Physics 161: Black Holes: Lecture 10: 27 Jan 2010

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10 Distances and Times around and inside a Black Hole

Last time we used the metric to find distances around a black hole. We found that the space is flat in the θ and ϕ directions, but curved in the radial direction with proper distance $dl_r = dr/\sqrt{1 - r_S/r}$. Looking at this again we see that moving a small proper distance dl_r implies moving a smaller radial coordinate distance $dr = dl_p\sqrt{1 - r_S/r}$. This seems fine, but look at what happens right near the Schwarzschild radius r_S. When $r \to r_S$, then $dr \to 0$, that is you are not moving at all in the radial coordinate! Does this mean that you can't get into the black hole? No, because it is a square-root singularity, which implies that it is integrable.

Consider the area under the curve $y = 1/\sqrt{x}$. As $x \to 0$, $y \to \infty$, but still the area is $A = \int_0^{x_0} x^{-1/2}dx = 2x^{1/2}|_0^{x_0} = 2\sqrt{x_0}$ which is finite. Likewise $dl/dr = (1 - r_s/r)^{-1/2}$ is integrable and we gave the formula for the integration last lecture. We therefore could find the proper distance from $r = 30$ km to r_S as 35.98 km. So just because a function goes to infinity in a region of interest doesn't always mean there is a problem. Of course, sometimes it does mean there is a problem as we shall see.

10.1 Time to fall into a black hole

Next let's calculate the time to fall into a black hole. There are several ways that we will do this. First, let's return to our geodesics for radial infall. Remember we considered the case with angular momentum $l = 0$, and found that starting at rest from $r = \infty$ meant the conserved energy was $E = mc^2$, and the equation of motion is

$$\frac{1}{2}m(\frac{dr}{d\tau})^2 - \frac{GMm}{r} = 0.$$

Let's find the proper time τ it takes for this freely falling object to go from $r_0 = 30$km to $r = r_S = 8.85$ km, the Schwarzchild radius of a 3 $M_\odot$ black hole. This is the time as measured by the wristwatch of the falling guy. We take the negative square root of this equation since we want to fall in (r decreasing):

$$\frac{dr}{d\tau} = -\sqrt{2GM}r^{-1/2},$$

or

$$\int_{30}^{r} dr r^{1/2} = \int_0^\tau -\sqrt{2GM}d\tau,$$

or

$$\frac{2}{3}r^{3/2}|_{r_0}^{r} = -\tau\sqrt{2GM},$$

or

$$\tau = \frac{2}{3c}(1/\sqrt{2GM/c^2})(r_0^{3/2} - r^{3/2}).$$

(Note I put the c back in to get the units right and make calculation easier.)) So to go from $r_0 = 30$ km to $r = r_S = 8.85$ km in our $3M_\odot$ black hole takes $\tau = \frac{2}{3}(1/\sqrt{(2.95)(3)\text{km}})(30^{3/2} - 8.85^{3/2})/3 \times 10^5\text{km/s} = 1.03 \times 10^{-4}$ s. Thus it takes about 0.1 millisecond! It is not clear yet, but this same equation works inside the black hole, so we can also find how long the falling guy has to live before hitting the singularity at the center. Just taking $r = 0$ in the above equation gives

$$\tau = \frac{2}{3}\frac{r_0^{3/2}}{\sqrt{2GM}} = 0.124\text{ms}.$$

Thus our guy gets only an extra 0.021 ms to live inside the black hole!

Now this is the time freely falling starting at infinity. We could also find the time to fall in if we started from rest at $r = 30$ km. For this we go back to the more general geodesic equation before plugging in $E = mc^2$,

$$m(\frac{dr}{d\tau})^2 = \frac{E^2}{mc^2} - \left(1 - \frac{2GM}{rc^2}\right) mc^2 = 0.$$

Now starting at $\tau = 0$, $dr/d\tau = 0$, and $r = r_0$, we find the conserved energy is $E = \pm mc^2\sqrt{1 - 2GM/r_0c^2}$. For falling in we take $E < 0$ then integrate the above equation as before. This can be done and we find a more complicated formula. The time to fall into $r = 0$ is a little simplier:

$$\tau = \frac{\pi}{2} r_0 \sqrt{\frac{r_0}{r_S}},$$

for a time of 0.29 ms, about three times longer than when you start falling from far away.

Now what does this look like to someone watching from far away. They don't use the proper time τ, but use the coordinate or "far away" time t. We can convert the equations above to coordinate time t by using our time geodesic equation: $\left(1 - \frac{2GM}{r}\right)\dot{t} = \frac{E}{m}$, or $dt/d\tau = (E/m)/(1 - r_S/r)$. Then

$$dr/dt = (dr/d\tau)/(dt/d\tau) = -\frac{m}{E}(r_S/r)^{1/2}(1 - \frac{r_S}{r}).$$

This equation can be solved as before, but we will find some trouble in doing it as we get close to r_S. Consider that limit, $r \to r_S$. Then $1 - r_S/r \to 0$, and a tiny step in dr means an infinite step in dt. This is different than the case before with proper distance because this is not an integrable square root singularity. This is real infinity that cannot be integrated over. In fact, if you try to do the integral you will find you *never* get to $r = r_S$! Time slows down and motion ceases. Everything hangs up at the horizon.