \Rightarrow \boxed{r_m = 2a_0}$$

(c) Average of $1/r$:

$$\left\langle \frac{1}{r} \right\rangle = \int dr\, P(r) \frac{1}{r} = \frac{\int_0^\infty dr\, r^3 e^{-2r/a_0}}{\int_0^\infty dr\, r^4 e^{-2r/a_0}}$$

Use $\int_0^\infty dr\, r^s e^{-\lambda r} = \frac{s!}{\lambda^{s+1}} \Rightarrow \int_0^\infty dr\, r^3 e^{-2r/a_0} = \frac{3!}{\left(\frac{2}{a_0}\right)^4} = \frac{6}{2^4} a_0^4$

$$\int_0^\infty dr\, r^4 e^{-2r/a_0} = \frac{4!}{\left(\frac{2}{a_0}\right)^5} = \frac{24}{2^5} a_0^5 \Rightarrow \left\langle \frac{1}{r} \right\rangle = \frac{6 \cdot 2^{\cancel{5}1}}{\cancel{2^4} \cdot 24} \cdot \frac{a_0^4}{a_0^5} = \frac{1}{2a_0}$$

$$\Rightarrow \boxed{\left\langle \frac{1}{r} \right\rangle = \frac{1}{2a_0} = \frac{1}{r_m}}$$

(d) In Bohr atom, $r_n^{Bohr} = \frac{a_0}{Z} n^2 = \frac{a_0 \cdot 2^2}{2} = 2a_0$ for $Z=2, n=2$

hence $\boxed{r_m = r_2^{Bohr}, \quad \left\langle \frac{1}{r} \right\rangle = \frac{1}{r_2^{Bohr}}}$

Problem 6

The electron energy is

$$E_0 = \frac{\hbar^2 \pi^2}{2 m_e \cdot 1 \mathring{A}^2} = 3.81 \pi^2 \, eV = 37.6 \, eV$$

Tunneling probability:

$$T \simeq e^{-2\sqrt{\frac{2m}{\hbar^2}(V-E)}\,\Delta x} = e^{-2\sqrt{\frac{1}{3.81}(90-37.6)}\cdot 6} =$$

$$= e^{-44.50} = 4.71 \times 10^{-20}$$

Velocity of electron in well is given by

$$v = \frac{p}{m_e}, \quad p = \hbar \pi \Rightarrow \frac{v}{c} = \frac{\hbar \pi}{m_e c} = \frac{\hbar c \pi}{m_e c^2} \Rightarrow$$

$$\Rightarrow \frac{v}{c} = 0.0121 \Rightarrow v = 0.0121 \times 3 \times 10^{18} \frac{\mathring{A}}{s} = 3.64 \times 10^{16} \frac{\mathring{A}}{s}$$

the probability per second that the particle will escape: since it travels $2 \times 1\mathring{A} = 2\mathring{A}$ each time it bounces against the barrier, per second it hits the barrier 1.82×10^{16} times $\Rightarrow$ since $T =$ escape prob,

$$\text{prob escape per second} = \frac{1}{\tau} = \frac{1.82 \times 10^{16}}{s} \times T = \frac{8.57 \times 10^{-4}}{s}$$

$$\Rightarrow \boxed{\tau = 1{,}167 \, s = 19.5 \text{ minutes}}$$

Problem 7

$n=3,\ l=1.$ Can make transitions to $n=2,\ l=0$ or $n=1,\ l=0$.
Or, first to $n=2$, then $n=2 \to n=1$.

$$E_n = -\frac{E_0}{n^2}, \quad E_0 = 13.6\,eV$$

$$\Delta E_{31} = E_3 - E_1 = E_0\left(1-\frac{1}{9}\right) = \frac{8}{9}E_0 = 12.0889\,eV = \frac{hc}{\lambda_{31}}$$

$$\Delta E_{32} = E_3 - E_2 = E_0\left(\frac{1}{4}-\frac{1}{9}\right) = \frac{5}{36}E_0 = 1.8889\,eV = \frac{hc}{\lambda_{32}}$$ with $hc = 12{,}400\,eV\,\mathring{A}$ =

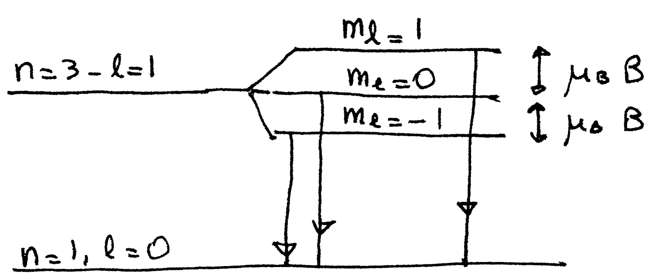
$$\Delta E_{21} = E_2 - E_1 = E_0\left(1-\frac{1}{4}\right) = \frac{3}{4}E_0 = 10.2\,eV = \frac{hc}{\lambda_{21}}$$

$$\Rightarrow \boxed{\lambda_{31} = 1025.73\,\mathring{A}\ ; \quad \lambda_{32} = 6564.67\,\mathring{A}\ ; \quad \lambda_{21} = 1215.69\,\mathring{A}}$$

(b) With magnetic field, $\Delta E = -\vec{\mu}\cdot\vec{B} = -\mu_z B$ for B in the z direction

$$\mu_z = -\frac{e}{2m_e}L_z = -\frac{e\hbar}{2m_e}m_\ell = -\mu_B m_\ell \Rightarrow \boxed{\Delta E(m_\ell) = \mu_B . B . m_\ell}$$

So energies of $n=3,\ l=1$ states are split



$$\boxed{\mu_B B = 5.79\times10^{-3}\,eV}$$

$\Delta E_{31}(m_\ell = 1) = 12.0889\,eV + 5.79\times10^{-3}\,eV = 12.0947\,eV$, $\lambda_{31}(m_\ell=1) = 1025.24\,\mathring{A}$

$\Delta E_{31}(m_\ell = 0) =$ – – – – – – $\lambda_{31}(m_\ell=0) = 1025.73\,\mathring{A}$

$\Delta E_{31}(m_\ell = -1) =$ – $= 12.0831\,eV$, $\lambda_{31}(m_\ell=-1) = 1026.23\,\mathring{A}$

Problem 7 (cont.)

Spin-orbit coupling: the internal magnetic field B_{eff} couples to the spin magnetic moment, giving rise to energy shift

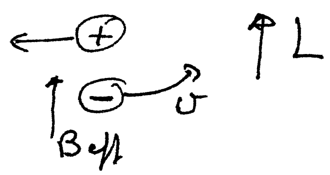
$$\Delta E_{spin} = -\vec{\mu}_s \cdot \vec{B}_{eff}$$

$$\mu_s = -m_s g_s \mu_B = \begin{cases} -\mu_B & \text{for } m_s = +1/2 \\ +\mu_B & \text{for } m_s = -1/2 \end{cases} \qquad (g_s = 2)$$

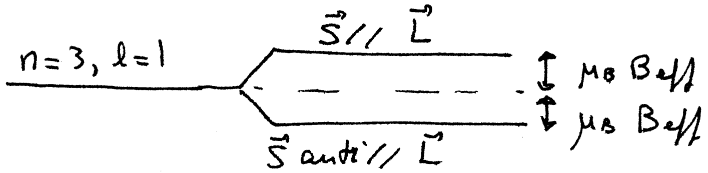
$$\Rightarrow \quad \Delta E = +\mu_B B_{eff} \quad \text{for } m_s = 1/2$$

$$\Delta E = -\mu_B B_{eff} \quad \text{for } m_s = -1/2$$

B_{eff} is parallel to orbital angular momentum



So energy level $n=3,\ l=1$ splits:



$\vec{J} = \vec{L} + \vec{S} =$ total angular momentum.

For $\vec{S} \parallel \vec{L}$, $\quad J = 3/2 \longrightarrow \Delta E_{s.o.} = +\mu_B B_{eff} = +2.895 \times 10^{-6}\ eV$

For $\vec{S} \nparallel \vec{L}$, $\quad J = 1/2 \longrightarrow \Delta E_{s.o.} = -\mu_B B_{eff} = -2.895 \times 10^{-6}\ eV$

$J = 3/2$ gives higher energy than $J = 1/2$.

Problem 8

$\Psi(x,y)$ is wave function. Assume $\boxed{\Psi(x,y) = \Psi_1(x)\Psi_2(y)}$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi_1(x)\Psi_2(y) +$$

$$+ \frac{1}{2}K(x^2+y^2)\Psi_1(x)\Psi_2(y) = E\Psi_1(x)\Psi_2(y) \Rightarrow$$

$$\Rightarrow \underbrace{-\frac{\hbar^2}{2m\Psi_1(x)}\frac{\partial^2\Psi_1(x)}{\partial x^2} + \frac{1}{2}Kx^2}_{\substack{\text{depends on } x \\ \Rightarrow = \text{constant} = C_1}} \underbrace{- \frac{\hbar^2}{2m\Psi_2(y)}\frac{\partial^2\Psi_2(y)}{\partial y^2} + \frac{1}{2}Ky^2}_{\substack{\text{depends on } y \\ = \text{constant} = C_2}} = E$$

$E = C_1 + C_2$, C_1 and C_2 are energies for 1 dim harmonic oscillator.

(b) Ground state wave function:

$$\Psi_0(x,y) = \Psi_{10}(x)\Psi_{20}(y) = Ce^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

(c) For 1d oscillator, $\mathcal{E}_n = \hbar\omega(n+\frac{1}{2})$, for 2d,

$$\mathcal{E}_{n_1,n_2} = \hbar\omega(n_1+\tfrac{1}{2}) + \hbar\omega(n_2+\tfrac{1}{2}) = \hbar\omega(n_1+n_2+1)$$

First excited state is $n_1=0, n_2=1$ or $n_1=1, n_2=0$

$\Rightarrow \mathcal{E} = \hbar\omega(1+1) = 2\hbar\omega$, degeneracy $= 2$.

(d) For 4 fermions

$n_1 = n_2 = 1$ ↑ $3\hbar\omega$

$n_1 = 0, n_2 = 1$ ↑ ↑ $n_1 = 1, n_2 = 0$ $2\hbar\omega$

$n_1 = n_2 = 0$ ↑ $\hbar\omega$

total energy $= \hbar\omega + 4\hbar\omega + 3\hbar\omega = \boxed{8\hbar\omega}$