

Physics 161: Black Holes: Lecture 7: 20 Jan 2010

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7 More Euler-Lagrange plus Gravitational redshift

7.1 Conserved quantities in the Euler-Lagrange formalism: Energy and Momentum

Note that as we did the derivations of the geodesics above we came across quantities that did not change as we traced along the affine parameter, that is we found conserved quantities. This is a general and important thing to watch for in using the Euler-Lagrange formalism. In general there will be a conserved quantity whenever the Lagrangian L does not depend explicitly on one of the variables. Actually there will be one conserved quantity per variable. This is easy to see. To be general, let's use the 4-vector notation x_μ , where $x_0 = t$, $x_1 = x$, $x_2 = y$, and $x_3 = z$, so μ runs from 0 to 3 and x_μ can represent any of the spacetime variables. If $\frac{\partial L}{\partial x_\mu} = 0$, then the Euler-Lagrange equation reads: $\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}_\mu} = 0$. Thus along the geodesic (running over values of the affine parameter λ), the quantity $\frac{\partial L}{\partial \dot{x}_\mu}$ is a constant.

In general we define p_μ , the **conjugate momentum**, of a variable x_μ , as

$$p_\mu = \frac{\partial L}{\partial \dot{x}_\mu}.$$

Thus we see that if the Lagrangian does not depend explicitly on a variable, then that variable's conjugate momentum is conserved. This p_μ may be an actual momentum, but it could be some other conserved quantity. If one takes $L = \frac{1}{2}m\dot{x}^2 - V(x)$, then in fact $p_x = mv_x$.

For the Minkowski metric, let's see what these conjugate momenta are. We saw above that the metric did not depend explicitly on any of the x_μ ; t, x, y , or z , thus we expect to have 4 conserved quantities. Consider the t equation with affine parameter proper time τ . We have $p_t = \frac{\partial L}{\partial \dot{t}} = \dot{t}$. Likewise for the x equation we find $p_x = \dot{x}$, $p_y = \dot{y}$, etc. During the above calculation we found that this constant for the Euler-Lagrange equations (which we called c_t) was equal to the Lorentz factor γ . Thus we see that

$$p_0 = p_t = \gamma = E/m,$$

where we noticed that in special relativity the energy of a particle is $E = m\gamma$, where m is the rest mass. Thus the conserved quantity associated with the t variable is nothing other than the energy. (It is energy per unit mass actually, but since γ is a constant we can multiply by another constant m and still have a constant.)

We also previously found the momentum conjugate to the x variable, $p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x} = c_x$. Using $c_x/c_t = v$ from before, we have $p_x = v_x c_t = v_x p_t = v_x \gamma$. But in special relativity the momentum is just $mv\gamma$, so see that $p_x = P_x/m$, is the momentum per unit mass. (We call the actual momentum P to

distinguish it from the momentum per unit mass.) Thus the 4 conserved quantities we discovered are just the energy and 3 components of momentum.

We can find the relation between these quantities from our definition of the Lagrangian: $L = 1 = \sqrt{\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2}$. Substituting in $p_t = \dot{t} = E/m$, $p_x = \dot{x} = P_x/m$, etc. we find

$$1 = \sqrt{(E/m)^2 - (P_x/m)^2 - (P_y/m)^2 - (P_z/m)^2}.$$

Squaring, multiplying through by m^2 , and using $P^2 = P_x^2 + P_y^2 + P_z^2$, we find the well known result $E^2 = P^2 + m^2$. Finally using units to put back the c 's, $E^2 = P^2 c^2 + m^2 c^4$, which reduces to the famous $E = mc^2$ in the limit of zero velocity (zero momentum).

7.2 Gravitational Redshift from the Schwarzschild metric

We already discussed the curvature of time in the Schwarzschild metric. Clocks run slower in a gravitation field (at smaller values of r). This is one of the more measurable effects of GR. Measurements are possible because atomic transitions act like small clocks which send messengers (photons) to us here at Earth. Recall the proper time in the Schwarzschild metric:

$$d\tau = dt \sqrt{1 - \frac{2GM}{rc^2}},$$

where $d\tau$ is tick of a clock measured at r , and dt is the length of that same tick measured far away ($r \rightarrow \infty$).

Now consider some atomic transition, for example, the $n = 3$ to $n = 2$ transition of atomic Hydrogen. This is called the H-alpha transition and results in emission of a photon with energy $E_\gamma = 13.6\text{eV}(\frac{1}{3^2} - \frac{1}{2^2}) = 1.89\text{ eV}$. Using $\lambda = c/\nu$, and $E_\gamma = h\nu$, we find the corresponding wavelength $H\alpha$: 6563.5 **Angstroms**. Now in the frame at r , the photon has a fixed frequency ν_0 , and each oscillation is like a little clock ticking with period $d\tau_0 = 1/\nu_0$. Thus the time between crests in the outgoing photon measures $d\tau$. That same photon travels out to us at $r = \infty$. Then that same period is observed as dt . Thus the $d\tau_\infty \equiv d\tau_{obs} = dt = d\tau_0/\sqrt{1 - 2GM/rc^2}$ is longer. Using the definitions above of ν and λ we see that the frequency decreases by the same factor and the more easily observed wavelength is longer at when observed at infinity than the emitted wavelength at r by

$$\lambda_{obs} = \frac{\lambda_0}{\sqrt{1 - \frac{2GM}{rc^2}}} = \frac{\lambda_0}{\sqrt{1 - \frac{r_S}{r}}},$$

where we used the definition of the Schwarzschild radius r_S .

The redshift of an emission line is defined as

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda_{obs} - \lambda_0}{\lambda_0} = (1 - \frac{r_S}{r})^{-1/2} - 1,$$

where λ_0 is the emitted wavelength in the rest frame of the atom. Note for small values of $2GM/rc^2 = r_S/r$, this expression can be Taylor expanded as $z \approx GM/rc^2 = \frac{1}{2}r_S/r$.

This gravitational redshift is totally different from either the normal Doppler shift redshift caused by an atom moving away from the observer, or the cosmological redshift caused by the expanding Universe. Let's see how big the effect is by calculating it for some typical astronomical objects. First how about

emission from the surface of the Sun? We expect the wavelengths received by us on Earth to be longer than the emitted wavelengths since the emitted photons have to climb out of the potential well of the Sun where clocks run slower. Using the approximation above $z_{\odot} \approx \frac{1}{2}3\text{km}/7 \times 10^5\text{km} \approx 2 \times 10^{-6}$. This is not a very big effect; a clock on the surface of the Sun loses about 30 seconds/year. For H-alpha the line redshift would be only 0.013 Angstroms.

How about for a white dwarf star? The mass of a white dwarf is about $0.6M_{\odot}$, and the radius is about 5500 km, near to the Earth radius. Thus $z \approx \frac{1}{2}(.6)(3\text{km})/5500\text{km} \approx 1.6 \times 10^{-4}$, and the wavelength of H-alpha emission received on Earth would be $6563.5(1+z) = 6564.6$ Angstroms, quite easily distinguishable from local H-alpha lines. This result agrees well with the measurements from the nearby white dwarf star Sirius B. A clock on the surface of Sirius B would lose about 0.6 second/hour.

Next consider the case of a neutron star, with $M = 1.4M_{\odot}$ and $R = 10$ km. Now we really should not use the approximation, but calculate $z = [1 - (1.4)(3\text{km}/10\text{km})]^{-1/2} - 1 = 0.313$. This is big shift: the H-alpha line would appear at 8618.2 Angstroms and clocks would run 31% slower.

Finally what about light emitted from the Schwarzschild radius of a black hole? With $r = r_S$, $z \rightarrow \infty$! Light would be invisible since it would be redshifted to infinitely low energies. Clocks sitting at that location would appear to have stopped!