Physics 161: Black Holes: Lecture 9/10: 25/27 Jan 2010

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9 More practice with metrics: Distances around a Black Hole

Suppose you have a very powerful spaceship and fly near to a black hole. Can you notice anything different from deep space? (Actually the calculations and results we get here are also true around the Earth on a very small scale.)

Suppose we fly close to a small black hole of mass $M = 3M_\odot$. (Actually, the smallest black hole we expect to find in nature is around $3M_\odot$, so we pick this number.) We know the Schwarzschild radius for this black hole is about 3×2.95 km = 8.85km, so we are careful to stay farther away from the hole than this. Let's suppose we fly all the way around the hole and measure a distance around (circumference) of $C = 2\pi 30$km. How far are we from the hole? Naively, we expect we are 30 km from the hole, but we should check this by using the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

To find the proper distance in the θ direction, we set $dt = dr = d\phi = 0$, and the find the proper length

$$dl_\theta = \sqrt{ds^2} = r d\theta.$$

Note that this is the same as the proper distance in flat space! So there is no curvature in the θ (or ϕ) direction! To find the distance around the black hole we integrate

$$C = \int_0^{2\pi} dl_\theta = 2\int_0^{\pi} r d\theta = 2\pi r,$$

just as in flat space. Thus if we have measured the distance around as 2π 30 km, we know we are at radial coordinate $r = 30$ km.

But does that mean we are 30 km from the center of the hole? We have to use the metric again to find out. This time we want the radial direction, so we set $dt = d\theta = d\phi = 0$, and we get

$$dl_r = \sqrt{ds^2} = \frac{dr}{\left(1 - \frac{2GM}{r}\right)^{1/2}}.$$

To find the distance from radial coordinate r_1 to radial r_2 we integrate from r_1 to r_2. If we set $r_1 = 8.85$ km and $r_2 = 30$ km, we can find our distance to the black hole horizon itself.

$$\Delta l_r = \int_{r_1}^{r_2} dr \left(1 - \frac{2GM}{r}\right)^{-1/2}.$$

Defining

$$A_i \equiv \sqrt{1 - \frac{2GM}{r_i c^2}},$$

we evaluate the integral as

$$\Delta l_r = \int_{r_1}^{r_2} dl_r = r_2 A_2 - r_1 A_1 + \frac{r_S}{2} \ln\left(\frac{r_2 A_2 + r_2 - r_S/2}{r_1 A_1 + r_1 - r_S/2}\right),$$

where we used the Schwarzschild radius $r_S = 2GM/c^2 = 2.95326 M/M_\odot$ km.

Using this formula we can find the distances. For example starting at r=30 km, and moving in to r = 20 km, we we naively expect to move 10 km, but we actually move 12.51 km. This is very weird, since after moving 12.51 km inward from $r = 30$km, the distance around the black hole would be $2\pi 20$ km *not* $2\pi 17.49$ km. Thus we are seeing directly the curving of space. It is just like the example of the two surveyors, but now the curvature is not in any direction we can experience! In the above, the way we tell what value of r we are at is to travel around the hole and use the circumference. Using the above formula we also find the distance from $r_1 = 10$ km to $r_2 = 30$ is 29.50 km, and the distance from the horizon at $r_S = 8.86$ km to $r = 30$ km is 35.98 km.

Fig: embedding diagram of curved space near a black hole

We can visualize the spatial curvature around a black hole by an embedding diagram. The key in this diagram is that the radial coordinate is just the straight 3-D distance, while the proper distance is measured along the curved surface. We continue the embedding diagram even inside the black hole horizon, which turns out to be correct, though we will have to think carefully before understanding why.

Finally, note that this metric and embedding diagram work not only for black holes, but also for the Earth! If the distance to the center of the Earth is r_{Earth}, then the distance around the Earth is really not equal to $2\pi r_{\mathrm{Earth}}$! Can you figure the difference?