Homework 7: Solutions

1. Since $\lambda = \dfrac{1}{\sqrt{2}\,\pi\, d^2\, n} = \dfrac{kT}{\sqrt{2}\,\pi\, d^2 \cdot P}$,

$$d = \text{const.} \sqrt{\frac{T}{\lambda P}} \,.$$

It follows that

$$\frac{d_B}{d_A} = \sqrt{\frac{T_B / T_A}{(\lambda_B / \lambda_A)(P_B / P_A)}}$$

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$$= \sqrt{\frac{2}{3 \times (1/6)}} = 2.$$

2. Collision frequency $= \pi d^2 n \bar{V} = \pi d^2 n \sqrt{\frac{16 kT}{\pi m}}$

$$= 4 d^2 P \sqrt{\frac{\pi}{m kT}} \qquad (\because P = nkT)$$

With the given data, remembering that $m = 44$ amu,

(4) we get: $8.7 \times 10^9 \text{ sec}^{-1}$.

Mean free path $= \frac{1}{\sqrt{2} \pi d^2 n} = \frac{kT}{\sqrt{2} \pi d^2 P}$

(4) $= 4.35 \times 10^{-8}$ m.

(2) The ratio $\lambda / d \simeq 95$.

3. The M.F.P. $\lambda = \dfrac{1}{\sqrt{2}\,\pi d^2 \cdot n}$.

It is given that, for air at 1 atmosphere and 300 K $\lambda \simeq 1.02 \times 10^{-7}$ m. In the present problem, $T = 300$ K and P is whatever it is at a height of 10 km above the Earth's surface. For m, we take 29 amu (20% O_2 & 80% N_2) i.e. 4.8×10^{-26} kg. Also, neglecting the variation of T,

$$n(z) = n(0)\exp\left(-\frac{mgz}{kT}\right), \quad \text{where } n(0) = \frac{P(0)}{kT}.$$

Now, first of all

$$n(0) = \frac{1.013 \times 10^{5}\ N/m^2}{1.38 \times 10^{-23}\ J/K \cdot 300\ K} = 2.45 \times 10^{25}\ m^{-3}.$$

Next,

$$\exp\left(-\frac{4.8 \times 10^{-26}\ kg \cdot 9.8\ m\,s^{-2} \cdot 10^{4}\ m}{1.38 \times 10^{-23}\ J/K \cdot 300\ K}\right) = e^{-1.136} = 0.321$$

So, $n(z) = 2.45 \times 10^{25}\ m^{-3} \times 0.321 = 7.87 \times 10^{24}\ m^{-3}$.

Thus

$$\lambda = \frac{1}{\sqrt{2}\,\pi\,(3 \times 10^{-10}\ m)^2 \cdot 7.87 \times 10^{24}\ m^{-3}}$$

$= 3.18 \times 10^{-7}$.

4. We make use of the result

$$R_{rms} = \sqrt{2 \lambda \bar{v} t} \,,$$

whence

$$t = \frac{R_{rms}^2}{2 \lambda \bar{v}} .$$

Substituting $R_{rms} = 10^{-1}$ m, $\lambda = 1.0 \times 10^{-7}$ m and $\bar{v} = 460$ m/s, we get

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$$t = 109 \text{ s} ,$$

fairly long compared to $R / \bar{v}$!

5. (a) We have to show that the given function $n(r,t)$ satisfies the diffusion equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n = D\,\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n}{\partial r}\right).$$

Since $n(r,t) = A\,t^{-3/2}\exp\left(-\frac{r^2}{4Dt}\right)$, where A is a constant, we have

$$\frac{\partial n}{\partial t} = A\left(-\frac{3}{2}t^{-5/2}\right)\exp(\cdots) + A\,t^{-3/2}\left(+\frac{r^2}{4Dt^2}\right)\exp(\cdots)$$

$$= A\cdot\frac{1}{2t^{5/2}}\cdot\left(-3+\frac{r^2}{2Dt}\right)\exp(\cdots). \qquad (1)$$

Next,

$$r^2\frac{\partial n}{\partial r} = r^2\cdot A\,t^{-3/2}\left(\frac{-2r}{4Dt}\right)\exp(\cdots)$$

$$= -A\,\frac{r^3}{2Dt^{5/2}}\exp\left(-\frac{r^2}{4Dt}\right),$$

whence

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dn}{dr}\right) = -\frac{A}{r^2}\,\frac{3r^2}{2Dt^{5/2}}\exp(\cdots)$$
$$-\frac{A}{r^2}\,\frac{r^3}{2Dt^{5/2}}\left(\frac{-2r}{4Dt}\right)\exp(\cdots$$

$$= A\,\frac{1}{2Dt^{5/2}}\cdot\left(-3+\frac{r^2}{2Dt}\right)\exp(\cdots). \qquad (2)$$

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Since expression (1) is D times expression (2), the diffusion equation is indeed satisfied.

(b) $$\langle r^2 \rangle = \frac{\int r^2 \cdot n(r,t)\, d^3r}{\int n(r,t)\, d^3r}$$, where integrations go over all $\underline{r}$.

Cancelling the r-independent factor, we are left with

$$\langle r^2 \rangle = \frac{\int_0^\infty r^2 e^{-\alpha r^2} \cdot 4\pi r^2\, dr}{\int_0^\infty e^{-\alpha r^2} \cdot 4\pi r^2\, dr} \qquad \left(\alpha = \frac{1}{4Dt}\right).$$

Cancel (4π) and recall that

$$\int_0^\infty e^{-\alpha r^2} r^4\, dr = \frac{3}{8}\sqrt{\pi}\, \alpha^{-5/2}$$

& $$\int_0^\infty e^{-\alpha r^2} r^2\, dr = \frac{1}{4}\sqrt{\pi}\, \alpha^{-3/2}.$$

It follows that

4 $$\langle r^2 \rangle = \frac{3}{2}\alpha^{-1} = 6Dt.$$