Physics 140B, Winter 2010
Homework 3 --- due Feb 04

1. Solve Problem 19-7 of Carter. For part (b), use equation (19.15).
[Note that the answers to this problem, as given at the back of the book, are all wrong. So, just ignore them].

2. The speed of sound, w, in any medium is given by the formula

$$w = (1/\rho\kappa_S)^{1/2},$$

where ρ is the mass density and κ_S the adiabatic compressibility of the medium. Show that, for an ideal Fermi gas at $T = 0$ K, $w = (1/\sqrt{3})v_F$, where v_F ($= p_F/m$) is the Fermi velocity of the gas particles.

3. The internal energy of a non-relativistic Fermi gas at low temperatures is given by the expression

$$U = \frac{3}{5} N \varepsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right],$$

where ε_F is the Fermi energy of the gas; note that $\varepsilon_F \propto (N/V)^{2/3}$.

(a) Using this expression (and your knowledge of Thermodynamics), derive the corresponding expressions for the free energy F and the pressure P of the gas.

(b) Using these expressions for F and P, show that the corresponding expression for the chemical potential μ of the gas is

$$\mu = \varepsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{kT}{\varepsilon_F} \right)^2 + \cdots \right].$$

4. Consider an electron gas with particle density n. Determine the *numerical* value of n for which the Fermi energy ε_F of the gas is equal to the rest energy, mc^2, of the electron. What is the corresponding value of the Fermi velocity v_F of the electrons?

5. Re-visit the problem of the "statistical equilibrium of white dwarf stars", for which it is essential that the motion of electrons be treated as relativistic.

(a) Using the approximation

$$\varepsilon = c\sqrt{p^2 + m_e^2 c^2} - m_e c^2$$

$$\simeq cp - m_e c^2 + \frac{m_e^2 c^3}{2p},$$

calculate the ground-state energy, U_0, of the electron gas in the star. Express your result in terms of the mass M and the radius R of the star; for this, remember that the total number of electrons $N \simeq 2M / m_{He}$.

(b) Combining the energy calculated in part (a) with the gravitational energy of the star, show that, in equilibrium, the mass-radius relationship for such stars is given by the formula

$$R \sim \frac{h}{m_e c}\left(\frac{M}{m_{He}}\right)^{1/3}\left[1 - \left(\frac{M}{M_0}\right)^{2/3}\right]^{1/2},$$

where

$$M_0 \sim \frac{(hc/G)^{3/2}}{m_{He}^2}.$$