Physics 161: Black Holes: Lecture 14: 5/8 Feb 2008

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14 Light bending and gravitational lenses

14.1 Formulas

Let's calculate the result of the most famous GR experiment. The simple formula we are going to derive for light bending is also the basis of all gravitational lensing, which has become a major industry in modern astrophysics. We start with the geodesics we found previously for light.

$$\frac{dr}{d\lambda} = \pm\sqrt{E^2 - (1 - \frac{r_S}{r})\frac{l^2}{r^2}},$$

and

$$\frac{d\phi}{d\lambda} = \frac{l}{r^2}.$$

Here we haven't really discussed what λ is, so let's get rid of it by dividing the second equation by the first: $d\phi/dr = (d\phi/d\lambda)/(dr/d\lambda)$:

$$\frac{d\phi}{dr} = \pm\frac{l}{r^2}\left(E^2 - (1 - \frac{r_S}{r})\frac{l^2}{r^2}\right)^{-1/2} = \frac{\pm 1}{r^2\sqrt{\frac{E^2}{l^2} - (1 - \frac{r_S}{r})\frac{1}{r^2}}}.$$

This is an equation for the angular deviation of the light ray ϕ, as a function of the radial coordinate, r. The good thing is that only the combination l/E appears. We can define this constant as

$$b = \frac{l}{E},$$

and call it the **impact parameter**, or distance of closest approach. This makes sense since for light we expect the angular momentum to be $l = pr\sin\phi$, and $E = p$, so $b = l/E = r\sin\phi$, which is just the normal definition of the impact parameter for something being fired at an object from far away.

<div align="center">Fig: Definition of Impact parameter,light bending trajectory</div>

This equation tells us how the angle of the light ϕ changes as we approach and then recede from a spherical mass. There are several possible trajectories, including circular, and inspiralling, but for now let's consider trajectories that start far away, come close, and then leave to far away on the other side. We would like to calculate by angle the light is bent because of its passage near to the massive object. The

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first step is a change of variables from r to $u = r^{-1}$. Then $r = 1/u$ and $dr = -du/u^2$. The differential equation is changed using $d\phi/du = (d\phi/dr)(dr/du)$, with $dr/du = -1/u^2$. Thus we get

$$\frac{d\phi}{du} = \pm \left(\frac{1}{b^2} - u^2 + r_S u^3 \right)^{-1/2}.$$

To find the angle we integrate this starting at $r = \infty$ ($u = 0$) with $\phi = \pi$, going to $r = b$ ($u = 1/b$), and ending at $r = \infty$ ($u = 0$) with $\phi \approx 0$. We see a problem, in that the integration variable goes from 0 to 0. We solve this problem by noting that by symmetry the angle of light bending from infinity to b, must be the same as the angle of light bending from b to infinity. So we can just go from $u = 0$ to $u = 1/b$, and then multiply our final answer by two.

$$\phi = \int_0^{1/b} \left(\frac{1}{b^2} - u^2 + r_S u^3 \right)^{-1/2} du.$$

This is the complete answer. We can do this integral numerically for any values of b and get all the possible light orbits. However, for almost all cases, the angle of light bending is small and we can find a very important analytic formula. This formula is the one almost always used in graviational lensing. To find it we note that usually the quantity $r_S/r \ll 1$, or equivalently $r_S u^3 \ll u^2$. This will be true if the impact parameter b is much larger tha r_S which is usually true. In this case we can turn this integral into one we can do analytically. We change variables again to y, where

$$y^2 = u^2 - r_S u^3,$$

so $y = (u^2 - r_S u^3)^{1/2} = u(1 - r_S u)^{1/2} \approx u(1 - r_S u/2)$. Then $dy \approx du - r_S u \, du = (1 - r_S u) du$, and $du/dy \approx (1 - r_S u)^{-1} \approx 1 + r_S u \approx 1 + r_S y$. We do all this so that we can make the integration variable y so we can do the integral, but also keep the first order term that has an r_S in it. The integral becomes

$$\phi = \int_0^{1/b} \left(\frac{1}{b^2} - y^2 \right)^{-1/2} \frac{du}{dy} dy \approx \int_0^{1/b} \frac{1 + r_S y}{(\frac{1}{b^2} - y^2)^{1/2}} dy.$$

This integral can be easily evaluated or looked up in a table to give

$$\phi \approx \sin^{-1}(by)|_0^{\frac{1}{b}} - r_S \sqrt{\frac{1}{b^2} - y^2}\Big|_0^{\frac{1}{b}},$$

or $\phi \approx \frac{\pi}{2} + \frac{r_S}{b}$. The total change in ϕ is twice this or $\pi + 2r_S/b$, for a total light bending angle of

$$\Delta\phi = \frac{2r_S}{b} = \frac{4GM}{bc^2}.$$

We can compare this answer with the experiment Eddington first did. During a total solar eclipse one can see stars near the Sun. For a light ray going near the limb of the Sun the impact parameter would be nearly the Sun's radius, $b = 7 \times 10^5$ km, and we know $r_S = 3$ km for the Sun, so $\Delta\phi = 8.5 \times 10^{-6}$ radians, or 1.75 arcsec. This is close to the value found by Eddingon, and was the first real test of GR. This formula is the basis of all gravitational lensing!

How is this test done? Thing of a bunch of stars about 1/2 degree apart on the sky. At night you carefully measure the angular distance between them. Now suppose the Sun, during the day, comes

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through that area. Well you can't see the stars because the Sun lights up the sky. But during a total eclipse of the Sun, the Moon blocks out the Sun and you can see the stars. Thus you can measure the effect on the starlight of passing by the Sun. How do the stars appear? They appear farther apart than when the Sun is not present, because the light bends around the Sun. We always think in our minds that light goes straight, so we extrapolate back to the wrong position when the light bends. We see from this that the 1.75 arcsec is not really right. The stars farther from the limb of the Sun bend less than 1.75 arcsec. In fact, what would a single circularly blob of light (e.g. a galaxy) look like when it is gravitationally lensed like this? The light from the part of the blob closest to the Sun will bend more than the light from the far edge of the blob. Thus the circular blob of light will be squashed radially. It will also be stretched tangentially. This is the hallmark of gravitationally lensed objects and can be used to tell whether or not something has been lensed. The limit of this is when the blob of light is directly behind a very small lens which is powerful enough. Then the circular blob of light is stretched into a ring of light around the lens. This is called the Einstein ring, and has been observed. What determines the radius of the ring? It is easily derived from the light bending formula above. It depends on r_S which depends only on the mass of the lens. Thus if we see an Einstein ring, we can know the mass of the lens, even if the lens mass itself is not shining and is invisible. This is one was to search for the dark matter. There is one complication, in that the Einstein radius also depends on the distance of the lens. We may come back to this.

14.2   Powerpoint presentation on Gravitational Lensing

Gravitational Lensing is divided into divided into two basic catagories: strong lensing when two or more images of the source appears and weak lensing, where the image is squashed and magnified, but no additional image is made. These are used for various purposes in astrophysics.

- Strong lensing of galaxies: Einstein rings

- Strong lensing of quasars, Einstein crosses, time delays, dark matter substructure

- weak lensing and dark matter

- weak lensing and the bullet cluster merger proof of dark matter

- Microlensing search for MACHOs

- Microlensing search for extra-solar planets