

Physics 161: Black Holes: Lecture 21: 25 Feb 2010

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21 Rotating Black Holes: the Kerr Metric

21.1 Kerr Metric

Almost all stars rotate. Thus when massive stars collapse into black holes at the end of their lives the resulting black holes should also be rotating. Thus the Schwarzschild metric we have been using is probably not the metric of most real black holes! This was known for a long time, but not until 1963 was the metric for a rotating black hole found by Kerr. For a black hole rotating in the ϕ direction the metric in Boyer-Lindquist coordinates is:

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - \frac{4aGMr \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{(r^2 + a^2)^2 + a^2 \Delta \sin^2 \theta}{\Sigma} \sin^2 \theta d\phi^2,$$

where

$$a = J/M$$

is the angular momentum per unit mass of the spinning hole,

$$\Sigma = r^2 + a^2 \cos^2 \theta,$$

and

$$\Delta = r^2 - 2GMr + a^2.$$

Actually the charged rotating black hole metric is nearly the same. Just replace Δ by

$$\Delta = r^2 - 2GMr + a^2 + Q^2/G,$$

where Q is the charge of the hole (in the proper units (esu)), and one has to add the c 's back in to use this equation.

Let's analyze this metric some. First note what happens if we take $a = 0$. Then $\Sigma = r^2$, and $\Delta = r^2 - rr_s$, so the metric becomes

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2,$$

which is just the Schwarzschild metric.

Next, notice the cross term, $dt d\phi$. In the tensor formulation this would correspond to a non-diagonal term in $g_{\mu\nu}$. It means that there is a blending of the the coordinates ϕ and t ; the forced motion we all must take in time will somehow be blended with motion in the ϕ direction. We will come back to this in a minute. Finally, note one needs caution using these coordinates since for $a \neq 0$ surfaces of constant t and constant r do not have the metric of a 2-sphere!

21.2 Horizon of Kerr metric: maximum rotation and charge of black holes

Next, how do we find the horizon, that is, the trapped surface or distance from which light can't escape? The proper way is to find the regions of the metric where all future lightcones converge. This requires more differential geometry than we have, but earlier for the Schwarzschild metric we found the horizon by when the sign of the dr term changed. Looking at the metric above, this will happen when the sign of Δ changes. Thus the horizon is at $\Delta = 0$. Solving $\Delta = r^2 - 2GMr + a^2 = 0$, we find two solutions for the horizon of a Kerr hole:

$$r_{\pm} = GM \pm \sqrt{(GM)^2 - a^2}.$$

These are called the inner and outer horizons, and the larger one r_+ , turns out to be the trapped surface. The smaller one is called the **inner or Cauchy horizon**. Note that if we take $a = 0$, $r_+ = r_S$ as expected, and that for $a \neq 0$, r_+ is always smaller than the Schwarzschild radius. Thus rotating black holes are smaller! Since there is no θ dependence in this equation we see that the horizons are both spherical.

Also note that if $a^2 > (GM)^2$, then there is no solution. A more careful analysis shows that when $a^2 > (GM)^2$ a black hole (trapped surface) cannot form. Thus there is a maximum angular momentum that a black hole can have:

$$a_{max} = GM = \frac{r_S}{2}, \text{ or } J_{max} = \frac{GM^2}{c}$$

This is called the **maximal black hole**. Also note that from the brief description of the charged rotating black hole above, there is also a maximal charge, which can be found from the condition that a horizon exists.

$$\left(\frac{GM}{c^2}\right)^2 \geq G\left(\frac{Q}{c^2}\right)^2 + \left(\frac{J}{Mc}\right)^2,$$

where I put the c 's back in to help with the units, each term is in units of meter², and cgs units must be used (with charge in esu). If this inequality is violated no horizon can form. The maximally charged black hole is therefore non-rotating, and the maximally rotating black hole is non-charged. In practice most of the Universe is not heavily charged, and for real black holes we expect the Kerr metric to obtain, and the maximally rotating black hole to be given by J_{max} above. Note for maximal black holes the horizon is a sphere of radius $r_+ = GM = r_S/2$, half the size of a non-rotating black hole of the same mass.

21.3 Singularity of Kerr metric

We would like to find the singularity in this metric, that is, the point that corresponds to $r = 0$ in the Schwarzschild metric. The proper way to do this is to calculate the curvature, as I discussed in a previous lecture, and then look for the places where it goes to infinity. The answer, which makes sense from inspection of the metric is that the singularity occurs at $\Sigma = 0$, or $r_{\text{sing}}^2 + a^2 \cos^2 \theta = 0$. If $a = 0$, we have $r_{\text{sing}} = 0$ as we should, but if $a \neq 0$, we have the singularity only at $\theta = \pi/2$ and $r = 0$. This is peculiar, since why should it matter what θ is when $r = 0$? Actually there is a problem with the coordinates inside the black hole, which means that this is not really the right answer. Consider the region $\Sigma = 0$ in limit $M \rightarrow 0$, and $a \neq 0$. Then since there is no mass we know the metric must be equivalent to the Minkowski flat space metric; which it is in fact in spheroidal coordinates. Then $\Sigma = 0$ is just a coordinate singularity of the type we have seen before. So it takes some coordinate transformations

and fairly subtle analysis to show that the actual singularity at $\Sigma = 0$ is actually a ring of radius a in the x-y plane. For maximal black holes both the horizon and $a = r_S/2$. Thus the ring singularity is just inside the horizon. If one entered the black hole at the equator one would hit the singularity immediately, but one entered from the north pole, one could all the way to the center without hitting the singularity.